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LOCAL COMPACTNESS AND CARTESIAN PRODUCTS OF QUOTIENT MAPS AND k -SPACES

by Ernest MICHAEL ⁽¹⁾

1. Introduction.

In 1948, J.H.C. Whitehead [8; Lemma 4] proved that, if X is locally compact Hausdorff, then the Cartesian product ⁽²⁾ $i_X \times g$ is a quotient map ⁽³⁾ for every quotient map g . Using this result, D.E. Cohen proved in [1; 3.2] that, if X is locally compact Hausdorff, then $X \times Y$ is a k -space ⁽⁴⁾ for every k -space Y . The principal purpose of this note is to show that these results are the best possible, in the sense that, if a regular space X is not locally compact, then the conclusions of both results are false. (That the conclusions are false without *some* restrictions on X is well known; see, for instance, Bourbaki [2, p. 151, Exercise 6] and C.H. Dowker [4; p. 563]).

Our main results are formally stated and proved in sections 2 and 3, while section 4 contains analogous results for sequential spaces, and section 5 considers the special case where X is metrizable.

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⁽²⁾ If $f_i: X_i \rightarrow Y_i$ ($i = 1, 2$), the product $f_1 \times f_2: X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is defined by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$. We use i_X to denote the identity map on X .

⁽³⁾ A map $f: X \rightarrow Y$ is a *quotient* map if a set $V \subset Y$ is open in Y if and only if $f^{-1}(V)$ is open in X .

⁽⁴⁾ A topological space X is a k -space if a subset A of X is closed whenever $A \cap K$ is closed in K for every compact $K \subset X$. All locally compact spaces and all first-countable spaces are k -spaces.

I am grateful to S.P. Franklin and A.H. Stone for a valuable conversation over a Mexican dinner during an Arizona sandstorm.

2. Products of quotient maps.

THEOREM 2.1. — *The following properties of a regular ⁽⁵⁾ space X are equivalent.*

- (a) X is locally compact.
- (b) $i_X \times g$ is a quotient map for every quotient map g .
- (c) $i_X \times g$ is a quotient map for every closed compact-covering ⁽⁶⁾ map g with domain and range paracompact k -spaces.

Proof. — The implication (a) \rightarrow (b) is the theorem of J.H.C. Whitehead quoted in the introduction, and (b) \rightarrow (c) is obvious because continuous closed maps are quotient maps. It remains to prove (c) \rightarrow (a).

Suppose X is not locally compact at some $x_0 \in X$. Let $\{U_\alpha\}_{\alpha \in A}$ be a local base at x_0 . Then, for all $\alpha \in A$, the closure \bar{U}_α is not compact, and thus has a well ordered family $\{F_\lambda\}_{\lambda < \lambda(\alpha)}$ of non-empty closed subsets whose intersection is empty ⁽⁷⁾. We assume that the collection of all the well-ordered index sets $\Lambda_\alpha = \{\lambda : \lambda \leq \lambda(\alpha)\}$, with $\alpha \in A$, is disjoint. Topologize each Λ_α with the order topology, which makes it compact Hausdorff. Let Λ denote the topological sum $\sum_{\alpha \in A} \Lambda_\alpha$, and let Y be the space obtained from Λ by identifying all the final points $\lambda(\alpha) \in \Lambda_\alpha$ to a single point $y_0 \in Y$. Clearly Λ is a paracompact k -space, and it is easy to check directly that so is Y . Let $g: \Lambda \rightarrow Y$ be the quotient map. Clearly g is closed, and g is compact-covering because every compact subset of Y is contained in the union of

⁽⁵⁾ I am grateful K. A. Baker for informing me that, while our proof of (c) \rightarrow (a) makes essential use of regularity, (b) \rightarrow (a) can nevertheless be proved for all Hausdorff spaces X by constructing a separate proof in case X is not regular. I don't know whether (c) \rightarrow (a) remains true for all Hausdorff X .

⁽⁶⁾ A continuous map $f: X \rightarrow Y$ is *compact-covering* if every compact subset of Y is the image of some compact subset of X .

⁽⁷⁾ This follows from [6; p. 163 H] and the fact that every simply ordered set has a cofinal well-ordered subset.

finitely many $g(\Lambda_\alpha)$. It remains to show that $h = i_X \times g$ is not a quotient map.

For each $\alpha \in A$ and $\lambda \in \Lambda_\alpha$, let $E_\lambda = \bigcap_{\nu < \lambda} F_\nu$. Then $E_{\lambda(\alpha)} = \emptyset$, and $E_\lambda \supset F_\lambda \neq \emptyset$ if $\lambda < \lambda(\alpha)$. For each $\alpha \in A$, define $S_\alpha \subset X \times \Lambda_\alpha$ by

$$S_\alpha = \bigcup \{E_\lambda \times \{\lambda\} : \lambda \in \Lambda_\alpha\}.$$

Then S_α is clearly closed in $X \times \Lambda_\alpha$. Define $S \subset X \times Y$ by

$$S = \bigcup_{\alpha \in A} h(S_\alpha).$$

Let us show that $h^{-1}(S)$ is closed in $X \times \Lambda$, but that S is not closed in $X \times Y$.

To see that $h^{-1}(S)$ is closed in $X \times \Lambda$, it suffices to check that $h^{-1}(S) \cap (X \times \Lambda_\alpha)$ is closed in $X \times \Lambda_\alpha$ for all α . But, since $E_{\lambda(\alpha)} = \emptyset$ for all α ,

$$h^{-1}(S) \cap (X \times \Lambda_\alpha) = S_\alpha,$$

and S_α is indeed closed in $X \times \Lambda_\alpha$.

To see that S is not closed in $X \times Y$, note first that $(x_0, y_0) \notin S$. However, if $U \times V$ is a neighborhood of (x_0, y_0) in $X \times Y$, then $\bar{U}_\beta \subset U$ for some $\beta \in A$; if we pick $\lambda \in g^{-1}(V) \cap \Lambda_\beta$ with $\lambda \neq \lambda_\beta$, then

$$\emptyset \neq h(E_\lambda \times \{\lambda\}) \subset (U \times V) \cap S.$$

Hence $(x_0, y_0) \in \bar{S}$, and that completes the proof.

3. Products of k-spaces.

THEOREM 3.1. — *The following properties of a regular ⁽⁵⁾ space X are equivalent.*

- (a) X is locally compact.
- (b) $X \times Y$ is a k-space for every k-space Y .
- (c) $X \times Y$ is a k-space for every paracompact k-space Y .

Proof. — The implication (a) \rightarrow (b) is the result of D.E. Cohen quoted in the introduction, and (b) \rightarrow (c) is obvious. It remains to prove (c) \rightarrow (a).

Suppose X is not locally compact. Then Theorem 2.1 implies that there exists a compact-covering map $g: \Lambda \rightarrow Y$, with Y a paracompact k -space, such that $i_X \times g$ is not a quotient map. Since g is compact-covering, so is $i_X \times g$. Now it is easy to show [7; Lemma 11.2] that any compact-covering map whose range is a Hausdorff k -space must be a quotient map. Since $i_X \times g$ is not a quotient map, it follows that $X \times Y$ is not a k -space. That completes the proof.

4. Two analogous results.

S. P. Franklin has pointed out that Theorems 2.1 and 3.1 have simple analogues in case the domain of g in Theorem 2.1, or the space Y in Theorem 3.1, are assumed to be sequential. Recall that a space Y is called *sequential* [5] if a subset A of Y is closed whenever $A \cap S$ is closed in S in for every convergent sequence (including the limit) S in Y . Since such S are compact, every sequential space is clearly a k -space. Moreover, quotients of sequential spaces are always sequential, and sequential spaces are precisely the quotients of (locally compact) metrizable spaces (see [5]).

For each infinite cardinal m , let D_m denote the discrete space of cardinality m , let Y_m be the quotient space obtained from $D_m \times [0, 1]$ by identifying all points in $D_m \times \{0\}$ (i.e. Y_m is the cone over D_m), and let $g_m: D_m \times [0, 1] \rightarrow Y_m$ be the quotient map.

By the *pointwise weight* of a space X we will mean the smallest cardinal m such that each $x \in X$ has a neighborhood base of cardinality $\leq m$.

THEOREM 4.1. — *The following properties of a regular space X are equivalent.*

- a) X is locally countably compact.
- b) $i_X \times g$ is a quotient map for every quotient map g with sequential domain.
- c) $i_X \times g_m$ is a quotient map, where m is the pointwise weight of X .

Proof. — (a) \rightarrow (b). This proof goes just like J. H. C. Whitehead's proof [8; Lemma 4] that (a) \rightarrow (b) in Theorem 2.1. In fact, Whitehead's proof is based on the fact that if U is an open subset of a product space $E \times F$, and if $C \subset F$ is compact, then $\{x \in E : \{x\} \times C \subset U\}$ is open in E . It is easy to check that, if E is sequential, this conclusion remains valid if C is only assumed to be countably compact.

(b) \rightarrow (c) Obvious.

(c) \rightarrow (a) Suppose X is not locally countably compact. Examining the proof of Theorem 2.1, one sees that then there are only m space Λ_α , and each Λ_α can be chosen to be a convergent sequence or, if one prefers, a closed interval. In the latter case, the map g constructed in the proof of Theorem 2.1 is precisely g_m . That completes the proof.

THEOREM 4.2. — *The following properties of a regular sequential space X are equivalent.*

- a) X is locally countably compact.
- b) $X \times Y$ is sequential for every sequential space Y .
- c) $X \times Y_m$ is a k -space, where m is the pointwise weight of X .

Proof. — (a) \rightarrow (b). This follows immediately from T. K. Boehme [1; Theorem] and S. P. Franklin [5; Proposition 1.10].

(b) \rightarrow (c). Obvious.

(c) \rightarrow (a). This follows from 4.1 (c) \rightarrow (a) in the same way that 3.1 (c) \rightarrow (a) followed from 2.1 (c) \rightarrow (a). That completes the proof.

BIBLIOGRAPHIE

- [1] T. K. BOEHME, Linear s -spaces, Proc. Symp. Convergence Structures, U. of Oklahoma, 1965.
- [2] N. BOURBAKI, Topologie Générale, Chapters 1 and 2, 3rd ed., Hermann, 1961.
- [3] D. E. COHEN, Spaces with weak topology, *Quart. J. Math.*, Oxford Ser. (2) 5 (1954), 77-80.

- [4] C. H. DOWKER, Topology of metric complexes, *Amer. J. Math.*, 74 (1952), 555-577.
- [5] S. P. FRANKLIN, Spaces in which sequences suffice, *Fund. Math.*, 57 (1965), 107-115.
- [6] J. L. KELLEY, *General Topology*, VanNostrand, (1955).
- [7] E. MICHAEL, \aleph_0 -spaces, *J. Math. Mech.*, 15 (1966), 983-1002.
- [8] J. H. C. WHITEHEAD, A note on a theorem of Borsuk, *Bull. Amer. Math. Soc.*, 54 (1958), 1125-1132.

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