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## ON THE ABSOLUTE CESARO SUMMABILITY FACTORS OF FOURIER SERIES (\*)

by NIRANJAN SINGH

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**1.1. DEFINITIONS.** — Let  $\sum a_n$  be a given infinite series with  $S_n$  as its  $n$ -th partial sum. The series  $\sum a_n$  is said to be absolutely summable  $(C, \alpha)$ , or summable  $|C, \alpha|$ , if the sequence  $\{\sigma_n^\alpha\}$  is of bounded variation, that is

$$\sum_n |\sigma_n^\alpha - \sigma_{n-1}^\alpha| < \infty,$$

where  $\{\sigma_n^\alpha\}$  is the  $n$ -th Cesàro mean of order  $\alpha$ ,  $\alpha > -1$ , of the sequence  $\{S_n\}$ .

If  $\{t_n^\alpha\}$  be the  $n$ -th Cesàro mean of order  $\alpha$  of the sequence  $\{na_n\}$ , then we have the following identity [6].

$$t_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha).$$

For any sequence  $\{u_n\}$ , we write

$$\Delta u_n = u_n - u_{n+1}$$

and

$$\Delta^r u_n = \sum_{p=0}^{\infty} A_p^{-r-1} u_{n+p},$$

provided the series on the right converges.

If  $S$  is a +ve integer, then

$$\Delta^S (u_n v_n) = \sum_{r=0}^S \binom{S}{r} \Delta^r u_n \Delta^{S-r} v_{n+r}$$

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By repeated partial summation, we observe that, for  $k = 0, 1, \dots$

$$\sum_{p=0}^q A_{n-p}^{r-1} u_p a_p = \sum_{p=0}^q S_p^k \Delta^{k+1} (A_{n-p}^{r-1} u_p) + \sum_{j=0}^k \Delta^j (A_{n-q-1}^{r-1} u_{q+1}) S_q^j$$

where  $S_n^k$  denotes the  $n$ -th Cesàro sum of order  $k$  of the sequence  $\{S_n\}$ . Hence, putting  $q = n$ , we get

$$(1.1.1) \quad \sum_{p=0}^n A_{n-p}^{r-1} u_p a_p = \sum_{p=0}^n S_p^k \Delta^{k+1} (A_{n-p}^{r-1} u_p).$$

A sequence  $\{\lambda_n\}$  is said to be convex, if  $\Delta^2 \lambda_n \geq 0$ , and it is said to be hyper-convex of order  $h$ , if

$$\Delta^{h+2} \lambda_n \geq 0, \quad (h = 0, 1, 2, \dots).$$

By definition hyper-convexity of order zero is the same as convexity.

Let  $f(t)$  be a periodic function with period  $2\pi$  and integrable in the sense of Lebesgue over  $(-\pi, \pi)$ . Without any loss of generality we may assume that the constant term in the Fourier series of  $f(t)$  is zero, that is

$$\int_{-\pi}^{\pi} f(t) dt = 0,$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t).$$

We use the following notations :

$$\begin{aligned} \Phi(t) &= \frac{1}{2} \{f(x+t) + f(x-t) - 2f(x)\}, \\ \Phi_\alpha(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \Phi(u) du, \quad \alpha > 0, \\ \Phi_0(t) &= \Phi(t), \\ \varphi_\alpha(t) &= \Gamma(\alpha+1) t^{-\alpha} \Phi_\alpha(t), \quad \alpha \geq 0, \\ (F(t))_h &= \frac{\partial^h F(t)}{\partial t^h}, \\ \varepsilon(n) &= (\log n)^{-\beta}, \quad \beta \geq 0, \\ \varepsilon^{-1}(n) &= \frac{1}{\varepsilon(n)}. \end{aligned}$$

1.2. Various results on summability factors of Fourier series due to Prasad [12], Izumi and Kwata [5], Cheng [3], Pati [8] and Dikshit [4] were generalized by Pati and Sinha [11] in the form of the following theorem.

**THEOREM A.** — *Let  $h$  be an integer  $\geq 0$ , and let  $\{\lambda_n\}$  be a monotonic non-increasing sequence when  $h = 0$ , and a hyper-convex sequence of order  $(h - 1)$  when  $h \geq 1$ , such that*

$$(i) \quad \sum \frac{\lambda_n}{n} < \infty, \quad (ii) \quad \sum n^h \Delta^{h+1} \lambda_n < \infty.$$

If

$$\int_0^t |\varphi_h(u)| du = o(t),$$

as  $t \rightarrow 0$ , then  $\sum_{n=1}^{\infty} \lambda_n A_n(x)$  is summable  $|C, h + 1 + \delta|$  for every  $\delta > 0$ .

Later on Ahmad [1] obtained the following theorem which includes as a special case for  $\beta = 0$  the above theorem of Pati and Sinha.

**THEOREM B.** — *Let  $\{\lambda_n\}$  be a sequence such that for all non-negative integral values of  $h$ ,  $\Delta^{h+1} \lambda_n \geq 0$ , and  $\sum \frac{\lambda_n}{n} < \infty$ . If*

$$\int_0^t |\varphi_h(u)| du = o\left\{t^{\varepsilon-1} \left(\frac{1}{t}\right)\right\},$$

as  $t \rightarrow 0$ , then  $\sum_{n=1}^{\infty} \varepsilon(n+1) \lambda_n A_n(x)$  is summable  $|C, h + 1 + \delta|$  for every  $\delta > 0$ .

In this paper we prove the following theorem for summability  $|C, 1 + h|$  by imposing suitable conditions on the sequence  $\{\lambda_n\}$ .

We prove the following theorem.

**THEOREM.** — *Let  $\{\lambda_n\}$  be a sequence such that for non-negative integral values of  $h$ ,  $\Delta^{h+1} \lambda_n \geq 0$ , and*

$$(1.2.1) \quad \sum \frac{\lambda_n}{n} (\log n)^{\frac{1}{2}} < \infty.$$

If

$$(1.2.2) \quad \int_0^t |\varphi_h(u)| du = O\left\{t\varepsilon^{-1}\left(\frac{1}{t}\right)\right\}, \quad t \rightarrow 0,$$

then,  $\sum_1^\infty \varepsilon(n+1)\lambda_n A_n(x)$  is summable  $|C, h+1|$ .

It may be remarked that this theorem generalizes the following theorem of the author [14] which in turn, includes a theorem of Pati [10].

**THEOREM C.** — Let  $\{\lambda_n\}$  be a convex sequence such that  $\sum \frac{\lambda_n}{n} (\log n)^{\frac{1}{2}} < \infty$ .

If

$$\int_0^t |\Phi(u)| du = O\left\{t\varepsilon^{-1}\left(\frac{1}{t}\right)\right\},$$

as  $t \rightarrow 0$ , then  $\sum_1^\infty \varepsilon(n+1)\lambda_n A_n(x)$  is summable  $|C, 1|$ .

**1.3.** For the proof of our theorem we require the following lemmas :

**LEMMA 1** [9]. — Let  $C_{n,\rho}^k$  and  $S_n^k(t)$  denote the  $n$ -th Cesàro-sums of order  $k$  corresponding to the series  $\sum_1^\infty (-1)^n n^\rho$  and  $\sum_1^\infty (\sin nt)_{n+1}$  ( $h \geq 0$ ), respectively, then

$$\begin{aligned} \text{(i)} \quad & C_{n,\rho}^k = O(n^k) \quad k \geq \rho \\ \text{(ii)} \quad & S_n^k(t) = O(n^{k+h+2}) \quad \left(0 < t \leq \frac{1}{n}\right), \quad k \geq 0 \\ & = O(n^{h+1}t^{-k-1}) + O(n^k t^{-h-2}), \quad (n^{-1} < t \leq \pi)k \geq 0. \end{aligned}$$

**LEMMA 2** [2]. — If  $k \geq -1, r \geq 0$ , necessary and sufficient conditions for  $\sum a_n \varepsilon_n$  to be summable  $|C, r|$  whenever

$$S_n = a_0 + a_1 + \dots + a_n = O(1)(C, k)$$

are

$$\begin{aligned} \text{(i)} \quad & \sum n^{k-r} |\varepsilon_n| < \infty, \\ \text{(ii)} \quad & \sum n^{-1} |\varepsilon_n| < \infty, \\ \text{(iii)} \quad & \sum n^k \left| \Delta^{k+1} \varepsilon_n \right| < \infty. \end{aligned}$$

LEMMA 3 [1]. — Let  $R_n^k(t)$  denote the  $n$ -th Cesàro sum of order  $k$  ( $0 \leq k < h + 1$ ) of the series  $\sum_1^\infty \varepsilon(n + 1) (\sin nt)_{h+1}$  ( $h \geq 0$ ), then

- (i)  $R_n^k(t) = O\{\varepsilon(n + 1)n^{k+h+2}\} \quad \left(0 < t \leq \frac{1}{n}\right),$
- (ii)  $R_n^k(t) = O\{\varepsilon(n + 1)n^{h+1}t^{-k-1}\} \quad (n^{-1} < t \leq \pi).$

LEMMA 4 [1]. — If (1.2.2) holds, then

$$\int_{\frac{1}{n}}^\pi t^{-1} |\varphi_h(t)| dt = O\{\varepsilon^{-1}(n + 1) \log n\}.$$

LEMMA 5 [1]. — Let  $h$  be a positive integer, and  $\{\lambda_n\}$  be a sequence such that  $\Delta^h \lambda_n \geq 0$ , and  $\sum \frac{\lambda_n}{n} < \infty$ , then

- (a)  $\Delta^r \lambda_n \downarrow \quad (r = 0, 1, \dots, h - 1).$
- (b)  $\lambda_n = \begin{cases} \sum_{m=n}^\infty \Delta \lambda_m & \text{for } h = 1 \\ \left(\frac{1}{h-1}\right)^{-1} \sum_{m=n}^\infty (m - n + 1)(m - n + 2) \dots \\ \quad (m - n + h - 1) \Delta^h \lambda_m & (h > 1) \end{cases}$
- (c)  $\sum m^{r-1} \Delta^r \lambda_m < \infty \quad (r = 1, 2, \dots, h - 1).$

LEMMA 6 [11]. — Let  $\{\lambda_n\}$  be a hyper-convex sequence of order  $(h - 1)$  when  $h \geq 1$ , or monotonic non-increasing when  $h = 0$ , such that

$$\sum \frac{\lambda_n}{n} < \infty.$$

If

$$\sum n^h \Delta^{h+1} \lambda_n < \infty,$$

then

$$\sum \log(n + 1) n^h \Delta^{h+1} \lambda_n < \infty.$$

LEMMA 7 [13]. — If

$$\int_0^t |\varphi_\alpha(u)| du = O\left\{t \left(\log \frac{1}{t}\right)^\beta\right\},$$

then

$$\sum_{m=0}^n |\sigma_m^{\alpha}|^2 = O\{n(\log n)^{2\beta+1}\} \quad \text{for } \beta > -\frac{1}{2}$$

and  $\alpha \geq 0$  where  $\sigma_m^{\alpha}$  is the  $m$ -th  $(C, \alpha)$  mean of the series  $\sum A_n(x)$ .

LEMMA 8. — We have for  $r = 0, 1, \dots, h$

$$\Delta^{h+1-r} \{(\mu + r)\varepsilon_{\mu+r+1}\} = O\left\{\frac{(\mu + 1)^{r-h}\varepsilon_{\mu+1}}{\log(\mu + 1)}\right\}.$$

*Proof.* — Since  $\Delta^p(\mu + r) = 0$  for  $p \geq 2$  we have

$$\begin{aligned} \Delta^{h+1-r} \{(\mu + r)\varepsilon_{\mu+r+1}\} &= \sum_{p=0}^{h+1-r} \binom{h+1-r}{p} \Delta^p(\mu + r) \Delta^{h+1-r-p} \varepsilon_{\mu+r+1} \\ &= (\mu + r) \Delta^{h+1-r} \varepsilon_{\mu+r+1} - (h+1-r) \Delta^{h-r} \varepsilon_{\mu+r+2} \\ &= O\left\{\frac{(\mu + 1)^{r-h}\varepsilon_{\mu+1}}{\log(\mu + 1)}\right\}. \end{aligned}$$

1.4. *Proof of the Theorem.* — Since

$$\begin{aligned} A_n(x) &= \frac{2}{\pi} \int_0^\pi \Phi(t) \cos nt \, dt \\ &= \frac{2}{\pi} \left[ \sum_{\rho=1}^h (-1)^{\rho-1} \Phi_\rho(t) (\cos nt)_{\rho-1} \right]_0^\pi \\ &\quad + (-1)^h \frac{2}{\pi} \int_0^\pi \Phi_h(t) (\cos nt)_h \, dt \\ &= A_{n,1}(x) + A_{n,2}(x), \quad \text{say.} \end{aligned}$$

Thus by virtue of the consistency theorem for absolute Cesàro-summability, it is sufficient for our purpose, to prove that each of the series

$$(1.4.1) \quad \sum_{n=1}^{\infty} \varepsilon(n+1) \lambda_n A_{n,1}(x),$$

and

$$(1.4.2) \quad \sum_{n=1}^{\infty} \varepsilon(n+1) \lambda_n A_{n,2}(x),$$

is summable  $[C, h+1]$ .

Now since  $\sin n\pi = 0$  and  $\cos n\pi = (-1)^n$ , for proving

the summability  $|C, h + 1|$  of (1.4.1), it is enough to show that if  $\rho$  is an odd integer,  $1 \leq \rho \leq h$ ,

$$\sum_{n=1}^{\infty} \varepsilon(n + 1)\lambda_n(-1)^n n^{\rho-1} \quad \text{is summable} \quad |C, h + 1|.$$

Taking the series  $\sum a_n$  in lemma 2 to be  $\sum(-1)^n n^{\rho-1}$ ,  $r = h, k = h - 1$ , we have from lemma 1.

$$C_{n,\rho-1}^{h-1} = O(n^{h-1}).$$

Also by taking  $\varepsilon_n$  to be  $\lambda_n \varepsilon_{n+1}$  we find that conditions (i) and (ii) of lemma 2 are satisfied. Also

$$\Sigma n^{h-1} \left| \Delta^h \left( \frac{\lambda_n}{(\log n + 1)} \beta \right) \right| = O \left\{ \sum_{n=1}^{\infty} \sum_{r=0}^h n^{r-1} \Delta^r \lambda_n \right\} = O(1),$$

by virtue of part (c) of lemma 5. Finally applying lemma 2 we find that  $\sum \lambda_n \varepsilon_{n+1} (-1)^n n^{\rho-1}$  is summable  $|C, h|$  and consequently summable  $|C, h + 1|$ .

Also the summability  $|C, h + 1|$  of the series (1.4.2) is equivalent to the assertion that

$$(1.4.3) \quad \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^{\pi} \phi_h(t) L_n^{h+1}(t) dt \right| < \infty,$$

where

$$L_n^{h+1}(t) = \frac{t^h}{A_{n+1}^{h+1}} \sum_{\nu=0}^n A_{n-\nu}^h \varepsilon(\nu + 1) \lambda_{\nu} (\sin \nu t)_{h+1}.$$

*Proof of (1.4.3).* — We have

$$\Sigma \equiv \sum_{\nu=1}^n A_{n-\nu}^h \varepsilon(\nu + 1) \lambda_{\nu} (\sin \nu t)_{h+1}.$$

Applying the process of repeated summation we have in the notation of Lemma 3,

$$\begin{aligned} \Sigma &= \sum_{\nu=1}^n R_{\nu}^h(t) \Delta^{h+1} (A_{n-\nu}^h \lambda_{\nu}) \\ &= \sum_{r=0}^h \binom{h+1}{r} \sum_{\nu=1}^n A_{n-\nu}^{h-r} \Delta^{h+1-r} \lambda_{\nu+r} R_{\nu}^h(t) \\ &\quad + \sum_{\nu=1}^n A_{n-\nu}^{-1} \lambda_{\nu+h+1} R_{\nu}^h(t) \\ &= \Sigma_1 + \Sigma_2, \quad \text{say.} \end{aligned}$$



Hence we need to prove that

$$\sum_{n=1}^{\infty} n^{-1} \left| \int_0^{\pi} \phi_h(t) \frac{t^h}{A_n^{h+1}} (\Sigma_1 + \Sigma_2) dt \right| < \infty,$$

for which it is sufficient to show that

$$(1.4.4) \quad \sum_{n=1}^{\infty} n^{-h-2} \int_0^{\pi} |\phi_h(t)| t^h |\Sigma_1| dt < \infty,$$

and

$$(1.4.5) \quad \sum_{n=1}^{\infty} n^{-h-2} \left| \int_0^{\pi} \phi_h(t) t^h \Sigma_2 dt \right| < \infty.$$

*Proof of (1.4.4).* — It suffices, for our purpose, to show that for  $0 \leq r \leq h$ ,

$$\sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^n A_{n-\nu}^{h-r} \Delta^{h+1-r} \lambda_{\nu+r} \int_0^{\pi} t^h |\phi_h(t)| |R_{\nu}^h(t)| dt < \infty.$$

The above expression is

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^n A_{n-\nu}^{h-r} \Delta^{h+1-r} \lambda_{\nu+r} \left( \int_0^{\frac{1}{\nu}} + \int_{\frac{1}{\nu}}^{\infty} \right) |\phi_h(t)| t^h |R_{\nu}^h(t)| dt \\ = \Sigma_{11} + \Sigma_{12}, \quad \text{say.} \end{aligned}$$

Now by lemma 3 and the hypothesis we have

$$\begin{aligned} \Sigma_{11} &\leq K^{(1)} \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^n A_{n-\nu}^{h-r} \Delta^{h+1-r} \lambda_{\nu+r} (\nu^{2h+2} \varepsilon(\nu+1)) \left( \frac{\nu^{-h-1}}{\varepsilon_{\nu+1}} \right), \\ &\leq K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^n \nu^{h+1} (n+1-\nu)^{h-r} \Delta^{h+1-r} \lambda_{\nu+r}, \\ &\leq K \sum_{\nu=1}^{\infty} \nu^{h+1} \Delta^{h+1-r} \lambda_{\nu+r} \sum_{n=\nu}^{\infty} (n+1-\nu)^{h-r} n^{-h-2} \\ &\leq K \sum_{\nu=1}^{\infty} \nu^{h+1} \Delta^{h+1-r} \lambda_{\nu+r} \nu^{-r-1} \\ &\leq K \sum_{\nu=1}^{\infty} \nu^{h-r} \Delta^{h-r+1} \lambda_{\nu+r} \leq K. \end{aligned}$$

By lemma 5 and the fact that

$$\begin{aligned} \sum_{n=\nu}^{\infty} (n+1-\nu)^{h-r} n^{-h-2} &= 0 \left( \int_{\nu}^{\infty} x^{-h-2} (x-\nu)^{h-r} dx \right) \\ &= 0(\nu^{-r-1}) \end{aligned}$$

(1) K is a constant not necessarily the same at each occurrence.

Also by lemmas 3 and 4 we get

$$\begin{aligned}
 \Sigma_{12} &\leq K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^n A_{n-\nu}^{h-r} \Delta^{h+1-r} \lambda_{\nu+r} \nu^{h+1} \varepsilon(\nu+1) \\
 &\qquad \int_{\frac{1}{\nu}}^{\pi} t^h |\varphi_h(t)| t^{-h-1} dt \\
 &\leq K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^n A_{n-\nu}^{h-r} \Delta^{h+1-r} \lambda_{\nu+r} \nu^{h+1} \varepsilon(\nu+1) \int_{\frac{1}{\nu}}^{\pi} t^{-1} |\varphi_h(t)| dt, \\
 &\leq K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^n A_{n-\nu}^{h-r} \Delta^{h+1-r} \lambda_{\nu+r} \nu^{h+1} \log(\nu+1), \\
 &\leq K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^n (n+1-\nu)^{h-r} \Delta^{h+1-r} \lambda_{\nu+r} \nu^{h+1} \log(\nu+1), \\
 &\leq K \sum_{\nu=1}^{\infty} \nu^{h+1} \log(\nu+1) \Delta^{h+1-r} \lambda_{\nu+r} \sum_{n=\nu}^{\infty} (n+1-\nu)^{h-r} n^{-h-2}, \\
 &\leq K \sum_{\nu=1}^{\infty} \log(\nu+1) \nu^{h-r} \Delta^{h-r+1} \lambda_{\nu+r}, \\
 &\leq K,
 \end{aligned}$$

by lemmas 5 and 6.

This completes the proof of (1.4.4).

*Proof of (1.4.5).* — Now we have to show that

$$\sum_{n=1}^{\infty} n^{-h-2} \left| \int_0^{\pi} t^h \varphi_h(t) \Sigma_2 dt \right| < \infty.$$

Since

$$\Sigma_2 = \lambda_{n+h+1} R_n^h(t),$$

substituting the value of  $\Sigma_2$ , we find that the above expression is

$$\begin{aligned}
 &\sum_{n=1}^{\infty} n^{-h-2} \left| \int_0^{\pi} t^h \varphi_h(t) \lambda_{n+h+1} R_n^h(t) dt \right| \\
 &\leq K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \int_0^{\pi} \Phi_h(t) R_n^h(t) dt \right| \\
 &= K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \sum_{\nu=1}^n A_{n-\nu}^{h-r} \varepsilon(\nu+1) \cdot \nu \cdot \int_0^{\pi} \Phi_h(t) (\cos \nu t)_h dt \right| \\
 &= K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \sum_{\nu=1}^n A_{n-\nu}^{h-r} \varepsilon(\nu+1) \cdot \nu \cdot (-1)^h \right|
 \end{aligned}$$

$$\begin{aligned}
& \left\{ (-1)^h \frac{2}{\pi} \int_0^\pi \Phi_h(t) (\cos \nu t)_h dt \right. \\
& + \frac{2}{\pi} \left[ \sum_{\rho=1}^h (-1)^{\rho-1} \Phi_\rho(t) (\cos \nu t)_{\rho-1} \right]_0^\pi \\
& - \frac{2}{\pi} \left[ \sum_{\rho=1}^h (-1)^{\rho-1} \Phi_\rho(t) (\cos \nu t)_{\rho-1} \right]_0^\pi \left. \right\} \\
& \leq K \sum_{n=1}^\infty n^{-h-2} \lambda_{n+h+1} \left| \sum_{\nu=1}^n A_{n-\nu}^h \varepsilon(\nu+1) \cdot \nu \cdot A_\nu(x) \right| \\
& + K \sum_{n=1}^\infty n^{-h-2} \lambda_{n+h+1} \\
& \left| \sum_{\nu=1}^n A_{n-\nu}^h \varepsilon(\nu+1) \cdot \nu \cdot \left[ \sum_{\rho=1}^h (-1)^{\rho-1} \Phi_\rho(t) (\cos \nu t)_{\rho-1} \right]_0^\pi \right| \\
& = I_1 + I_2, \text{ say.}
\end{aligned}$$

By repeated partial summation we have

$$\sum_{\nu=0}^n A_{n-\nu}^h \varepsilon(\nu+1) \cdot \nu \cdot A_\nu(x) = \sum_{\nu=0}^n \check{S}_\nu^{h+1} \Delta (A_{n-\nu}^h \nu \cdot \varepsilon_{\nu+1}),$$

where  $\check{S}_n^h$  denotes the  $n$ -th Cesàro-sum of order  $h$  of the series  $\sum A_n(x)$ .

Now since

$$\begin{aligned}
\Delta (A_{n-\nu}^h \nu \cdot \varepsilon(\nu+1)) &= \sum_{r=0}^{h+1} \binom{h+1}{r} \Delta^r (A_{n-\nu}^h) \Delta^{h+1-r} \{(\nu+r) \varepsilon_{\nu+r+1}\} \\
&= \sum_{r=0}^{h+1} \binom{h+1}{r} A_{n-\nu}^{h-r} \Delta^{h+1-r} \{(\nu+r) \varepsilon_{\nu+r+1}\} \\
&= \sum_{r=0}^h \binom{h+1}{r} A_{n-\nu}^{h-r} \Delta^{h+1-r} \{(\nu+r) \varepsilon_{\nu+r+1}\} \\
&\quad + A_{n-\nu}^{-1} (\nu+h+1) \varepsilon_{\nu+h+2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{\nu=1}^n A_{n-\nu}^h \varepsilon(\nu+1) \nu A_\nu(x) &= \sum_{r=0}^h \binom{h+1}{r} \sum_{\nu=0}^n \check{S}_\nu^{h+1} A_{n-\nu}^{h-r} \Delta^{h+1-r} \{(\nu+r) \varepsilon_{\nu+r+1}\} \\
&\quad + \check{S}_n^h (n+h+1) \varepsilon_{n+h+2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
 I_1 &\leq K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \sum_{r=0}^h \binom{h+1}{r} \\
 &\quad \left| \sum_{\nu=0}^n |\dot{S}_{\nu}^{*h}| A_{n-\nu}^{h-r} \Delta^{n+1-r} \{(\nu+r)\varepsilon_{\nu+r+1}\} \right| \\
 &\quad + K \sum n^{-h-2} \lambda_{n+h+1} (n+h+1) \varepsilon_{n+h+2} |\dot{S}_n^{*h}| \\
 &= I_{11} + I_{12}, \quad \text{say.}
 \end{aligned}$$

Now

$$\begin{aligned}
 I_{12} &= K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} (n+h+1) A_n^h |\dot{\sigma}_n^{*h}| \varepsilon_{n+h+2} \\
 &= 0 \left[ \sum_{n=1}^{\infty} |\dot{\sigma}_n^{*h}| \lambda_n \frac{\varepsilon_{n+1}}{n} \right].
 \end{aligned}$$

Applying Abel's transformation we have by Lemma 7.

$$\begin{aligned}
 \sum_{n=1}^m |\dot{\sigma}_n^{*h}| \frac{\lambda_n \varepsilon_{n+1}}{n} &= \sum_{n=1}^{m-1} \Delta \left( \frac{\lambda_n \varepsilon_{n+1}}{n} \right) \sum_{\nu=0}^n |\dot{\sigma}_\nu^{*h}| \\
 &\quad + \frac{\lambda_m \varepsilon_{m+1}}{m} \sum_{n=0}^m |\dot{\sigma}_n^{*h}| \\
 &= 0 \left[ \sum_{n=1}^{m-1} \Delta \left( \frac{\lambda_n \varepsilon_{n+1}}{n} \right) n \varepsilon^{-1} (n+1) (\log(n+1))^{\frac{1}{2}} \right] \\
 &\quad + 0 \left[ \frac{\lambda_m \varepsilon_{m+1}}{m} \cdot m \varepsilon^{-1} (m+1) (\log(m+1))^{\frac{1}{2}} \right] \\
 &= 0 \left[ \sum_1^{m-1} \Delta \lambda_n (\log n + 1)^{\frac{1}{2}} \right] \\
 &\quad + 0 \left[ \sum_1^{m-1} \frac{\lambda_{n+1}}{n+1} (\log n + 1)^{\frac{1}{2}} \right] \\
 &= 0(1) + 0(1) = 0(1).
 \end{aligned}$$

Since  $\Delta \varepsilon_n = 0 \left( \frac{\varepsilon_n}{n} \right)$  and  $\lambda_m \log(m+1) = 0(1)$ .

Now in order to show that  $I_{11} = 0(1)$  it is sufficient to prove that

$$\begin{aligned}
 \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \sum_{\nu=0}^n |\dot{S}_{\nu}^{*h}| A_{n-\nu}^{h-r} \Delta^{h+1-r} \{(\nu+r)\varepsilon_{\nu+r+1}\} \\
 = 0(1) \quad \text{for } r = 0, 1, \dots, h.
 \end{aligned}$$

The above expression is by lemma 8

$$\begin{aligned}
 & \left| \sum_{v=1}^{\infty} |\dot{S}_v^h| \left| \Delta^{h+1-r} \{(\nu+r)\varepsilon_{\nu+r+1}\} \right| \left| \sum_{n=v}^{\infty} (n-\nu+1)^{h-r} n^{-h-2} \lambda_{n+h+1} \right| \right| \\
 & \leq K \sum_{v=1}^{\infty} |\dot{S}_v^h| \lambda_{\nu+h+1} \left| \Delta^{h+1-r} \{(\nu+r)\varepsilon_{\nu+r+1}\} \right| \left| \sum_{n=v}^{\infty} (n-\nu+1)^{h-r} \cdot n^{-h-2} \right| \\
 & = 0 \left( \sum_{v=0}^{\infty} |\dot{S}_v^h| \lambda_{\nu+h+1} \frac{(\nu+1)^{r-h} \varepsilon_{\nu+1}}{\log(\nu+1)} \nu^{-r-1} \right), \\
 & = 0 \left( \sum_{v=0}^{\infty} |\dot{\sigma}_v^h| (\nu+1)^{r-h} \frac{\varepsilon_{\nu+1}}{\log(\nu+1)} \nu^{h-r-1} \lambda_{\nu+h+1} \right) \\
 & = 0 \left( \sum_{v=0}^{\infty} |\sigma_v^h| \lambda_{\nu+h+1} \frac{\varepsilon(\nu+1)}{\nu \log \nu + 1} \right) \\
 & = 0(1),
 \end{aligned}$$

as shown in the proof of  $I_{12} = 0(1)$ .

Hence

$$I_1 = 0(1).$$

Now we proceed to show that  $I_2 = 0(1)$ .

If  $\rho$  is an odd integer, then it is sufficient to show that

$$(1.4.6) \quad k \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \sum_{v=1}^n A_{n-v}^{h-\nu} \varepsilon_{\nu+1} (-1)^{\nu} \nu^{\rho} \right| < \infty,$$

for  $1 \leq \rho \leq h$ .

By repeated partial summation we have

$$\sum_{v=1}^n A_{n-v}^{h-\nu} \varepsilon_{\nu+1} (-1)^{\nu} \nu^{\rho} = \sum_{v=0}^n C_{v,\rho}^h \Delta^{h+1} (A_{n-v}^h \varepsilon(\nu+1)),$$

where  $C_{n,\rho}^h$  is the  $n$ -th Cesàro sum of order  $h$  of the series  $\sum (-1)^{\nu} \nu^{\rho}$ .

Also

$$\begin{aligned}
 \Delta^{h+1} (A_{n-v}^{h-\nu} \varepsilon_{\nu+1}) &= \sum_{r=0}^{h+1} \binom{h+1}{r} \Delta^r (A_{n-v}^{h-\nu}) \Delta^{h+1-r} \varepsilon_{\nu+r+1} \\
 &= \sum_{r=0}^{h+1} \binom{h+1}{r} A_{n-v}^{h-r} \Delta^{h+1-r} \varepsilon_{\nu+r+1} \\
 &= \sum_{r=0}^h \binom{h+1}{r} A_{n-v}^{h-r} \Delta^{h+1-r} \varepsilon_{\nu+r+1} + A_{n-v}^{-1} \varepsilon_{\nu+h+2}.
 \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{\nu=1}^n A_{n-\nu}^h \varepsilon_{\nu+1} (-1)^\nu \rho^{\nu} &= \sum_{r=0}^h \binom{h+1}{r} \sum_{\nu=0}^n C_{\nu,\rho}^h A_{n-\nu}^{h-r} \Delta^{h+1-r} \varepsilon_{\nu+r+1} \\ &+ \sum_{\nu=0}^n C_{\nu,\rho}^h A_{n-\nu}^{-1} \varepsilon_{\nu+h+2} \\ &= 0 \left( \sum_{\nu=0}^n \rho^h (n - \nu + 1)^{h-r} \left| \Delta^{h+1-r} \varepsilon_{\nu+r+1} \right| \right) \\ &+ 0(n^h \varepsilon_{n+h+2}), \end{aligned}$$

by lemma 1.

Therefore the expression in (1.4.6) is

$$\begin{aligned} &= 0 \left( \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \sum_{\nu=0}^n \rho^h (n - \nu + 1)^{h-r} \left| \Delta^{h-r+1} \varepsilon_{\nu+r+1} \right| \right) \\ &+ 0 \left( \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} n^h \varepsilon_{n+h+2} \right) \\ &= 0 \left( \sum_{\nu=1}^{\infty} \rho^h \left| \Delta^{h-r+1} \varepsilon_{\nu+r+1} \right| \sum_{n=\nu}^{\infty} (n - \nu + 1)^{h-r} n^{-h-2} \lambda_{n+h+1} \right) \\ &+ 0 \left( \sum_{n=1}^{\infty} \frac{\lambda_n}{n} \right). \\ &= 0 \left( \sum_{\nu=1}^{\infty} \rho^h \left| \Delta^{h-r+1} \varepsilon_{\nu+r+1} \right| \lambda_{\nu+h+1} \sum_{n=\nu}^{\infty} (n - \nu + 1)^{h-r} n^{-h-2} \right) + 0(1) \\ &= 0 \left( \sum_{\nu=1}^{\infty} \rho^h \left| \Delta^{h-r+1} \varepsilon_{\nu+r+1} \right| \lambda_{\nu+h+1} \cdot \rho^{-r-1} \right) + 0(1) \\ &= 0 \left( \sum_{\nu=1}^{\infty} \rho^{h-r-1} \frac{\varepsilon_{\nu+1} \lambda_{\nu}}{(\nu + 1)^{h-r+1}} \right) + 0(1) \\ &= 0 \left( \sum_{\nu=1}^{\infty} \frac{\lambda_{\nu}}{\nu} \right) + 0(1) \\ &= 0(1). \end{aligned}$$

This completes the proof of the theorem.

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