SINGH NIRANJAN

On the absolute Cesáro summability of factors of Fourier series

Annales de l’institut Fourier, tome 18, n° 2 (1968), p. 17-30

<http://www.numdam.org/item?id=AIF_1968__18_2_17_0>
ON THE ABSOLUTE CESARO SUMMABILITY FACTORS OF FOURIER SERIES (*)

by NIRANJAN SINGH

1.1. DEFINITIONS. — Let $\Sigma a_n$ be a given infinite series with $S_n$ as its $n$-th partial sum. The series $\Sigma a_n$ is said to be absolutely summable $(C, \alpha)$, or summable $|C, \alpha|$, if the sequence $\{\sigma_n^\alpha\}$ is of bounded variation, that is

$$\sum_n |\sigma_n^\alpha - \sigma_{n-1}^\alpha| < \infty,$$

where $\{\sigma_n^\alpha\}$ is the $n$-th Cesaro mean of order $\alpha$, $\alpha > -1$, of the sequence $\{S_n\}$.

If $\{t_n^\alpha\}$ be the $n$-th Cesàro mean of order $\alpha$ of the sequence $\{na_n\}$, then we have the following identity [6].

$$t_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha).$$

For any sequence $\{u_n\}$, we write

$$\Delta u_n = u_n - u_{n+1}$$

and

$$\Delta^r u_n = \sum_{p=0}^{\infty} A_p^{r-1} u_{n+p},$$

provided the series on the right converges.

If $S$ is $a + ve$ integer, then

$$\Delta^s (u_n v_n) = \sum_{r=0}^{s} \left( \begin{array}{c} s \\ r \end{array} \right) \Delta^r u_n \Delta^{s-r} v_{n+r}$$

(*) Acknowledgement. — I take this opportunity to express my sincerest thanks to Dr. S. M. Mazhar for his constant encouragement and able guidance during the preparation of this paper.
By repeated partial summation, we observe that, for $k = 0, 1, \ldots$

$$
\sum_{p=0}^{q} A_{n-p}^{-1} u_p a_p = \sum_{p=0}^{q} S_{p}^k \Delta (A_{n-p}^{-1} u_p) + \sum_{j=0}^{k} \Delta (A_{n-j-1}^{-1} u_{q+1}) S_{q}^j
$$

where $S_{n}^k$ denotes the $n$th Cesàro sum of order $k$ of the sequence $\{S_{n}\}$. Hence, putting $q = n$, we get

$$
(1.1.1) \quad \sum_{p=0}^{n} A_{n-p}^{-1} u_p a_p = \sum_{p=0}^{n} S_{p}^k \Delta (A_{n-p}^{-1} u_p).
$$

A sequence $\{\lambda_{n}\}$ is said to be convex, if $\Delta^{2} \lambda_{n} \geq 0$, and it is said to be hyper-convex of order $h$, if

$$
\Delta^{h+2} \lambda_{n} \geq 0, \quad (h = 0, 1, 2, \ldots).
$$

By definition hyper-convexity of order zero is the same as convexity.

Let $f(t)$ be a periodic function with period $2\pi$ and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, that is

$$
\int_{-\pi}^{\pi} f(t) \, dt = 0,
$$

and

$$
f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t).
$$

We use the following notations:

$$
\Phi(t) = \frac{1}{2} \{f(x + t) + f(x - t) - 2f(x)\},
$$

$$
\Phi_{\alpha}(t) = \frac{1}{(\alpha)} \int_{0}^{t} (t - u)^{\alpha-1} \Phi(u) \, du, \quad \alpha > 0,
$$

$$
\Phi_{0}(t) = \Phi(t),
$$

$$
\varphi(t) = \int (\alpha + 1) t^{-\alpha} \Phi_{\alpha}(t), \quad \alpha \geq 0,
$$

$$
(F(t))_{n} = \frac{d^{n} F(t)}{dt^{n}},
$$

$$
\epsilon(n) = (\log n)^{-\beta}, \quad \beta \geq 0,
$$

$$
\epsilon^{-1}(n) = \frac{1}{\epsilon(n)}.
$$

**Theorem A.** — Let $h$ be an integer $\geq 0$, and let $\{\lambda_n\}$ be a monotonic non-increasing sequence when $h = 0$, and a hyper-convex sequence of order $(h - 1)$ when $h \geq 1$, such that

(i) $\sum \frac{\lambda_n}{n} < \infty$,  
(ii) $\sum n^h \lambda_n < \infty$.

If

$$\int_0^t |\varphi_n(u)| \, du = 0(t),$$

as $t \to 0$, then $\sum_{n=1}^{\infty} \lambda_n A_n(x)$ is summable $|C, h + 1 + \delta|$ for every $\delta > 0$.

Later on Ahmad [1] obtained the following theorem which includes as a special case for $\beta = 0$ the above theorem of Pati and Sinha.

**Theorem B.** — Let $\{\lambda_n\}$ be a sequence such that for all non-negative integral values of $h$, $\Delta^{h+1} \lambda_n \geq 0$, and $\sum \frac{\lambda_n}{n} < \infty$.

If

$$\int_0^t |\varphi_n(u)| \, du = 0\left(t^{\varepsilon^{-1}}\left(\frac{1}{t}\right)\right),$$

as $t \to 0$, then $\sum_{n=1}^{\infty} \varepsilon(n + 1)\lambda_n A_n(x)$ is summable $|C, h + 1 + \delta|$ for every $\delta > 0$.

In this paper we prove the following theorem for summability $|C, 1 + h|$ by imposing suitable conditions on the sequence $\{\lambda_n\}$.

We prove the following theorem.

**Theorem.** — Let $\{\lambda_n\}$ be a sequence such that for non-negative integral values of $h$, $\Delta^{h+1} \lambda_n \geq 0$, and

$$\sum \frac{\lambda_n}{n} (\log n)^{\frac{1}{2}} < \infty.$$
If
\[
(1.2.2) \quad \int_0^t |\varphi_n(u)| \, du = \frac{t e^{-1} \left( \frac{1}{t} \right)}{1}, \quad t \to 0,
\]
then, \( \sum_{i=1}^{\infty} \varepsilon(n + 1)\lambda_n A_n(x) \) is summable \( |C, h + 1| \).

It may be remarked that this theorem generalizes the following theorem of the author [14] which in turn, includes a theorem of Pati [10].

**Theorem C.** — Let \( \{\lambda_n\} \) be a convex sequence such that
\[
\sum_{n=1}^{\infty} \frac{\lambda_n}{n} \left( \log n \right) \frac{1}{2} < \infty.
\]
If
\[
\int_0^t |\Phi(u)| \, du = 0 \left( t e^{-1} \left( \frac{1}{t} \right) \right),
\]
as \( t \to 0 \), then \( \sum_{n=1}^{\infty} \varepsilon(n + 1)\lambda_n A_n(x) \) is summable \( |C, 1| \).

1.3. For the proof of our theorem we require the following lemmas:

**Lemma 1** [9]. — Let \( C_{n, p}^k \) and \( S_n^k(t) \) denote the \( n \)-th Cesàro-sums of order \( k \) corresponding to the series \( \sum_{n=1}^{\infty} (-1)^n n^p \) and \( \sum_{n=1}^{\infty} (\sin nt)^{(h+1)}(h \geq 0) \), respectively, then
\[
(i) \quad C_{n, p}^k = 0(n^k) \quad k \geq p
\]
\[
(ii) \quad S_n^k(t) = 0(n^{k+h+2}) \quad \left( 0 < t \leq \frac{1}{n} \right), \quad k \geq 0
\]
\[
= 0(n^{h+1}t^{k-1}) + 0(n^{k}t^{-2}) \quad \left( n^{-1} < t \leq \pi \right) k \geq 0.
\]

**Lemma 2** [2]. — If \( k \gg -1, r \geq 0 \), necessary and sufficient conditions for \( \sum a_n \varepsilon_n \) to be summable \( |C, r| \) whenever
\[
S_n = a_0 + a_1 + \cdots + a_n = 0(1)(C, k)
\]
are
\[
(i) \quad \Sigma n^{k-r} |\varepsilon_n| < \infty,
\]
\[
(ii) \quad \Sigma n^{-1} |\varepsilon_n| < \infty,
\]
\[
(iii) \quad \Sigma n^k |_{\Delta \varepsilon_n}^{k+1} | < \infty
\]
Lemma 3 [1]. — Let \( R_n^k(t) \) denote the \( n \)th Cesaro sum of order \( k \) (\( 0 \leq k < h + 1 \)) of the series \( \sum_1^\infty \varepsilon(n + 1) (\sin nt)_{n+1} \) \((h \geq 0)\), then

(i) \( R_n^k(t) = 0 \{ \varepsilon(n + 1)n^{k+h+1} \} \quad (0 < t \leq \frac{1}{n}) \),

(ii) \( R_n^k(t) = 0 \{ \varepsilon(n + 1)n^{h+1}t^{-k-1} \} \quad (\frac{1}{n} < t \leq \pi) \).

Lemma 4 [1]. — If (1.2.2) holds, then

\[
\int_1^\pi t^{-1} |\varphi_h(t)| \, dt = 0 \{ \varepsilon^{-1}(n + 1) \log n \}.
\]

Lemma 5 [1]. — Let \( h \) be a positive integer, and \( \{\lambda_n\} \) be a sequence such that \( \Delta \lambda_n \geq 0 \), and \( \sum \frac{\lambda_n}{n} < \infty \), then

(a) \( \Delta \lambda_n^0 \) \( (r = 0, 1, \ldots, h - 1) \).

(b) \( \lambda_n = \begin{cases} \sum_{m=n}^{\infty} \Delta \lambda_m & \text{for} \quad h = 1 \\ (h-1)^{-1} \sum_{m=n}^{\infty} (m-n+1)(m-n+2) \ldots \\ (m-n+h-1) \Delta \lambda_m & (h>1) \end{cases} \)

(c) \( \sum m^{-1} \Delta \lambda_m < \infty \) \( (r = 1, 2, \ldots, h - 1) \).

Lemma 6 [11]. — Let \( \{\lambda_n\} \) be a hyper-convex sequence of order \( (h - 1) \) when \( h \geq 1 \), or monotonic non-increasing when \( h = 0 \), such that

\[
\sum \frac{\lambda_n}{n} < \infty.
\]

If

\[
\sum n^h \Delta \lambda_n < \infty,
\]

then

\[
\sum \log (n + 1)n^h \Delta \lambda_n < \infty.
\]

Lemma 7 [13]. — If

\[
\int_0^t |\varphi_x(u)| \, du = 0 \{ t \left( \log \frac{1}{t} \right)^\beta \},
\]
then
\[ \sum_{m=0}^{n} |\sigma_{m}^{a}|^2 = O\{n (\log n)^{2\beta+1}\} \quad \text{for} \quad \beta > -\frac{1}{2} \]
and \( \alpha \geq 0 \) where \( \sigma_{m}^{a} \) is the \( m \)-th \( (C, \alpha) \) mean of the series \( \sum A_{n}(x) \).

**Lemma 8.** — We have for \( r = 0, 1, \ldots, h \)
\[ \Delta \{ (\mu + r)\xi_{\mu+r+1} \} = O\{ (\mu + 1)^{-h}\xi_{\mu+1} \} \quad \text{for} \quad p > 2. \]

**Proof.** — Since \( \Delta (\mu + r) = 0 \) for \( p \geq 2 \) we have
\[ \Delta \{ (\mu + r)\xi_{\mu+r+1} \} = \sum_{p=0}^{h+1-r} \binom{h+1-r}{p} (\mu + 1)^{h+1-r-p} \Delta (\mu + r) \Delta \xi_{\mu+r+p+1} = 0\{ (\mu + 1)^{-h}\xi_{\mu+1} \} . \]

**1.4. Proof of the Theorem.** — Since
\[ A_{n}(x) = \frac{2}{\pi} \int_{0}^{\pi} \Phi(t) \cos nt \ dt \]
\[ = \frac{2}{\pi} \left[ \sum_{r=1}^{h} (-1)^{r-1} \Phi_{r}(t) (\cos nt)_{r-1} \right]_{0}^{\pi} + (-1)^{h} \frac{2}{\pi} \int_{0}^{\pi} \Phi_{h}(t) (\cos nt)_{h} \ dt \]
\[ = A_{n,1}(x) + A_{n,2}(x) , \quad \text{say.} \]

Thus by virtue of the consistency theorem for absolute Cesàro-summability, it is sufficient for our purpose, to prove that each of the series
\[ (1.4.1) \quad \sum_{n=1}^{\infty} \xi(n + 1)\lambda_{n}A_{n,1}(x) , \]
and
\[ (1.4.2) \quad \sum_{n=1}^{\infty} \xi(n + 1)\lambda_{n}A_{n,2}(x) , \]
is summable \( |C, h + 1| \).
Now since \( \sin n\pi = 0 \) and \( \cos n\pi = (-1)^{n} \), for proving
the summability $|C, h + 1|$ of (1.4.1), it is enough to show that if $\rho$ is an odd integer, $1 \leq \rho \leq h$,

$$\sum_{n=1}^{\infty} \epsilon(n + 1)\lambda_n(-1)^n n^{\rho-1}$$

is summable $|C, h + 1|$.

Taking the series $\Sigma a_n$ in lemma 2 to be $\Sigma(-1)^n n^{\rho-1}$, $r = h$, $k = h - 1$, we have from lemma 1.

$$C_{n, h + 1} = 0(n^{h-1}).$$

Also by taking $\epsilon_n$ to be $\lambda_n \epsilon_{n+1}$ we find that conditions (i) and (ii) of lemma 2 are satisfied. Also

$$\Sigma n^{h-1} \left| \Delta \left( \frac{\lambda_n}{(\log n + 1) \beta} \right) \right| = 0 \left\{ \sum_{n=1}^{\infty} \sum_{r=0}^{h} n^{r-1} \Delta \lambda_n \right\} = O(1),$$

by virtue of part (c) of lemma 5. Finally applying lemma 2 we find that $\Sigma \lambda_n \epsilon_{n+1}(-1)^n n^{\rho-1}$ is summable $|C, h|$ and consequently summable $|C, h + 1|$.

Also the summability $|C, h + 1|$ of the series (1.4.2) is equivalent to the assertion that

$$(1.4.3) \quad \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_{0}^{\pi} \varphi_n(t) L_{n+1}^h(t) \, dt \right| < \infty,$$

where

$$L_{n+1}^h(t) = \frac{t^h}{A_{n+1}^h} \sum_{v=0}^{n} A_v^h \epsilon(v + 1)\lambda_v (\sin vt)_{h+1}.$$

Proof of (1.4.3). — We have

$$\Sigma \equiv \sum_{v=1}^{n} A_v^h \epsilon(v + 1)\lambda_v (\sin vt)_{h+1}.$$

Applying the process of repeated summation we have in the notation of Lemma 3,

$$\Sigma = \sum_{v=1}^{n} R_v^h(t) \Delta (A_v^h)$$

$$= \sum_{r=0}^{h} \left( \frac{h + 1}{r} \right) \sum_{v=1}^{n} A_{v-r}^h \Delta \lambda_{v+r} R_v^h(t)$$

$$+ \sum_{v=1}^{n} A_v^h \lambda_{v+h+1} R_v^h(t)$$

$$= \Sigma_1 + \Sigma_2, \quad \text{say}.\]
Hence we need to prove that
\[
\sum_{n=1}^{\infty} n^{-1} \left| \int_{0}^{\pi} \phi(t) \frac{t^{h}}{A_{n+1}^{h+1}} (\Sigma_{1} + \Sigma_{2}) \, dt \right| < \infty,
\]
for which it is sufficient to show that
\[
(1.4.4) \quad \sum_{n=1}^{\infty} n^{-h-2} \int_{0}^{\pi} |\phi(t)| t^{h} |\Sigma_{1}| \, dt < \infty,
\]
and
\[
(1.4.5) \quad \sum_{n=1}^{\infty} n^{-h-2} \left| \int_{0}^{\pi} \phi(t) t^{h} \Sigma_{2} \, dt \right| < \infty.
\]

Proof of (1.4.4). — It suffices, for our purpose, to show that for \( 0 < r < h, \)
\[
\sum_{n=1}^{\infty} n^{-h-2} \sum_{v=1}^{n} A_{v-r}^{h+1-r} \Delta \lambda_{v+r} \int_{0}^{\pi} t^{h} |\phi(t)||R_{v}(t)| \, dt < \infty.
\]

The above expression is
\[
\sum_{n=1}^{\infty} n^{-h-2} \sum_{v=1}^{n} A_{v-r}^{h+1-r} \Delta \lambda_{v+r} \left( \int_{0}^{\pi} t^{h} + \int_{1}^{\pi} t^{h} |\phi(t)||R_{v}(t)| \, dt \right)
\]
\[= \Sigma_{11} + \Sigma_{12}, \text{ say.}\]

Now by lemma 3 and the hypothesis we have
\[
\Sigma_{11} \leq K (1) \sum_{n=1}^{\infty} n^{-h-2} \sum_{v=1}^{n} A_{v-r}^{h+1-r} \Delta \lambda_{v+r} (\phi^{2h+2\lambda}(v + 1)) \left( \frac{\phi^{h-1}}{\varepsilon_{v+1}} \right),
\]
\[
\leq K \sum_{n=1}^{\infty} n^{-h-2} \sum_{v=1}^{h+1} (n + 1 - \phi)^{h-r} \Delta \lambda_{v+r}
\]
\[
\leq K \sum_{v=1}^{\infty} \phi^{h+1} \Delta \lambda_{v+r} \sum_{n=v}^{\infty} (n + 1 - \phi)^{h-r} n^{-h-2}
\]
\[
\leq K \sum_{v=1}^{\infty} \phi^{h+1} \Delta \lambda_{v+r} \phi^{r-1}
\]
\[
\leq K \sum_{v=1}^{\infty} \phi^{h-r} \Delta \lambda_{v+r} \leq K.
\]

By lemma 5 and the fact that
\[
\sum_{n=v}^{\infty} (n + 1 - \phi)^{h-r} n^{-h-2} = 0 \left( \int_{v}^{\infty} x^{-h-2} (x - \phi)^{h-r} \, dx \right)
\]
\[= 0(\phi^{r-1})
\]

(1) \( K \) is a constant not necessarily the same at each occurrence.
Also by lemmas 3 and 4 we get

\[ \Sigma_{12} \leq K \sum_{n=1}^{\infty} n^{-h-2} \sum_{v=1}^{n} A_{n-v}^{h-r} \Delta \lambda_{v+r} v^{h+1} \epsilon (\nu + 1) \int_{\frac{1}{1}}^{\pi} t^{h} | \varphi(t) | t^{-h-1} \, dt \]

\[ \leq K \sum_{n=1}^{\infty} n^{-h-2} \sum_{v=1}^{n} A_{n-v}^{h-r} \Delta \lambda_{v+r} v^{h+1} \epsilon (\nu + 1) \int_{\frac{1}{1}}^{\pi} t^{-1} | \varphi(t) | \, dt, \]

\[ \leq K \sum_{n=1}^{\infty} n^{-h-2} \sum_{v=1}^{n} A_{n-v}^{h-r} \Delta \lambda_{v+r} v^{h+1} \log (\nu + 1), \]

\[ \leq K \sum_{n=1}^{\infty} n^{-h-2} \sum_{v=1}^{n} (n + 1 - \nu)^{h-r} \Delta \lambda_{v+r} v^{h+1} \log (\nu + 1), \]

\[ \leq K \sum_{v=1}^{\infty} v^{h+1} \log (\nu + 1) \Delta \lambda_{v+r} \sum_{n=v}^{\infty} (n + 1 - \nu)^{h-r} n^{-h-2}, \]

\[ \leq K, \]

by lemmas 5 and 6.

This completes the proof of (1.4.4).

**Proof of (1.4.5).** — Now we have to show that

\[ \sum_{n=1}^{\infty} n^{-h-2} \left| \int_{0}^{\pi} t^{h} \varphi(t) \Sigma_{2} \, dt \right| < \infty. \]

Since

\[ \Sigma_{2} = \lambda_{n+h+1} R_{n}^{h}(t), \]

substituting the value of \( \Sigma_{2} \), we find that the above expression is

\[ \sum_{n=1}^{\infty} n^{-h-2} \left| \int_{0}^{\pi} t^{h} \varphi(t) \lambda_{n+h+1} R_{n}^{h}(t) \, dt \right| \]

\[ \leq K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \int_{0}^{\pi} \Phi_{h}(t) R_{n}^{h}(t) \, dt \right| \]

\[ = K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \sum_{v=1}^{n} A_{n-v}^{h} \epsilon (\nu + 1) \cdot \nu \cdot \int_{0}^{\pi} \Phi_{h}(t) (\cos \nu t)_{h} \, dt \right| \]

\[ = K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \sum_{v=1}^{n} A_{n-v}^{h} \epsilon (\nu + 1) \cdot \nu \cdot (-1)^{h} \right| \]
\[
\left\{ (-1)^h \frac{2}{\pi} \int_0^\pi \Phi_h(t) (\cos \nu t) \, dt \right. \\
+ \frac{2}{\pi} \left[ \sum_{\varphi = 1}^h (-1)^{\varphi-1} \Phi_\varphi(t) (\cos \nu t) \right]_0^\pi \\
- \frac{2}{\pi} \left[ \sum_{\varphi = 1}^h (-1)^{\varphi-1} \Phi_\varphi(t) (\cos \nu t) \right]_0^\pi \\
\leq K \sum_{n=1}^\infty n^{-h-2} \lambda_n + 1 \left| \sum_{\nu=1}^n A_n^h \varepsilon(\nu + 1) \cdot \nu \cdot A_\nu(x) \right| \\
+ K \sum_{n=1}^\infty n^{-h-2} \lambda_n + 1 \\
\left| \sum_{\nu=1}^n A_n^h \varepsilon(\nu + 1) \cdot \nu \cdot \left[ \sum_{\varphi = 1}^h (-1)^{\varphi-1} \Phi_\varphi(t) (\cos \nu t) \right]_0^\pi \right| \\
= I_1 + I_2, \text{ say.}
\]

By repeated partial summation we have
\[
\sum_{\nu=0}^n A_n^h \varepsilon(\nu + 1) \cdot \nu \cdot A_\nu(x) = \sum_{\nu=0}^n A_n^h \Delta \left( A_n^h \varepsilon(\nu + 1) \cdot A_\nu(x) \right),
\]
where \( A_n^h \) denotes the \( n \)-th Cesàro-sum of order \( h \) of the series \( \Sigma A_n(x) \).

Now since
\[
\sum_{\nu=0}^n A_n^h \varepsilon(\nu + 1) \cdot \nu \cdot A_\nu(x) = \sum_{\nu=0}^n A_n^h \varepsilon(\nu + 1) \cdot \nu \cdot A_\nu(x)
\]

It follows that
\[
\sum_{\nu=1}^n A_n^h \varepsilon(\nu + 1) \cdot \nu \cdot A_\nu(x)
\]

\[
= \sum_{\nu=0}^h \binom{h + 1}{r} \sum_{\nu=0}^n A_n^{h+r} \varepsilon(\nu + 1) \cdot \nu \cdot A_\nu(x)
\]

\[
= \hat{S}_n^h(n + h + 1) \varepsilon_{n+h+2}.
\]
Therefore
\[ I_1 \leq K \sum_{n=1}^{\infty} n^{-h} \lambda_{n+h+1} \sum_{r=0}^{h} \binom{h+1}{r} \sum_{\nu=0}^{n+1-r} |S_{\nu}\| A_n^{h-r} \{ (\nu + r) \varepsilon_{\nu+r+1} \} \]
\[ + K \sum_{n=1}^{\infty} n^{-h} \lambda_{n+h+1} (n + h + 1) \varepsilon_{n+h+2} |S_n^h| \]
\[ = I_{11} + I_{12}, \]
say.

Now
\[ I_{12} = K \sum_{n=1}^{\infty} n^{-h} \lambda_{n+h+1} (n + h + 1) A_n^h \sigma_n^h \varepsilon_{n+h+2} \]
\[ = 0 \left[ \sum_{n=1}^{\infty} |\sigma_n^h| \frac{\varepsilon_{n+1}}{n} \right]. \]

Applying Abel's transformation we have by Lemma 7.
\[ \sum_{n=1}^{m} |\sigma_n^h| \lambda_n \varepsilon_{n+1} = \sum_{n=1}^{m-1} \Delta \left( \frac{\lambda_n \varepsilon_{n+1}}{n} \right) \sum_{\nu=0}^{n} |\sigma_n^h| \]
\[ + \frac{\lambda_m \varepsilon_{m+1}}{m} \sum_{n=0}^{m} |\sigma_n^h| \]
\[ = 0 \left[ \sum_{n=1}^{m-1} \Delta \left( \frac{\lambda_n \varepsilon_{n+1}}{n} \right) n \varepsilon^{-1} (n + 1) (\log (n + 1))^{1/2} \right] \]
\[ + 0 \left[ \frac{\lambda_m \varepsilon_{m+1}}{m} m \varepsilon^{-1} (m + 1) (\log (m + 1))^{1/2} \right] \]
\[ = 0 \left[ \sum_{n=1}^{m-1} \Delta \lambda_n (\log n + 1)^{1/2} \right] \]
\[ + 0 \left[ \sum_{n=1}^{m-1} \frac{\lambda_{n+1}}{n+1} (\log n + 1)^{1/2} \right] \]
\[ = 0(1) + 0(1) = 0(1). \]

Since \( \Delta \varepsilon_n = 0 \left( \frac{\varepsilon_n}{n} \right) \) and \( \lambda_m \log (m + 1) = 0(1) \).

Now in order to show that \( I_{11} = 0(1) \) it is sufficient to prove that
\[ \sum_{n=1}^{\infty} n^{-h} \lambda_{n+h+1} \sum_{\nu=0}^{n} |S_{\nu}\| A_n^{h-r} \{ (\nu + r) \varepsilon_{\nu+r+1} \} \]
\[ = 0(1) \]
for \( r = 0, 1, \ldots, h. \)
The above expression is by lemma 8

\[ \sum_{\nu=1}^{\infty} \left| \delta_{\nu} \right| \triangle \{(\nu + r)\varepsilon_{\nu+r+1}\} \left| \sum_{n=\nu}^{\infty} (n - \nu + 1)^{h-r} n^{-h-2} \lambda_{n+h+1} \right| \leq K \sum_{\nu=1}^{\infty} \left| \delta_{\nu} \right| \lambda_{\nu+h+1} \triangle \{(\nu + r)\varepsilon_{\nu+r+1}\} \left| \sum_{n=\nu}^{\infty} (n - \nu + 1)^{h-r} n^{-h-2} \right| \]

\[ = 0 \left( \sum_{\nu=0}^{\infty} \left| \delta_{\nu} \right| \lambda_{\nu+h+1} \frac{(\nu + 1)^{r-h} \varepsilon_{\nu+1}}{\log (\nu + 1)} \varepsilon_{\nu-r-1} \right), \]

\[ = 0 \left( \sum_{\nu=0}^{\infty} \left| \delta_{\nu} \right| \lambda_{\nu+h+1} \frac{\varepsilon_{\nu+1}}{\log (\nu + 1)} \varepsilon_{\nu-r-1}^{h-r-1} \lambda_{\nu+h+1} \right) \]

\[ = 0 \left( \sum_{\nu=0}^{\infty} \left| \delta_{\nu} \right| \lambda_{\nu+h+1} \frac{\varepsilon (\nu + 1)}{\nu \log (\nu + 1)} \varepsilon_{\nu-r-1} \right) \]

\[ = 0(1), \]

as shown in the proof of \( I_{12} = 0(1) \).

Hence \( I_1 = 0(1) \).

Now we proceed to show that \( I_2 = 0(1) \).

If \( \rho \) is an odd integer, then it is sufficient to show that

\[ (1.4.6) \quad k \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \sum_{\nu=1}^{n} A_{n-\nu}^h \varepsilon_{\nu+1} (-1)^{\nu} \nu^\rho \right| < \infty, \]

for \( 1 \leq \rho \leq h \).

By repeated partial summation we have

\[ \sum_{\nu=0}^{n} A_{n-\nu}^h \varepsilon_{\nu+1} (-1)^{\nu} \nu^\rho = \sum_{\nu=0}^{n} C_h^\rho \Delta (A_{n-\nu}^h \varepsilon (\nu + 1)), \]

where \( C_h^\rho \) is the \( n-th \) Cesàro sum of order \( h \) of the series \( \sum (-1)^{\nu} \nu^\rho \).

Also

\[ \Delta (A_{n-\nu}^h \varepsilon_{\nu+1}) = \sum_{r=0}^{h+1} \binom{h+1}{r} \Delta (A_{n-\nu}^h) \Delta \varepsilon_{\nu+r+1} \]

\[ = \sum_{r=0}^{h+1} \binom{h+1}{r} A_{n-\nu}^{h-r} \Delta \varepsilon_{\nu+r+1} \]

\[ = \sum_{r=0}^{h} \binom{h+1}{r} A_{n-\nu}^{h-r} \Delta \varepsilon_{\nu+r+1} + A_{n-\nu}^{-1} \varepsilon_{\nu+h+2}. \]
Therefore
\[
\sum_{v=1}^{n} A_{n-v}^h (-1)^v \nu^r = \sum_{r=0}^{h} \binom{h+1}{r} \sum_{v=0}^{n} C_{v-1}^h A_{n-v}^h \sum_{r=0}^{h+1-r} \Delta \epsilon_{v+r+1} \\
+ \sum_{v=0}^{n} C_{v-1}^h A_{n-v}^h \epsilon_{v+h+2} \\
= 0 \left( \sum_{v=0}^{n} \nu^h (n - \nu + 1)^{h-r} \left| \Delta \epsilon_{v+r+1} \right| \right) \\
+ 0(n^h \epsilon_{n+h+2}),
\]
by lemma 1.

Therefore the expression in (1.4.6) is
\[
= 0 \left( \sum_{n=1}^{\infty} n^{-h+2} \lambda_{n+h+1} \sum_{v=0}^{n} \nu^h (n - \nu + 1)^{h-r} \left| \Delta \epsilon_{v+r+1} \right| \right) \\
+ 0 \left( \sum_{n=1}^{\infty} n^{-h+2} \epsilon_{n+h+2} \right) \\
= 0 \left( \sum_{v=0}^{n} \nu^h \left| \Delta \epsilon_{v+r+1} \right| \sum_{n=v}^{\infty} (n - \nu + 1)^{h-r} n^{-h-2} \lambda_{n+h+1} \right) \\
+ 0 \left( \sum_{n=1}^{\infty} \lambda_{n} \right) \\
= 0 \left( \sum_{v=0}^{n} \nu^h \left| \Delta \epsilon_{v+r+1} \right| \lambda_{v+h+1} \sum_{n=v}^{\infty} (n - \nu + 1)^{h-r} n^{-h-2} \right) + 0(1) \\
= 0 \left( \sum_{v=0}^{n} \nu^h \left| \Delta \epsilon_{v+r+1} \right| \lambda_{v+h+1} \nu^{r-1} \right) + 0(1) \\
= 0 \left( \sum_{v=0}^{n} \nu^h \left( \epsilon_{v+1} \lambda_{v} \right) \right) + 0(1) \\
= 0(1).
\]

This completes the proof of the theorem.

BIBLIOGRAPHY


Manuscrit reçu le 18 décembre 1967.

NIRANJAN SINGH
Department of Mathematics and Statistics
Aligarh Muslim University,
Aligarh. (UP), India.