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SINGH NIRANJAN

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ON THE ABSOLUTE CESARO SUMMABILITY FACTORS OF FOURIER SERIES (*)

by NIRANJAN SINGH

1.1. DEFINITIONS. — *Let* Σa_n *be a given infinite series with* S_n *as its* $n - th$ *partial sum. The series* Σa_n *is said to be absolutely summable* (C, α), *or summable* $|C, \alpha|$, *if the sequence* $\{\sigma_n^\alpha\}$ *is of bounded variation, that is*

$$\sum_n |\sigma_n^\alpha - \sigma_{n-1}^\alpha| < \infty,$$

where $\{\sigma_n^\alpha\}$ *is the* $n - th$ *Cesàro mean of order* α, $\alpha > -1$, *of the sequence* $\{S_n\}$.

If $\{t_n^\alpha\}$ be the $n - th$ Cesàro mean of order α of the sequence $\{na_n\}$, then we have the following identity [6].

$$t_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha).$$

For any sequence $\{u_n\}$, we write

$$\Delta u_n = u_n - u_{n+1}$$

and

$$\overset{r}{\Delta} u_n = \sum_{p=0}^{\infty} A_p^{-r-1} u_{n+p},$$

provided the series on the right converges.

If S is $a + ve$ integer, then

$$\overset{S}{\Delta}(u_n v_n) = \sum_{r=0}^{S} \binom{s}{r} \overset{r}{\Delta} u_n \overset{S-r}{\Delta} v_{n+r}$$

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By repeated partial summation, we observe that, for $k = 0, 1, \ldots$

$$\sum_{p=0}^{q} A_{n-p}^{r-1} u_p a_p = \sum_{p=0}^{q} S_p^k \overset{k+1}{\Delta} (A_{n-p}^{r-1} u_p) + \sum_{j=0}^{k} \overset{j}{\Delta} (A_{n-q-1}^{r-1} u_{q+1}) S_q^j$$

where S_n^k denotes the $n-th$ Cesàro sum of order k of the sequence $\{S_n\}$. Hence, putting $q = n$, we get

$$(1.1.1) \qquad \sum_{p=0}^{n} A_{n-p}^{r-1} u_p a_p = \sum_{p=0}^{n} S_p^k \overset{k+1}{\Delta} (A_{n-p}^{r-1} u_p).$$

A sequence $\{\lambda_n\}$ is said to be convex, if $\overset{2}{\Delta} \lambda_n \geqslant 0$, and it is said to be hyper-convex of order h, if

$$\overset{h+2}{\Delta} \lambda_n \geqslant 0, \qquad (h = 0, 1, 2, \ldots).$$

By definition hyper-convexity of order zero is the same as convexity.

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, that is

$$\int_{-\pi}^{\pi} f(t)\, dt = 0,$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t).$$

We use the following notations:

$$\begin{aligned}
\Phi(t) &= \frac{1}{2} \{f(x+t) + f(x-t) - 2f(x)\}, \\
\Phi_\alpha(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \Phi(u)\, du, \qquad \alpha > 0, \\
\Phi_0(t) &= \Phi(t), \\
\varphi_\alpha(t) &= \Gamma(\alpha+1) t^{-\alpha} \Phi_\alpha(t), \qquad \alpha \geqslant 0, \\
(F(t))_h &= \frac{\partial^h F(t)}{\partial t^h}, \\
\varepsilon(n) &= (\log n)^{-\beta}, \qquad \beta \geqslant 0, \\
\varepsilon^{-1}(n) &= \frac{1}{\varepsilon(n)}.
\end{aligned}$$

1.2. Various results on summability factors of Fourier series due to Prasad [12], Izumi and Kwata [5], Cheng [3], Pati [8] and Dikshit [4] were generalized by Pati and Sinha [11] in the form of the following theorem.

THEOREM A. — *Let* h *be an integer* $\geqslant 0$, *and let* $\{\lambda_n\}$ *be a monotonic non-increasing sequence when* $h = 0$, *and a hyper-convex sequence of order* $(h-1)$ *when* $h \geqslant 1$, *such that*

$$\text{(i)} \quad \Sigma \frac{\lambda_n}{n} < \infty, \qquad \text{(ii)} \quad \Sigma n^h \overset{h+1}{\Delta} \lambda_n < \infty.$$

If

$$\int_0^t |\varphi_h(u)|\, du = 0(t),$$

as $t \to 0$, *then* $\sum\limits_{n=1}^{\infty} \lambda_n A_n(x)$ *is summable* $|C, h + 1 + \delta|$ *for every* $\delta > 0$.

Later on Ahmad [1] obtained the following theorem which includes as a special case for $\beta = 0$ the above theorem of Pati and Sinha.

THEOREM B. — *Let* $\{\lambda_n\}$ *be a sequence such that for all non-negative integral values of* h, $\overset{+1}{\Delta} \lambda_n \geqslant 0$, *and* $\Sigma \frac{\lambda_n}{n} < \infty$. *If*

$$\int_0^t |\varphi_h(u)|\, du = 0 \left\{ t\varepsilon^{-1}\left(\frac{1}{t}\right)\right\},$$

as $t \to 0$, *then* $\sum\limits_{n=1}^{\infty} \varepsilon(n+1)\lambda_n A_n(x)$ *is summable* $|C, h + 1 + \delta|$ *for every* $\delta > 0$.

In this paper we prove the following theorem for summability $|C, 1 + h|$ by imposing suitable conditions on the sequence $\{\lambda_n\}$.

We prove the following theorem.

THEOREM. — *Let* $\{\lambda_n\}$ *be a sequence such that for non-negative integral values of* h, $\overset{h+1}{\Delta} \lambda_n \geqslant 0$, *and*

$$\Sigma \frac{\lambda_n}{n} (\log n)^{\frac{1}{2}} < \infty. \tag{1.2.1}$$

If

$$(1.2.2)\qquad \int_0^t |\varphi_h(u)|\, du = 0\left\{t\varepsilon^{-1}\left(\frac{1}{t}\right)\right\}, \qquad t \to 0,$$

then, $\sum_1^\infty \varepsilon(n+1)\lambda_n A_n(x)$ *is summable* $|C, h+1|$.

It may be remarked that this theorem generalizes the following theorem of the author [14] which in turn, includes a theorem of Pati [10].

THEOREM C. — *Let* $\{\lambda_n\}$ *be a convex sequence such that* $\sum \frac{\lambda_n}{n} (\log n)^{\frac{1}{2}} < \infty$.

If

$$\int_0^t |\Phi(u)|\, du = 0\left\{t\varepsilon^{-1}\left(\frac{1}{t}\right)\right\},$$

as $t \to 0$, *then* $\sum_1^\infty \varepsilon(n+1)\lambda_n A_n(x)$ *is summable* $|C, 1|$.

1.3. For the proof of our theorem we require the following lemmas:

LEMMA 1 [9]. — *Let* $C_{n,\rho}^k$ *and* $S_n^k(t)$ *denote the* n — *th Cesàro-sums of order* k *corresponding to the series* $\sum_1^\infty (-1)^n n^\rho$ *and* $\sum_1^\infty (\sin nt)_{h+1} (h \geqslant 0)$, *respectively, then*

(i) $$C_{n,\rho}^k = 0(n^k) \quad k \geqslant \rho$$

(ii) $$S_n^k(t) = 0(n^{k+h+2}) \quad \left(0 < t \leqslant \frac{1}{n}\right), \qquad k \geqslant 0$$
$$= 0(n^{h+1}t^{-k-1}) + 0(n^k t^{-h-2}), \quad (n^{-1} < t \leqslant \pi) k \geqslant 0.$$

LEMMA 2 [2]. — *If* $k \geqslant -1, r \geqslant 0$, *necessary and sufficient conditions for* $\Sigma a_n \varepsilon_n$ *to be summable* $|C, r|$ *whenever*

$$S_n = a_0 + a_1 + \cdots + a_n = 0(1)(C, k)$$

are

(i) $$\Sigma n^{k-r}|\varepsilon_n| < \infty,$$

(ii) $$\Sigma n^{-1}|\varepsilon_n| < \infty,$$

(iii) $$\Sigma n^k \left|\overset{k+1}{\Delta} \varepsilon_n\right| < \infty.$$

LEMMA 3 [1]. — *Let* $R_n^k(t)$ *denote the* n — *th Cesàro sum of order* k $(0 \leqslant k < h+1)$ *of the series* $\sum_1^\infty \varepsilon(n+1)(\sin nt)_{h+1}$ $(h \geqslant 0)$, *then*

(i) $\quad R_n^k(t) = 0\{\varepsilon(n+1)n^{k+h+2}\} \qquad \left(0 < t \leqslant \frac{1}{n}\right),$

(ii) $\quad R_n^k(t) = 0\{\varepsilon(n+1)n^{h+1}t^{-k-1}\} \quad (n^{-1} < t \leqslant \pi).$

LEMMA 4 [1]. — *If* (1.2.2) *holds, then*

$$\int_{\frac{1}{n}}^{\pi} t^{-1}|\varphi_h(t)|\, dt = 0\{\varepsilon^{-1}(n+1)\log n\}.$$

LEMMA 5 [1]. — *Let* h *be a positive integer, and* $\{\lambda_n\}$ *be a sequence such that* $\overset{h}{\triangle}\lambda_n \geqslant 0$, *and* $\Sigma\frac{\lambda_n}{n} < \infty$, *then*

$$(a) \qquad \overset{r}{\triangle}\lambda_n \underset{0}{\downarrow} \quad (r = 0, 1, \ldots, h-1).$$

$$(b) \quad \lambda_n = \begin{cases} \sum_{m=n}^{\infty} \Delta\lambda_m & \text{for} \quad h = 1 \\ \left(\underline{|h-1}\right)^{-1} \sum_{m=n}^{\infty} (m-n+1)(m-n+2)\ldots \\ \qquad\qquad (m-n+h-1)\overset{h}{\triangle}\lambda_m & (h > 1) \end{cases}$$

$$(c) \qquad \Sigma m^{r-1}\overset{r}{\triangle}\lambda_m < \infty \quad (r = 1, 2, \ldots, h-1).$$

LEMMA 6 [11]. — *Let* $\{\lambda_n\}$ *be a hyper-convex sequence of order* $(h-1)$ *when* $h \geqslant 1$, *or monotonic non-increasing when* $h = 0$, *such that*

$$\Sigma\frac{\lambda_n}{n} < \infty.$$

If

$$\Sigma n^h \overset{h+1}{\triangle}\lambda_n < \infty,$$

then

$$\Sigma \log(n+1)n^h \overset{h+1}{\triangle}\lambda_n < \infty.$$

LEMMA 7 [13]. — *If*

$$\int_0^t |\varphi_\alpha(u)|\, du = 0\left\{t\left(\log\frac{1}{t}\right)^\beta\right\},$$

then

$$\sum_{m=0}^{n} |\overset{*}{\sigma}{}_m^{\alpha}|^2 = 0\{n (\log n)^{2\beta+1}\} \quad \text{for} \quad \beta > -\frac{1}{2}$$

and $\alpha \geqslant 0$ *where* $\overset{*}{\sigma}{}_m^{\alpha}$ *is the* m — *th* (C, α) *mean of the series* $\Sigma A_n(x)$.

LEMMA 8. — *We have for* $r = 0, 1, \ldots, h$

$$\overset{h+1-r}{\Delta} \{(\mu + r)\varepsilon_{\mu+r+1}\} = 0\left\{\frac{(\mu + 1)^{r-h}\varepsilon_{\mu+1}}{\log (\mu + 1)}\right\}.$$

Proof. — Since $\overset{p}{\Delta} (\mu + r) = 0$ for $p \geqslant 2$ we have

$$\begin{aligned}
\overset{h+1-r}{\Delta} &\{(\mu + r)\varepsilon_{\mu+r+1}\} \\
&= \sum_{p=0}^{h+1-r} \binom{h+1-r}{p} \overset{p}{\Delta} (\mu + r) \overset{h+1-r-p}{\Delta} \varepsilon_{\mu+r+p+1} \\
&= (\mu + r) \overset{h+1-r}{\Delta} \varepsilon_{\mu+r+1} - (h + 1 - r) \overset{h-r}{\Delta} \varepsilon_{\mu+r+2} \\
&= 0\left\{\frac{(\mu + 1)^{r-h}\varepsilon_{\mu+1}}{\log (\mu + 1)}\right\}.
\end{aligned}$$

1.4. *Proof of the Theorem.* — Since

$$\begin{aligned}
A_n(x) &= \frac{2}{\pi} \int_0^{\pi} \Phi(t) \cos nt \, dt \\
&= \frac{2}{\pi} \left[\sum_{\rho=1}^{h} (-1)^{\rho-1}\Phi_\rho(t) (\cos nt)_{\rho-1}\right]_0^{\pi} \\
&+ (-1)^h \frac{2}{\pi} \int_0^{\pi} \Phi_h(t) (\cos nt)_h \, dt \\
&= A_{n,1}(x) + A_{n,2}(x), \quad \text{say.}
\end{aligned}$$

Thus by virtue of the consistency theorem for absolute Cesàro-summability, it is sufficient for our purpose, to prove that each of the series

$$(1.4.1) \qquad \sum_{n=1}^{\infty} \varepsilon(n + 1)\lambda_n A_{n,1}(x),$$

and

$$(1.4.2) \qquad \sum_{n=1}^{\infty} \varepsilon(n + 1)\lambda_n A_{n,2}(x),$$

is summable $|C, h + 1|$.

Now since $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$, for proving the summability $|C, h+1|$ of (1.4.1), it is enough to show that if ρ is an odd integer, $1 \leqslant \rho \leqslant h$,

$$\sum_{n=1}^{\infty} \varepsilon(n+1)\lambda_n(-1)^n n^{\rho-1} \qquad \text{is summable} \qquad |C, h+1|.$$

Taking the series Σa_n in lemma 2 to be $\Sigma(-1)^n n^{\rho-1}$, $r = h$, $k = h - 1$, we have from lemma 1.

$$C^{h-1}_{n,\rho-1} = 0(n^{h-1}).$$

Also by taking ε_n to be $\lambda_n \varepsilon_{n+1}$ we find that conditions (i) and (ii) of lemma 2 are satisfied. Also

$$\Sigma n^{h-1} \left| \overset{h}{\Delta} \left(\frac{\lambda_n}{(\log n+1)} \beta \right) \right| = 0 \left\{ \sum_{n=1}^{\infty} \sum_{r=0}^{h} n^{r-1} \overset{r}{\Delta} \lambda_n \right\}$$
$$= 0(1),$$

by virtue of part (*c*) of lemma 5. Finally applying lemma 2 we find that $\Sigma\lambda_n\varepsilon_{n+1}(-1)^n n^{\rho-1}$ is summable $|C, h|$ and consequently summable $|C, h+1|$.

Also the summability $|C, h+1|$ of the series (1.4.2) is equivalent to the assertion that

$$(1.4.3) \qquad \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^{\pi} \phi_h(t) L_n^{h+1}(t)\, dt \right| < \infty,$$

where

$$L_n^{h+1}(t) = \frac{t^h}{A_n^{h+1}} \sum_{\nu=0}^{n} A^h_{n-\nu}\, \varepsilon(\nu+1)\lambda_\nu\, (\sin \nu t)_{h+1}.$$

Proof of (1.4.3). — We have

$$\Sigma \equiv \sum_{\nu=1}^{n} A^h_{n-\nu} \varepsilon(\nu+1)\lambda_\nu\, (\sin \nu t)_{h+1}.$$

Applying the process of repeated summation we have in the notation of Lemma 3,

$$\Sigma = \sum_{\nu=1}^{n} R_\nu^h(t) \overset{h+1}{\Delta} (A^h_{n-\nu}\lambda_\nu)$$
$$= \sum_{r=0}^{h} \binom{h+1}{r} \sum_{\nu=1}^{n} A^{h-r}_{n-\nu} \overset{h+1-r}{\Delta} \lambda_{\nu+r} R_\nu^h(t)$$
$$+ \sum_{\nu=1}^{n} A^{-1}_{n-\nu} \lambda_{\nu+h+1} R_\nu^h(t)$$
$$= \Sigma_1 + \Sigma_2, \quad \text{say.}$$

Hence we need to prove that

$$\sum_{n=1}^{\infty} n^{-1} \left| \int_0^{\pi} \phi_h(t) \frac{t^h}{A_n^{h+1}} (\Sigma_1 + \Sigma_2)\, dt \right| < \infty,$$

for which it is sufficient to show that

$$(1.4.4) \qquad \sum_{n=1}^{\infty} n^{-h-2} \int_0^{\pi} |\phi_h(t)| t^h |\Sigma_1|\, dt < \infty,$$

and

$$(1.4.5) \qquad \sum_{n=1}^{\infty} n^{-h-2} \left| \int_0^{\pi} \phi_h(t) t^h\, \Sigma_2\, dt \right| < \infty.$$

Proof of (1.4.4). — It suffices, for our purpose, to show that for $0 \leqslant r \leqslant h$,

$$\sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^{n} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \lambda_{\nu+r} \int_0^{\pi} t^h |\phi_h(t)| |R_\nu^h(t)|\, dt < \infty.$$

The above expression is

$$\sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^{n} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \lambda_{\nu+r} \left(\int_0^{\frac{1}{\nu}} + \int_{\frac{1}{\nu}}^{\infty} \right) |\phi_h(t)| t^h |R_\nu^h(t)|\, dt$$
$$= \Sigma_{11} + \Sigma_{12}, \quad \text{say.}$$

Now by lemma 3 and the hypothesis we have

$$\begin{aligned}
\Sigma_{11} &\leqslant K\,(^1) \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^{n} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \lambda_{\nu+r} (\nu^{2h+2} \varepsilon(\nu + 1)) \left(\frac{\nu^{-h-1}}{\varepsilon_{\nu+1}} \right), \\
&\leqslant K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^{n} \nu^{h+1} (n + 1 - \nu)^{h-r} \overset{h+1-r}{\Delta} \lambda_{\nu+r} \\
&\leqslant K \sum_{\nu=1}^{\infty} \nu^{h+1} \overset{h+1-r}{\Delta} \lambda_{\nu+r} \sum_{n=\nu}^{\infty} (n + 1 - \nu)^{h-r} n^{-h-2} \\
&\leqslant K \sum_{\nu=1}^{\infty} \nu^{h+1} \overset{h+1-r}{\Delta} \lambda_{v+r} \nu^{-r-1} \\
&\leqslant K \sum_{\nu=1}^{\infty} \nu^{h-r} \overset{h-r+1}{\Delta} \lambda_{\nu+r} \leqslant K.
\end{aligned}$$

By lemma 5 and the fact that

$$\sum_{n=v}^{\infty} (n + 1 - \nu)^{h-r} n^{-h-2} = 0\left(\int_v^{\infty} x^{-h-2} (x - \nu)^{h-r}\, dx \right)$$
$$= 0(\nu^{-r-1})$$

(1) K is a constant not necess-arily the same at each occurrence.

Also by lemmas 3 and 4 we get

$$\Sigma_{12} \leqslant K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^{n} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \lambda_{\nu+r} \nu^{h+1} \varepsilon(\nu+1)$$

$$\int_{\frac{1}{\nu}}^{\pi} t^h |\varphi_h(t)| t^{-h-1}\, dt$$

$$\leqslant K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^{n} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \lambda_{\nu+r} \nu^{h+1} \varepsilon(\nu+1) \int_{\frac{1}{\nu}}^{\pi} t^{-1} |\varphi_h(t)|\, dt,$$

$$\leqslant K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^{n} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \lambda_{\nu+r} \nu^{h+1} \log(\nu+1),$$

$$\leqslant K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^{n} (n+1-\nu)^{h-r} \overset{h+1-r}{\Delta} \lambda_{\nu+r} \nu^{h+1} \log(\nu+1),$$

$$\leqslant K \sum_{\nu=1}^{\infty} \nu^{h+1} \log(\nu+1) \overset{h+1-r}{\Delta} \lambda_{\nu+r} \sum_{n=\nu}^{\infty} (n+1-\nu)^{h-r} n^{-h-2},$$

$$\leqslant K \sum_{\nu=1}^{\infty} \log(\nu+1) \nu^{h-r} \overset{h-r+1}{\Delta} \lambda_{\nu+r},$$

$$\leqslant K,$$

by lemmas 5 and 6.

This completes the proof of (1.4.4).

Proof of (1.4.5). — Now we have to show that

$$\sum_{n=1}^{\infty} n^{-h-2} \left| \int_0^{\pi} t^h \varphi_h(t)\, \Sigma_2\, dt \right| < \infty.$$

Since

$$\Sigma_2 = \lambda_{n+h+1} R_n^h(t),$$

substituting the value of Σ_2, we find that the above expression is

$$\sum_{n=1}^{\infty} n^{-h-2} \left| \int_0^{\pi} t^h \varphi_h(t) \lambda_{n+h+1} R_n^h(t)\, dt \right|$$

$$\leqslant K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \int_0^{\pi} \Phi_h(t) R_n^h(t)\, dt \right|$$

$$= K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \sum_{\nu=1}^{n} A_{n-\nu}^{h} \varepsilon(\nu+1) . \nu . \int_0^{\pi} \Phi_h(t)\, (\cos \nu t)_h\, dt \right|$$

$$= K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \sum_{\nu=1}^{n} A_{n-\nu}^{h} \varepsilon(\nu+1) . \nu . (-1)^h \right.$$

$$
\begin{aligned}
&\Big\{(-1)^h \frac{2}{\pi} \int_0^\pi \Phi_h(t) (\cos \nu t)_h \, dt \\
&+ \frac{2}{\pi} \Big[\sum_{\rho=1}^{h} (-1)^{\rho-1} \Phi_\rho(t) (\cos \nu t)_{\rho-1}\Big]_0^\pi \\
&- \frac{2}{\pi} \Big[\sum_{\rho=1}^{h} (-1)^{\rho-1} \Phi_\rho(t) (\cos \nu t)_{\rho-1}\Big]_0^\pi \Big\}\Big| \\
&\leqslant K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \Big| \sum_{\nu=1}^{n} A_{n-\nu}^{h} \varepsilon(\nu+1) \cdot \nu \cdot A_\nu(x) \Big| \\
&+ K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \\
&\quad \Big| \sum_{\nu=1}^{n} A_{n-\nu}^{h} \varepsilon(\nu+1) \cdot \nu \cdot \Big[\sum_{\rho=1}^{h} (-1)^{\rho-1} \Phi_\rho(t) (\cos \nu t)_{\rho-1}\Big]_0^\pi \Big| \\
&= I_1 + I_2, \quad \text{say.}
\end{aligned}
$$

By repeated partial summation we have

$$
\sum_{\nu=0}^{n} A_{n-\nu}^{h} \varepsilon(\nu+1) \cdot \nu \cdot A_\nu(x) = \sum_{\nu=0}^{n} \overset{*}{S}{}_\nu^{h} \overset{h+1}{\Delta} (A_{n-\nu}^{h} \nu \cdot \varepsilon_{\nu+1}),
$$

where $\overset{*}{S}{}_n^h$ denotes the $n-th$ Cesàro-sum of order h of the series $\Sigma A_n(x)$.

Now since

$$
\begin{aligned}
\overset{h+1}{\Delta} (A_{n-\nu}^{h} \nu \cdot \varepsilon(\nu+1)) &= \sum_{r=0}^{h+1} \binom{h+1}{r} \overset{r}{\Delta} (A_{n-\nu}^{h}) \overset{h+1-r}{\Delta} \{(\nu+r)\varepsilon_{\nu+r+1}\} \\
&= \sum_{r=0}^{h+1} \binom{h+1}{r} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \{(\nu+r)\varepsilon_{\nu+r+1}\} \\
&= \sum_{r=0}^{h} \binom{h+1}{r} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \{(\nu+r)\varepsilon_{\nu+r+1}\} \\
&\quad + A_{n-\nu}^{-1} (\nu+h+1) \varepsilon_{\nu+h+2}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\sum_{\nu=1}^{n} A_{n-\nu}^{h} \varepsilon(\nu+1) \nu A_\nu(x) \\
&\qquad = \sum_{r=0}^{h} \binom{h+1}{r} \sum_{\nu=0}^{n} \overset{*}{S}{}_\nu^{h} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \{(\nu+r)\varepsilon_{\nu+r+1}\} \\
&\qquad + \overset{*}{S}{}_n^{h} (n+h+1) \varepsilon_{n+h+2}.
\end{aligned}
$$

Therefore

$$I_1 \leqslant K \sum_{n=1}^{\infty} n^{-h-2}\lambda_{n+h+1} \sum_{r=0}^{h} \binom{h+1}{r} \sum_{\nu=0}^{n} |\overset{*}{S}{}_\nu^h| A_{n-\nu}^{h-r} \left| \overset{n+1-r}{\Delta} \{(\nu + r)\varepsilon_{\nu+r+1}\} \right|$$
$$+ K\Sigma n^{-h-2}\lambda_{n+h+1}(n + h + 1)\varepsilon_{n+h+2}|\overset{*}{S}{}_n^h|$$
$$= I_{11} + I_{12}, \quad \text{say.}$$

Now

$$I_{12} = K \sum_{n=1}^{\infty} n^{-h-2}\lambda_{n+h+1}(n + h + 1)A_n^h|\overset{*}{\sigma}{}_n^h|\varepsilon_{n+h+2}$$
$$= 0\left[\sum_{n=1}^{\infty} |\overset{*}{\sigma}{}_n^h|\lambda_n \frac{\varepsilon_{n+1}}{n}\right].$$

Applying Abel's transformation we have by Lemma 7.

$$\sum_{n=1}^{m} |\overset{*}{\sigma}{}_n^h| \frac{\lambda_n\varepsilon_{n+1}}{n} = \sum_{n=1}^{m-1} \Delta\left(\frac{\lambda_n\varepsilon_{n+1}}{n}\right) \sum_{\nu=0}^{n} |\overset{*}{\sigma}{}_n^h|$$
$$+ \frac{\lambda_m\varepsilon_{m+1}}{m} \sum_{n=0}^{m} |\overset{*}{\sigma}{}_n^h|$$
$$= 0\left[\sum_{n=1}^{m-1} \Delta\left(\frac{\lambda_n\varepsilon_{n+1}}{n}\right) n\varepsilon^{-1}(n + 1)(\log(n+1))^{\frac{1}{2}}\right]$$
$$+ 0\left[\frac{\lambda_m\varepsilon_{m+1}}{m} \cdot m\varepsilon^{-1}(m + 1)(\log(m + 1))^{\frac{1}{2}}\right]$$
$$= 0\left[\sum_{1}^{m-1} \Delta\lambda_n (\log n + 1)^{\frac{1}{2}}\right]$$
$$+ 0\left[\sum_{1}^{m-1} \frac{\lambda_{n+1}}{n+1} (\log n + 1)^{\frac{1}{2}}\right]$$
$$= 0(1) + 0(1) = 0(1).$$

Since $\Delta\varepsilon_n = 0\left(\frac{\varepsilon_n}{n}\right)$ and $\lambda_m \log(m + 1) = 0(1)$.

Now in order to show that $I_{11} = 0(1)$ it is sufficient to prove that

$$\sum_{n=1}^{\infty} n^{-h-2}\lambda_{n+h+1} \sum_{\nu=0}^{n} |\overset{*}{S}{}_\nu^h| A_{n-\nu}^{h-r}| \overset{h+1-r}{\Delta} \{(\nu + r)\varepsilon_{\nu+r+1}\}|$$
$$= 0(1) \qquad \text{for} \qquad r = 0, 1, \ldots, h.$$

The above expression is by lemma 8

$$\begin{aligned}
&\sum_{\nu=1}^{\infty} |\overset{*}{S}{}_\nu^h| \left| \overset{h+1-r}{\Delta} \{(\nu + r)\varepsilon_{\nu+r+1}\} \right| \left| \sum_{n=\nu}^{\infty} (n - \nu + 1)^{h-r} n^{-h-2} \lambda_{n+h+1} \right| \\
&\leqslant K \sum_{\nu=1}^{\infty} |\overset{*}{S}{}_\nu^h| \lambda_{\nu+h+1} \left| \overset{h+1-r}{\Delta} \{(\nu + r)\varepsilon_{\nu+r+1}\} \right| \left| \sum_{n=\nu}^{\infty} (n - \nu + 1)^{h-r} . n^{-h-2} \right| \\
&= 0\left(\sum_{\nu=0}^{\infty} |\overset{*}{S}{}_\nu^h| \lambda_{\nu+h+1} \frac{(\nu + 1)^{r-h} \varepsilon_{\nu+1}}{\log (\nu + 1)} \nu^{-r-1} \right), \\
&= 0\left(\sum_{\nu=0}^{\infty} |\overset{*}{\sigma}{}_\nu^h| (\nu + 1)^{r-h} \frac{\varepsilon_{\nu+1}}{\log (\nu + 1)} \nu^{h-r-1} \lambda_{\nu+h+1} \right) \\
&= 0\left(\sum_{\nu=0}^{\infty} |\overset{*}{\sigma}{}_\nu^h| \lambda_{\nu+h+1} \frac{\varepsilon(\nu + 1)}{\nu \log \overline{\nu + 1}} \right) \\
&= 0(1),
\end{aligned}$$

as shown in the proof of $I_{12} = 0(1)$.

Hence

$$I_1 = 0(1).$$

Now we proceed to show that $I_2 = 0(1)$.

If ρ is an odd integer, then it is sufficient to show that

$$(1.4.6) \quad k \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \sum_{\nu=1}^{n} A_{n-\nu}^h \varepsilon_{\nu+1} (-1)^\nu \nu^\rho \right| < \infty,$$

for $1 \leqslant \rho \leqslant h$.

By repeated partial summation we have

$$\sum_{\nu=1}^{n} A_{n-\nu}^h \varepsilon_{\nu+1} (-1)^\nu . \nu^\rho = \sum_{\nu=0}^{n} C_{\nu,\rho}^h \overset{h+1}{\Delta} (A_{n-\nu}^h \varepsilon(\nu + 1)),$$

where $C_{n,\rho}^h$ is the $n - th$ Cesàro sum of order h of the series $\Sigma(-1)^\nu . \nu^\rho$.

Also

$$\begin{aligned}
\overset{h+1}{\Delta} (A_{n-\nu}^h \varepsilon_{\nu+1}) &= \sum_{r=0}^{h+1} \binom{h+1}{r} \overset{r}{\Delta} (A_{n-\nu}^h) \overset{h+1-r}{\Delta} \varepsilon_{\nu+r+1} \\
&= \sum_{r=0}^{h+1} \binom{h+1}{r} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \varepsilon_{\nu+r+1} \\
&= \sum_{r=0}^{h} \binom{h+1}{r} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \varepsilon_{\nu+r+1} + A_{n-\nu}^{-1} \varepsilon_{\nu+h+2}.
\end{aligned}$$

Therefore

$$\sum_{\nu=1}^{n} A_{n-\nu}^{h}\varepsilon_{\nu+1}(-1)^{\nu}\nu^{\rho} = \sum_{r=0}^{h}\binom{h+1}{r}\sum_{\nu=0}^{n} C_{\nu,\rho}^{h}A_{n-\nu}^{h-r}\overset{h+1-r}{\Delta}\varepsilon_{\nu+r+1}$$
$$+\sum_{\nu=0}^{n} C_{\nu,\rho}^{h}A_{n-\nu}^{-1}\varepsilon_{\nu+h+2}$$
$$=0\left(\sum_{\nu=0}^{n}\nu^{h}(n-\nu+1)^{h-r}\left|\overset{h+1-r}{\Delta}\varepsilon_{\nu+r+1}\right|\right)$$
$$+0(n^{h}\varepsilon_{n+h+2}),$$

by lemma 1.

Therefore the expression in (1.4.6) is

$$=0\left(\sum_{n=1}^{\infty} n^{-h-2}\lambda_{n+h+1}\sum_{\nu=0}^{n}\nu^{h}(n-\nu+1)^{h-r}\left|\overset{h-r+1}{\Delta}\varepsilon_{\nu+r+1}\right|\right)$$
$$+0\left(\sum_{n=1}^{\infty} n^{-h-2}\lambda_{n+h+1}n^{h}\varepsilon_{n+h+2}\right)$$
$$=0\left(\sum_{\nu=1}^{\infty}\nu^{h}\left|\overset{h-r+1}{\Delta}\varepsilon_{\nu+r+1}\right|\sum_{n=\nu}^{\infty}(n-\nu+1)^{h-r}n^{-h-2}\lambda_{n+h+1}\right)$$
$$+0\left(\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\right).$$
$$=0\left(\sum_{\nu=1}^{\infty}\nu^{h}\left|\overset{h-r+1}{\Delta}\varepsilon_{\nu+r+1}\right|\lambda_{\nu+h+1}\sum_{n=\nu}^{\infty}(n-\nu+1)^{h-r}n^{-h-2}\right)+0(1)$$
$$=0\left(\sum_{\nu=1}^{\infty}\nu^{h}\left|\overset{h-r+1}{\Delta}\varepsilon_{\nu+r+1}\right|\lambda_{\nu+h+1}\cdot\nu^{-r-1}\right)+0(1)$$
$$=0\left(\sum_{\nu=1}^{\infty}\nu^{h-r-1}\frac{\varepsilon_{\nu+1}\lambda_{\nu}}{(\nu+1)^{h-r+1}}\right)+0(1)$$
$$=0\left(\sum_{\nu=1}^{\infty}\frac{\lambda_{\nu}}{\nu}\right)+0(1)$$
$$=0(1).$$

This completes the proof of the theorem.

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NIRANJAN SINGH
Department of Mathematics and Statistics
Aligarh Muslim University,
Aligarh. (UP), India.