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ON THE ANALYTICITY OF GENERALIZED EIGENFUNCTIONS (CASE OF REAL VARIABLES)

by Eberhard GERLACH

The present note is a direct continuation of Chapter III in our paper [1]; its purpose is to extend the results on analyticity of the generalized eigenfunctions to the case of proper functional Hilbert spaces consisting of functions which are (real-) analytic in a domain in Euclidean space. We continue to use the notation and numbering from Chap. III in [1].

Our basic tool will be the following.

PROPOSITION 4. — *Let* G *be a domain in Euclidean space* $\mathbf{R}^n$, *and* $\mathcal{B}$ *a class of functions defined everywhere in* G *and analytic there, and suppose that these form a proper functional Banach space* $\{\mathcal{B}, G\}$. *Then there exists a common domain* $\tilde{G}$ *in complex space* $\mathbf{C}^n$, *containing* G, *to which all* $f \in \mathcal{B}$ *can be extended analytically.*

Proof. — Since $\{\mathcal{B}, G\}$ is a p.f. Banach space, to every $x \in G$ there is an $L(x) \in \mathcal{B}'$ ($\mathcal{B}'$ is the continuous dual of $\mathcal{B}$) such that $f(x) = \langle f, L(x) \rangle$. This defines a function L from G into $\mathcal{B}'$ which is weakly-* real-analytic. It is well-known that Banachspace-valued functions defined on a complex domain which are weakly or weakly-* analytic are complex-analytic also in the strong topology. We shall show that L is strongly (real-) analytic; then it can be extended to a strongly analytic function $\tilde{L}$ (still into $\mathcal{B}'$) in some complex domain $\tilde{G}$ containing G. Finally each $f \in \mathcal{B}$ will be extended to an analytic function $\tilde{f}$ on $\tilde{G}$ by setting $f(z) = \langle f, \tilde{L}(z) \rangle$ for $z \in \tilde{G}$.

Recall (cf. for instance [2]) that for any function g which is analytic in the fixed domain $D \subset \mathbf{C}^1$ and for any compact $K \subset D$, there exists a finite number $M(g;\, K)$ such that for any choice of $\zeta,\ \zeta + \alpha,\ \zeta + \beta$ in K:

$$(6)\quad \left| \frac{1}{\alpha - \beta} \left\{ \frac{1}{\alpha} [g(\zeta + \alpha) - g(\zeta)] - \frac{1}{\beta} [g(\zeta + \beta) - g(\zeta)] \right\} \right| \leqslant M(g;\, K).$$

The same is true if instead of D one has a fixed open interval $I \subset \mathbf{R}^1$.

We shall establish existence of the strong derivatives $\frac{\partial}{\partial x_i} L(x)$ in $\mathcal{B}'$. These derivatives exist in the weak-* topology since for each $f \in B$

$$\frac{\partial}{\partial x_i} f(x) = \lim_{h \to 0} \frac{1}{h} (f(x + \varepsilon_i h) - f(x)) = \lim_{h \to 0} \frac{1}{h} \langle f,\ L(x + \varepsilon_i h) - L(x) \rangle.$$

Let N be a compact neighborhood of x; then there are numbers $M(f;\, N)$ so that for all sufficiently small h and k

$$\left\| \left\langle f,\ \frac{1}{h - k} \left\{ \frac{1}{h} [L(x + \varepsilon_i h) - L(x)] - \frac{1}{k} L(x + \varepsilon_i k) - L(x) \right\} \right\rangle \right\| \leqslant M(f;\, N).$$

Then by the uniform boundedness theorem, there is a constant $M(N)$ such that $\left\| \frac{1}{h - k} \{\ldots\} \right\| \leqslant M(N)$. Letting h and k tend to zero, one now obtains existence of the strong derivative $\frac{\partial}{\partial x_i} L(x)$. Since all derivatives of the $f \in \mathcal{B}$ are analytic, the preceding procedure can be repeated; thus L possesses strong derivatives of all orders. It is easy to check that L and all its derivatives are strongly continuous.

The Taylor series for L will converge strongly to the values of L if $\|(\alpha!)^{-1} D_\alpha L(x)\|^{\frac{1}{|\alpha|}}$ is uniformly bounded on compacts $K \subset G$, with a bound independent of α. (Here the α_i are non-negative integers, $\alpha = (\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n)$,

$|\alpha| = \Sigma\alpha_i$, $D_\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$, $\alpha ! = \alpha_1 ! \ldots \alpha_n !$). Since all $f \in \mathcal{B}$ are analytic,

$$|(\alpha !)^{-1} D_\alpha f(x)|^{\frac{1}{|\alpha|}} = |\langle f, (\alpha !)^{-1} D_\alpha L(x) \rangle|^{\frac{1}{|\alpha|}}$$

(for fixed f) is uniformly bounded on compacts $K \subset G$, independent of α. But for variable f, this expression is a sub-additive continuous functional on $\mathcal{B}$. By the uniform boundedness theorem then

$$\sup_{\|f\| \leqslant 1} |\langle f, (\alpha !)^{-1} D_\alpha L(x) \rangle|^{\frac{1}{|\alpha|}} = \Big(\sup_{\|f\|=1} |\langle \ldots \rangle|\Big)^{\frac{1}{|\alpha|}}$$
$$= \|(\alpha !)^{-1} D_\alpha L(x)\|^{\frac{1}{|\alpha|}}$$

is uniformly bounded on compacts $K \subset G$, independent of α. Thus L has a strongly convergent power series expansion in some neighborhood of any point $x \in G$.

For each $x \in G$, let $S(x)$ be the largest open ball in $\mathbf{C}^n$, centered at x, in which the Taylor series for L about x converges and set

$$\tilde{G} = \bigcup_{x \in G} S(x).$$

Then the series expansions yield an analytic continuation $\tilde{L}$ of L from G to $\tilde{G}$. Finally, for $f \in \mathcal{B}$, define

$$\tilde{f}(z) = \langle f, \tilde{L}(z) \rangle \quad \text{for} \quad z \in \tilde{G} \qquad \text{and} \qquad \|\tilde{f}\| = \|f\|;$$

this gives us a p.f. Banach space $\{\tilde{\mathcal{B}}, \tilde{G}\}$ which is isometrically isomorphic to $\{\mathcal{B}, G\}$. The proof of Proposition 4 is complete.

From now on, $\{\mathcal{F}, G\}$ will denote a p.f. Hilbert space consisting of analytic functions on a domain $G \subset \mathbf{R}^n$. Our aim is to extend the results of Corollary 2. III and Theorem 3. III in [1] to such spaces.

The anti-space $\overline{\mathcal{F}}$ of the Hilbert space $\mathcal{F}$ is identified with the dual $\mathcal{F}'$, and $\mathcal{F}$ itself with its continuous anti-dual $\mathcal{F}^*$ $(= \overline{\mathcal{F}}' = \overline{\mathcal{F}'})$ [1] by means of the canonical mappings J and θ:

$\mathcal{F}' = J\mathcal{F}$ where J is the anti-isomorphism $f \to Jf = (\cdot, f)$

(1) For these notations, cf. L. Schwartz [3].

and

$\mathcal{F}^* = \theta\mathcal{F}$ *where* θ *is the isomorphism* $f \to \theta f = (f, .)$.

If K is the reproducing kernel of $\mathcal{F}$ then for $f \in \mathcal{F}$

$$f(x) = (f, K(., x)) = \langle f, L(x) \rangle \quad \textit{for every } x \in G$$

where $\langle\ ,\ \rangle$ denotes the pairing of $\mathcal{F}$ and $\mathcal{F}'$. Thus $L(x) = JK_x$ and

$$K(x, y) = (K_y, K_x) = \langle K_y, JK_x \rangle = \langle J^{-1}L(y), L(x) \rangle.$$

By Proposition 4, L and F extend analytically to a complex domain $\tilde{G}$; we obtain the p.f. Hilbert space $\{\tilde{\mathcal{F}}, \tilde{G}\}$ with r.k. $\tilde{K}$:

$$\tilde{f}(z) = \langle f, \tilde{L}(z) \rangle = (f, J^{-1}\tilde{L}(z))$$

and

$$\tilde{K}(z, w) = \langle J^{-1}\tilde{L}(w), \tilde{L}(z) \rangle = (\tilde{K}_w, \tilde{K}_z).$$

Since the function $\tilde{L}$ is strongly analytic from $\tilde{G}$ into $\mathcal{F}'$ and $\tilde{K}_z = J^{-1}\tilde{L}(z)$, we note that $\tilde{K}(., z)$ is strongly anti-analytic for $z \in \tilde{G}$ (i.e. the function $\bar{z} \to \tilde{K}(., z)$ is strongly analytic from $\bar{\tilde{G}} = \{z | \bar{z} \in \tilde{G}\}$ into $\mathcal{F}'$). Let U denote the extension isomorphism $U : \mathcal{F} \to \tilde{\mathcal{F}}$ constructed by Proposition 4. If $\{g_k\}$ is a complete orthonormal system in $\mathcal{F}$, then so is $\{\tilde{g}_k = Ug_k\}$ in $\tilde{F}$ and $\tilde{K}(z, w) = \sum_{k=1}^{\infty} \tilde{g}_k(z)\overline{\tilde{g}_k(w)}$ for $z, w \in \tilde{G}$, i.e., $\tilde{K}$ is also a « direct » continuation of K.

Corollary 2'. — *Let* G *be an arbitrary domain in* $\mathbf{R}^n$ *and* $\{\mathcal{F}, G\}$ *any p.f. Hilbert space of functions (real-) analytic in* G. *Then* $\{\mathcal{F}, G\}$ *is Hilbert-Schmidt expansible.*

Proof. — By Corollary 2, there is an H.S. operator T in $\tilde{\mathcal{F}}$ such that $\tilde{K}_\zeta \in T\tilde{\mathcal{F}}$ for all $\zeta \in \tilde{G}$. Now $S = U^{-1}TU$ is H.S. in $\mathcal{F}$, and $K_\xi \in S\mathcal{F}$ for all $\xi \in G$.

Now let A be a selfadjoint operator in $\mathcal{F}$ with resolution of identity E(.) and spectral measure μ. Then $(f, g) = (Uf, Ug)^{\sim}$ for all $f, g \in \mathcal{F}$. The operator $\tilde{A} = UAU^{-1}$ is selfadjoint in $\tilde{\mathcal{F}}$ and unitarily equivalent to A; its resolution of identity is $\tilde{E}(.) = UE(.)U^{-1}$, and μ is also a spectral measure for $\tilde{A}$.

Both $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are H.S.-expansible. Let $\tilde{\Lambda}_{\tilde{G}}$ denote the complement in $\mathbf{R}^1$ of the set of all λ for which

$$\frac{d(\tilde{E}(\lambda)\tilde{K}_w, E(\lambda)\tilde{K}_z)}{d\mu(\lambda)} = \tilde{K}(z, w; \lambda) \quad \textit{exists and is finite for all} \quad z, w \in \tilde{G}$$

(similar definition for Λ_G, without tildas). Then $\tilde{\Lambda}_{\tilde{G}} \supset \Lambda_G$ and $\mu(\tilde{\Lambda}_{\tilde{G}}) = 0$. Let $\tilde{\mathcal{F}}_{\tilde{G}}^{(\lambda)}$ $(\mathcal{F}_G^{(\lambda)})$ be the p.f. Hilbert space on $\tilde{G}(G)$ defined by the r.k. $\tilde{K}(.,\ .;\lambda)$ $(K(.,\ .;\lambda))$. For $\tilde{f} \in \tilde{\mathcal{F}}$, let $\tilde{\Lambda}_{\tilde{f},(\tilde{G})}$ be the smallest set containing $\tilde{\Lambda}_{\tilde{G}}$ such that for all $\lambda \notin \tilde{\Lambda}_{\tilde{f},(\tilde{G})}$:

$$\begin{cases} \dfrac{d(\tilde{E}(\lambda)\tilde{f},\ \tilde{E}(\lambda)\tilde{K}_z)}{d\mu(\lambda)} = \tilde{f}(z;\lambda) \quad \textit{exists, is finite} \\ \textit{and} = 0 \quad \textit{whenever} \quad \tilde{K}(z, z; \lambda) = 0, \quad \textit{for all} \quad z \in \tilde{G} \end{cases}$$

and

$$\tilde{f}(.,\ \lambda) \in \tilde{\mathcal{F}}_{\tilde{G}}^{(\lambda)}, \ \frac{d\|\tilde{E}(\lambda)\tilde{f}\|^2}{d\mu(\lambda)} \quad \textit{exists and equals} \quad \|\tilde{f}(.\,;\lambda)\|^2_{\tilde{\mathcal{F}}_{\tilde{G}}^{(\lambda)}}$$

(similar definition for $\Lambda_{f,(G)}$, without tildas). The correspondence $\tilde{f} \to \tilde{f}(.\,;\lambda)$ defines $\tilde{P}_{\tilde{G}}^{(\lambda)}$ with domain $\tilde{\mathfrak{D}}_{\tilde{G}}^{(\lambda)} = \{\tilde{f} | \lambda \notin \tilde{\Lambda}_{\tilde{f},(\tilde{G})}\}$. For $\tilde{f} \in \tilde{\mathfrak{D}}_{\tilde{G}}^{(\lambda)}$, $f(.\,;\lambda)$ is just the restriction of $\tilde{f}(.\,;\lambda)$ to the domain G.

THEOREM 3'. — *Let* A *be an arbitrary selfadjoint operator in* $\{\mathcal{F}, G\}$ *with spectral measure* μ. *Then there is a set* Λ *on the real line,* $\mu(\Lambda) = 0$, *which is determined by Theorem* 3 (*and also Corollary* 2, *Theorem* 11. I, *and the above considerations*) *such that the generalized eignefunctions*

$$\frac{dE(\lambda)f(x)}{d\mu(\lambda)} = f(x;\lambda) \in \mathcal{F}_G^{(\lambda)} \quad \textit{for} \quad \lambda \notin \Lambda \quad \textit{and} \quad \tilde{f} \in \tilde{\mathfrak{D}}_{\tilde{G}}^{(\lambda)}$$

are real-analytic in the whole domain G.

Proof. — According to the preceding preparations, set $Uf = \tilde{f}$. If $\lambda \notin \Lambda$ and $\tilde{f} \in \tilde{\mathfrak{D}}_{\tilde{G}}^{(\lambda)}$ then $\tilde{f}(.\,;\lambda)$ is analytic in $\tilde{G}$ by Theorem 3, and consequently its restriction $f(.\,;\lambda)$ is (real-) analytic in G.

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