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## SMALL SUBSETS OF FINITE ABELIAN GROUPS

par Robert KAUFMAN

Varopoulos [2], in constructing certain sets in infinite compact abelian groups, made special use of various properties of finite groups. In this note a method is given by which many of the results of [2] can be recovered in a uniform manner. To explain the utility of the theorem stated below, let  $G = \prod_{i=1}^{\infty} G_i$  be the product of infinitely many finite abelian groups,  $\mu_i$  a probability measure on  $G_i$  ( $1 \leq i < \infty$ ). As usual, the Fourier-Stieltjes transform is defined for a measure  $\nu$  as

$$\hat{\nu}(x) = \int_G \overline{x(t)} \nu(dt), \quad x \in \hat{G}.$$

Then  $\hat{\mu}(x) \longrightarrow 0$  as  $x \longrightarrow \infty$  in  $\hat{G}$  if and only if

$$\max \{ |\hat{\mu}_i(x_i)| : x_i \in \hat{G}_i, x_i \neq 1 \}$$

converges to 0 as  $i \longrightarrow \infty$ .

**THEOREM.** — *For each  $\varepsilon > 0$  there is an  $M$  such that any finite abelian group  $G$  of order  $> M$  contains a subset  $S$  with the properties*

1)  $\log |S| \leq \varepsilon \log |G|$

2)  $|\sum_{s \in S} x(s)| < \varepsilon |S|$  for any character  $x \neq 1$  of  $G$ .

(Here  $|A|$  means the number of elements of a set  $A$ ).

Observe that by 2)  $S$  generates  $G$ , and by 1) that the  $r$ -fold sum  $\pm S \pm \dots \pm S$  contains at most  $2^r |S|^r$  elements ; this is the kind of estimate needed often in [2].

In the proof it is necessary to decompose finite subgroups of the circle  $T$ , the decomposition depending upon a positive integer  $K$ . If  $H$  is such a subgroup and  $H$  has order  $r \leq K^2$ ,  $H$  is partitioned into singletons. Otherwise, let  $\omega$  be an element of  $H - \{1\}$  for which the Euclidean distance  $|\omega - 1|$  is smallest. Set

$$H_i = \{\omega^j : (i-1)rK^{-1} \leq j < irK^{-1}\}, 1 \leq i < K,$$

$$H_K = \{\omega^j : rK^{-1}(K-1) \leq j < r\}.$$

Note that  $rK^{-1} \leq |H_i| + 1 \leq rK^{-1} + 1$ .

Each character  $x \neq 1$  maps  $G$  onto a finite subgroup  $H$  of  $T$ , decomposed into subsets depending on some  $K$  chosen in advance. Suppose that for some  $p > 0$  and every  $H_i$ ,  $1 \leq i \leq K^2$ , depending on any  $x \neq 1$ ,

$$p \left(1 - \frac{1}{2} \varepsilon\right) |x^{-1}(H_i)| \leq |S \cap x^{-1}(H_i)| \leq p \left(1 + \frac{1}{2} \varepsilon\right) |x^{-1}(H_i)|.$$

Then requirement 2) is met for large enough  $K$ . Because  $|S| = o(p) \cdot |G|$ , requirement 1) for large  $|G|$  follows from

$$\log p = \left(\frac{1}{2} \varepsilon - 1\right) \log |G|.$$

$S$  is chosen as a "random" element of  $2^G$ , each element being chosen with probability  $p$ . More precisely, let  $\{X_g : g \in G\}$  be a set of  $|G|$  independent random variables indexed by  $G$ , and

$$P\{X_g = 1\} = p, P\{X_g = 0\} = 1 - p, g \in G.$$

Of course  $g \in S$  means  $X_g = 1$  and so

$$|S \cap x^{-1}(H_i)| = \sum_{g \in x^{-1}(H_i)} X_g.$$

All of this can be stated without reference to any special properties of  $G$ . The sums in question are, in number,  $\leq K^2 |G|$ , and the number of independent random variables in each sum is  $\geq K^{-2} |G|$ . Write

$Y = \sum_1^N X_j$  for any of these sums ; it is enough to show that for all such  $Y$ ,

$$P\left\{|Y - pN| \geq \frac{1}{2} \varepsilon pN\right\} = o(|G|)$$

as  $|G| \rightarrow \infty$ .

Writing  $\bar{E}$  for expectation, for any number  $\lambda$

$$\bar{E}(e^{\lambda Y}) = \bar{E}^N(e^{\lambda X_1}) = (1 + p(e^\lambda - 1))^N.$$

If

$$\lambda = \log\left(1 + \frac{1}{2} \varepsilon\right) \text{ and } Y \geq \left(1 + \frac{1}{2} \varepsilon\right) pN, e^{\lambda Y} \geq \left(1 + \frac{1}{2} \varepsilon\right)^{(1 + \frac{1}{2} \varepsilon)pN},$$

so that

$$P\left\{Y \geq \left(1 + \frac{1}{2} \varepsilon\right) pN\right\} \leq \left(1 + \frac{1}{2} \varepsilon p\right)^N / \left(1 + \frac{1}{2} \varepsilon\right)^{(1 + \frac{1}{2} \varepsilon)pN},$$

$$\begin{aligned} N^{-1} \log P\left\{Y \geq \left(1 + \frac{1}{2} \varepsilon\right) pN\right\} &\leq \log\left(1 + \frac{1}{2} \varepsilon p\right) \\ &\quad - p\left(1 + \frac{1}{2} \varepsilon\right) \log\left(1 + \frac{1}{2} \varepsilon\right). \end{aligned}$$

Now  $\log\left(1 + \frac{1}{2} \varepsilon p\right) - p\left(1 + \frac{1}{2} \varepsilon\right) \log\left(1 + \frac{1}{2} \varepsilon\right) =$

$$O(p^2) + \frac{1}{2} \varepsilon p - p\left(1 + \frac{1}{2} \varepsilon\right) \log\left(1 + \frac{1}{2} \varepsilon\right),$$

while  $s < (1 + s) \log(1 + s)$  for  $s > 0$ .

Hence  $P\left\{Y \geq \left(1 + \frac{1}{2} \varepsilon\right) pN\right\} \leq c^{pN}$

for some  $c < 1$ .

This is clearly  $o(|G|)$ , and the magnitude of  $P\left\{Y \leq \left(1 - \frac{1}{2} \varepsilon\right) Np\right\}$  is subject to entirely similar estimates. The proof is complete.

We indicate briefly how the argument can be applied to the  $a$ -adic integers, defined by Hewitt and Ross ([1], pp. 108-111). The analogue of the first result is as follows.

For each  $\varepsilon > 0$  there is an  $M$  such that for any numbers

$$a_1 > 1, \dots, a_N > 1 \quad \text{with} \quad a_1 \dots a_N > M,$$

there is a subset  $S$  of the set  $\{(u_1, \dots, u_N), 0 \leq u_i < a_i\}$  with the properties

$$1') \log |S| \leq \varepsilon \log(a_1 \dots a_N).$$

2') For any complex number  $w \neq 1$ , but  $w^{a_1 a_2 \dots a_N} = 1$ ,

$$\left| \sum_{s \in S} w^{u_1 + a_1 u_2 + \dots + a_1 \dots a_{N-1} u_N} \right| < \varepsilon |S|.$$

In fact, the collection of all possible exponents

$$\{u_1 + a_1 u_2 + \dots + a_1 \dots a_{N-1} u_N : 0 \leq u_i < a_i\}$$

is a complete residue system modulo  $a_1 a_2 \dots a_N$ , so the previous arguments are valid.

#### BIBLIOGRAPHIE

- [1] E. HEWITT and K.A. ROSS, *Abstract Harmonic Analysis I*, (1963).  
 [2] N. Th. VAROPOULOS, Sets of multiplicity in locally compact abelian groups, *Ann. Inst. Fourier, Grenoble*, XVI (1966), 123-158.

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