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Theory of Bessel potentials. II

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THEORY OF BESSEL POTENTIALS. PART II.

by R. ADAMS, N. ARONSZAJN and K. T. SMITH

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Foreword.

This foreword is addressed to those who have read the first version of this paper issued as Technical Report 26. We decided to publish a second version for the following reason. Our main interest is the space $P^\alpha(D)$, the restriction of the space $P^\alpha(R^n)$ (cf. Part. I) to an open subset D of R^n. Since we do not have any general intrinsic definition of the space $P^\alpha(D)$, in Report 26 we introduced the space $\check{P}^\alpha(D)$ which is defined in an intrinsic and direct manner and which in general satisfies $\check{P}^\alpha(D) \supset P^\alpha(D)$. Our main concern was to define a class of open sets D, as large as possible, for which $\check{P}^\alpha(D) = P^\alpha(D)$. After the first version appeared we noticed that we could define a functional space (denoted here temporarily by $\check{\check{P}}^\alpha(D)$) by a definition as simple as that of $\check{P}^\alpha(D)$ and which better approximates the class $P^\alpha(D)$, more precisely, $\check{P}^\alpha(D) \supset \check{\check{P}}^\alpha(D) \supset P^\alpha(D)$. Furthermore, we noticed that all the theorems we obtained for $\check{P}^\alpha(D)$ have their analogues for the class $\check{\check{P}}^\alpha(D)$. In most cases these analogues are simpler to state, admit simpler proofs, and often are more general. We decided, therefore, to abandon the preceding definition of $\check{P}^\alpha(D)$ and, starting with the present paper, we deal only with the class $\check{\check{P}}^\alpha(D)$ which will be denoted from now on by $\check{P}^\alpha(D)$ (1).

Another change in the second version is due to the fact that we were able to apply the idea of Lichtenstein reflection of order p (in particular the reflection of order ∞ introduced by R. T. Seeley [11] in the case of hyperplanes) to the Lipschitzian graph domains. By such generalized Lichtenstein exten-

(1) This change of definition was already applied in some published papers and in papers which are to appear soon.

sion we obtain an unrestricted, *simultaneous* extension theorem for SLG-domains. In the previous version we used the Calderon extension method which gives only simultaneous extensions between two consecutive integers.

Other improvements and additions are described in the Introduction.

Introduction.

This second part of the « Theory of Bessel Potentials » contains only Chapter III, which deals with open subsets of a euclidean space. Chapter IV, which will appear subsequently will deal with potentials on manifolds.

The most natural way of defining potentials in a subdomain D of R^n is to define them as restrictions of potentials in the whole space. This class is denoted by $P^\alpha(D)$. However, this is not an intrinsic definition and it would be very inconvenient in applications. We therefore introduce the class $\check{P}^\alpha(D)$ formed by functions in $P^\alpha_{loc}(D)$ and having a finite standard norm $|u|_{\alpha,D}$. In the past this standard norm has been used for integral orders α. We now extend it to non-integral α by formula (2.1) of § 2. We also introduce the approximate α-norm, $|u\rceil_{\alpha,D}$, by formula (2.3) of § 2 which is equivalent to the standard α-norm but is simpler to use.

The main purpose of this paper is to investigate the class $\check{P}^\alpha(D)$ and its relations to $P^\alpha(D)$. Since the functions in $P^\alpha(D)$ are restrictions of those in P^α, the properties of $P^\alpha(D)$ are already very well determined. It is of importance therefore to investigate those domains where $\check{P}^\alpha(D) = P^\alpha(D)$. For all « regular » domains D this equality holds. It is one of the main purposes of the paper to find as general classes of domains as possible for which this equality still holds. The equality can be characterized by the fact that for the class $\check{P}^\alpha(D)$ there exists a linear continuous extension-mapping into the class $P^\alpha(R^n)$ assigning to each $u \in \check{P}^\alpha(D)$ an extension $\tilde{u}$, $\tilde{u} \in P^\alpha(R^n)$, defined in the whole space.

In addition to the main problem of extension we investigate many other properties of functions in $\check{P}^\alpha(D)$.

We give now a brief summary of the sections of this paper.

§ 0 takes up an idea proposed in the last remark of Chap. II, (going back to Lebesgue), i.e. to « correct » any given locally integrable function in a well determined way (e.g. by taking limits of the mean values) so that if the function is equivalent to a « nice » function, the corrected function will also be nice. We use this idea in many instances by assuming, a priori, that we are dealing with « corrected » functions and this makes for simpler statements and proofs.

§ 1 introduces the class $P^\alpha(D)$ and gives translations (for the most part obvious) of the properties of P^α into those of $P^\alpha(D)$.

§ 2. The class $\check{P}^\alpha(D)$ is defined. Among other things, we take up the problem of multipliers and show that if $u \in \check{P}^\alpha(D)$ and if φ and all of its derivatives up to order α^* (α^* is the greatest integer $< \alpha$) are Lipschitzian on D, then $u \to \varphi u$ is a bounded mapping of $\check{P}^\alpha(D)$ into $\check{P}^\alpha(D)$.

§ 3 deals with the problem of inessential singularities of functions in $\check{P}^\alpha(D)$ and shows under what circumstances it is possible to extend $u \in \check{P}^\alpha(D)$ to a function in $\check{P}^\alpha(D_1)$ when D_1 differs from D by a set of Lebesgue measure 0. This result is applied to develop the Lichtenstein extension method for hyperplanes.

In § 4 the behavior of the standard and approximate norms for a single function u when α varies is considered. In particular the properties of the standard and approximate norms are studied as α converges to an integer. It is shown that $|u|_{\alpha,D}$ is a continuous function of α when D is convex.

§ 5 introduces the idea of boundary properties and their localization. This is applied to convex domains to define the L-convex domains which have, essentially, the local structure of $C^{(0,1)}$-homeomorphic images of convex domains. This is the largest class in which we can prove essentially all the properties of functions in $P^\alpha(D)$ for the functions in $\check{P}^\alpha(D)$ (it is not known if for L-convex D, $P^\alpha(D) = \check{P}^\alpha(D)$ for all α).

In § 6 we define the graph domains and show that $P^\alpha(D)$ is dense in $\check{P}^\alpha(D)$ for this class of domains.

§ 7 introduces the class of domains $\mathcal{E}(I)$ where I is an interval in $[0, \infty)$. $D \in \mathcal{E}(I)$ if there is a linear extension mapping of the smallest linear class of functions containing $\bigcup_{\alpha \in I} \check{P}^\alpha(D)$

into the measurable functions on R^n such that the mapping carries $\check{P}^\alpha(D)$ boundedly (uniformly on every compact subinterval of I) into $P^\alpha(R^n)$ for $\alpha \in I$. A localization theorem is proved for the class $\mathcal{E}(I)$; in imprecise terms it can be stated as follows: if D is locally in $\mathcal{E}(I)$ then $D \in \mathcal{E}(I)$.

§ 8 and § 9 are preparatory sections for the main theorems of the paper which are proved in § 10 and § 11. In § 8 the regularized distance and singular multipliers are defined and a number of their properties are proved.

§ 9 gives the necessary theoretical background for the constructions in § 10. It is the most involved section of the present paper, made more complicated by the necessity of obtaining uniform bounds in different theorems in order to have uniform bounds in the simultaneous extension theorems of § 10.

§ 10. We introduce the notions of uniform and regular systems $\{Q_i\}$ where $Q_i = \overline{D}_i$ and the D_i are open sets satisfying certain properties. The main results of this section are concerned with the case where all the D_i's are in $\mathcal{E}(I)$ and we determine additional conditions under which $\left(\bigcup_i Q_i\right)^0$ — the interior of $\bigcup_i Q_i$ — also belongs to $\mathcal{E}(I)$.

In § 11 we define the generalized Lichtenstein extension. We use this extension to prove that if ∂D is locally Lipschitzian then $D \in \mathcal{E}([0, \infty))$ i.e. there is a simultaneous linear extension of $\check{P}^\alpha(D)$ into $P^\alpha(R^n)$ for all $\alpha \geq 0$.

In § 12 we apply the results of § 10 and § 11 to establish the extension theorem for some concrete classes of domains. First we find a necessary and sufficient condition for a convex domain D (bounded or unbounded) to be in $\mathcal{E}([0, \infty))$. The condition is that for some bounded cone C and for every $x \in \partial D$ there is a congruent cone with vertex x contained in $\overline{D}$. Secondly we prove that all finite geometric polyhedra which are not locally disconnected by their boundary belong to $\mathcal{E}([0, \infty))$.

§ 13 gives different examples and counter examples concerning the subjects treated in the paper.

In recent years a great amount of work was done concerning Bessel potentials connected with L^p classes in R^n [2, 3, 4, 7, 10, 12, 13]. Of the different classes considered only the classes $P^{\alpha,p}$ (2) lend themselves to a treatment in open subsets of R^n analogous to the one given in the present paper for the class P^α ($\check{P}^\alpha(D)$ is identified in this connection with $\check{P}^{\alpha,2}(D)$). It turns out that with very few exceptions, for all statements concerning $\check{P}^\alpha(D)$ there exist parallel statements concerning $\check{P}^{\alpha,p}(D)$. At the end of most of the sections of this paper remarks are made to show in what manner the results of the section can be extended to the class $\check{P}^{\alpha,p}(D)$. Since almost all the exceptions occur in the cases $p = 1$ and $p = \infty$, we restrict these statements essentially to the case $1 < p < \infty$.

Finally, we have added an Appendix where we treat the question of complete continuity for the standard norms of different orders. The problem for non-integral orders differs in many aspects from the one for integral orders which has already been investigated by many authors.

The bibliography contains only references which were not given in Part. I.

For the convenience of the reader a list of symbols, notations and terminology introduced in this paper is given after the bibliography.

(2) See [2]. These classes are perfect completions which correspond to the classes W_p^α [2, 3, 7, 10, 12, 13]; these latter classes appear as completions relative to the class of sets of Lebesgue measure 0.

CHAPTER III

THE SPACES $P^{\alpha}(D)$ AND $\check{P}^{\alpha}(D)$.

0. The corrected functions.

In the last remark of Part I (Chap. II, § 11) we introduced the corrected function u' for any function u locally integrable in an open set $D \subset R^n$ ($u \in L^1_{loc}(D)$) by defining

$$u'(x) = \lim_{\rho \searrow 0} \frac{1}{|S(x, \rho)|} \int_{S(x,\rho)} u(y)\, dy, \tag{0.1}$$

for all $x \in D$ for which the limit exists and is finite. All the other points of D form the exceptional set of u' : the corrected function is not defined there.

The idea of such a « correction » goes back to Lebesgue; $u'(x) = u(x)$ a.e. and the exceptional set is of measure 0. Since in some instances the use of corrected functions allows a simplification of statements and proofs we will review the main properties of corrected functions in this section.

If we introduce the function $\varphi^0(x) = \dfrac{n}{\omega_n} \chi^0(x)$, where χ^0 is the characteristic function of the unit sphere $S(0, 1)$ and ω_n is the area of $\partial S(0, 1)$, formula (0.1) can be written

$$u'(x) = \lim_{\rho \searrow 0} \int u(x - y) \rho^{-n}\, \varphi^0(y/\rho)\, dy. \tag{0.1'}$$

This formula suggests a more general procedure to define corrected functions. We consider any bounded measurable function $\varphi(x)$ vanishing outside of a compact such that

$$\int \varphi(x)\,dx = 1. \tag{0.2}$$

We put then

$$u^\varphi(x) = \lim_{\rho \searrow 0} \int u(x-y)\rho^{-n}\,\varphi(y/\rho)\,dy \tag{0.3}$$

for all $x \in \mathrm{D}$ for which the limit exists and is finite. The other points of D form the exceptional set of u^φ.

Obviously u^φ is the same for all u in the same equivalence class (relative to Lebesgue measure). The following proposition is also obvious.

1) *If* $u_k \in \mathrm{L}^1_{\mathrm{loc}}(\mathrm{D})$, $k = 1, 2$, α_k *are complex numbers,*

$$u = \alpha_1 u_1 + \alpha_2 u_2,$$

and if A^φ *and* A^φ_k *denote the exceptional sets of* u^φ *and* u^φ_k *respectively, then* $\mathrm{A}^\varphi \subset \mathrm{A}^\varphi_1 \cup \mathrm{A}^\varphi_2$ *and*

$$u^\varphi(x) = \alpha_1 u^\varphi_1(x) + \alpha_2 u^\varphi_2(x) \qquad \textit{for} \qquad x \in \mathrm{D} - (\mathrm{A}^\varphi_1 \cup \mathrm{A}^\varphi_2).$$

For $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathrm{D})$ the Lebesgue set of u is the set of all $x \in \mathrm{D}$ for which there exists a number $u^{\mathrm{L}}(x)$ such that

$$\lim_{\rho \searrow 0} \frac{1}{|\mathrm{S}(x,\rho)|} \int_{\mathrm{S}(x,\rho)} |u(y) - u^{\mathrm{L}}(x)|\,dy \equiv \lim_{\rho \searrow 0} \int |u(x-y) - u^{\mathrm{L}}(x)|\rho^{-n}\varphi^0(y/\rho)\,dy = 0. \tag{0.4}$$

A classical theorem of Lebesgue says that the exceptional set of u^{L} (i.e. D — the Lebesgue set) is of measure 0 and that

$$u^{\mathrm{L}}(x) = u(x)$$

a.e. in D.

As a counterpart of Prop. 1) we get immediately

1') *If* $u_k \in \mathrm{L}^1_{\mathrm{loc}}(\mathrm{D})$, $k = 1, 2$, $u = \alpha_1 u_2 + \alpha_2 u_2$ *and* A^{L} *and* $\mathrm{A}^{\mathrm{L}}_k$ *are the exceptional sets of* u^{L} *and* u^{L}_k *respectively then*

$$\mathrm{A}^{\mathrm{L}} \subset \mathrm{A}^{\mathrm{L}}_1 \cup \mathrm{A}^{\mathrm{L}}_2$$

and

$$u^{\mathrm{L}}(x) = \alpha_1 u^{\mathrm{L}}_1(x) + \alpha_2 u^{\mathrm{L}}_2(x) \qquad \textit{for} \qquad x \in \mathrm{D} - (\mathrm{A}^{\mathrm{L}}_1 \cup \mathrm{A}^{\mathrm{L}}_2).$$

2) *If* $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathrm{D})$, *then for any* φ, u^φ *is an extension of* u^{L}.

In fact, if x is in the Lebesgue set of u, M is the bound of φ and $\varphi(y) = 0$ for $|y| > \delta$, then

$$\begin{aligned}\left|\int u(x-y)\rho^{-n}\varphi(y/\rho)\,dy - u^{\mathrm{L}}(x)\right| &= \left|\int [u(x-y) - u^{\mathrm{L}}(x)]\rho^{-n}\,\varphi(y/\rho)\,dy\right| \\ &\leqslant \int |u(x-y) - u^{\mathrm{L}}(x)|\rho^{-n}|\varphi(y/\rho)|\,dy \\ &\leqslant \frac{\omega_n}{n}\mathrm{M}\delta^n \int |u(x-y) - u^{\mathrm{L}}(x)|(\rho\delta)^{-n}\,\varphi^0(y/\rho\delta)\,dy,\end{aligned}$$

and by (0.4) the last integral converges to 0 for $\rho \searrow 0$.

It follows from Prop. 2) that u, u^{L}, and all the u^{φ} are in the same equivalence class. We may call u^{L} the *minimal* corrected function. The introduction of the function u^{φ} besides u^{L} is justified by the fact that in many cases it is easier to find the set where u^{φ} is defined rather then the Lebesgue set. A case in point is the following application to Fourier transforms of L^2 functions.

3) *Let* $u \in \mathrm{L}^2(\mathrm{R}^n)$. *The corrected function* u^{φ} *is given pointwise by*

$$u^{\varphi}(x) = \lim_{\rho \searrow 0} \int e^{i(x,\xi)}\hat{u}(\xi)\,\hat{\varphi}(\rho\xi)\,d\xi \tag{0.5}$$

and the exceptional set of u^{φ} *is the set of* x*'s where the limit does not exist (or is infinite).*

The proof is immediate since

$$(2\pi)^{n/2}\hat{u}(\xi)\hat{\varphi}(\rho\xi) = (u(x) * \rho^{-n}\varphi(x/\rho))^{\wedge}$$

and $\hat{\varphi}(\xi)$ is an entire function of order 1, L^2 on R^n, so that $\hat{u}(\xi)\hat{\varphi}(\rho\xi) \in \mathrm{L}^1$ and the integral in (0.5) is an ordinary Lebesgue integral.

For different functions u we may choose different functions φ to simplify the integration in (0.5). We give here the transform of the simplest functions φ.

a) *Spherical means*, $\varphi = \varphi^0$, $\hat{\varphi}(\xi) = \dfrac{n}{\omega_n}\,|\xi|^{-n/2}\,\mathfrak{J}_{n/2}(|\xi|)$ where $\mathfrak{J}_{n/2}(|\xi|)$ is the Bessel function of first kind of order $n/2$;

b) *Cubic means*, $\varphi =$ characteristic function of the cube $|x_k| < 1/2$, $k = 1, \ldots, n$, $\hat{\varphi}(\xi) = \displaystyle\prod_{k=1}^{n}\sqrt{\frac{2}{\pi}}\,\frac{\sin\frac{1}{2}\xi_k}{\xi_k}$.

We add two more general propositions:

4) *Let* $u \in L^1_{loc}(D)$ *and* $\varphi \in L^\infty_{loc}(D)$. *The function* $(u\varphi)^L$ *is an extension of* $u^L \varphi^L$ (3).

In fact, let x be in the intersection of the Lebesgue sets of u and φ. Then

$$\frac{1}{|S(x,\rho)|}\int_{S(x,\rho)} |u(y)\varphi(y) - u^L(x)\varphi^L(x)|\,dy$$
$$\leqslant \frac{1}{|S(x,\rho)|}\Big[\int_{S(x,\rho)} |u(y) - u^L(x)|\,|\varphi(y)|\,dy + \int_{S(x,\rho)} |u^L(x)||\varphi(y) - \varphi^L(x)|\,dy\Big]$$

and the right side $\to 0$ for $\rho \downarrow 0$ by virtue of (0.4) since

$$\varphi \in L^\infty_{loc}(D).$$

5) *Let* T *be a homeomorphism of* D *onto* D* *such that* T *and* T^{-1} *are locally Lipschitzian. Then if* $u \in L^1_{loc}(D)$ *and*

$$u^*(x^*) = u(T^{-1}x^*),$$

the Lebesgue set of u *is mapped onto the Lebesgue set of* u^* *and* $u^{*L}(x^*) = u^L(T^{-1}x^*)$.

Proof. — For $x \in D$ there are constant M and ρ_0 such that for $|y - x| < M^{-1}\rho_0$, $M^{-1}|y - x| \leqslant |Ty - Tx| \leqslant M|y - x|$ and the Jacobian $\dfrac{\partial(y)}{\partial(Ty)}$ is majorated a.e. by M^n. Hence for $\rho \leqslant \rho_0$ and any number u_0 we have

$$\frac{M^{-2n}}{|S(x, M^{-1}\rho)|}\int_{|x-y|<M^{-1}\rho} |u(y) - u_0|\,dy$$
$$\leqslant \frac{1}{|S(x,\rho)|}\int_{|y^*-Tx|<\rho} |u(T^{-1}y^*) - u_0|\,dy^*$$
$$\leqslant \frac{M^{2n}}{|S(x, M\rho)|}\int_{|x-y|<M\rho} |u(y) - u_0|\,dy,$$

from which the proposition follows by letting $\rho \downarrow 0$.

The application of the corrected function to Bessel potentials is based on the following proposition.

(3) This proposition is not true in general if the hypothesis is changed to $u \in L^p_{loc}(D)$, $\varphi \in L^q_{loc}(D)$ with $1 < p, q < \infty$, $1/p + 1/q = 1$.

6) *Let $d\mu$ be a signed Borel measure such that*

$$\int G_\alpha(x-y)\, d\mu(y)$$

is defined and finite for at least one $x \in R^n$. Put

$$u(x) = \int G_\alpha(x-y)\, d\mu(y)$$

wherever the integral exists and is finite. Then $u \in L^1_{loc}(R^n)$ and u^L is an extension of u.

Proof. — By Prop. 1, § 6, II, we have $u \in L^1_{loc}(R^n)$ and for $\alpha > n$, $u^L(x) \equiv u(x)$.

For $\alpha \leq n$ let x be a point where $u(x)$ is defined. By (4.2), II it follows that $d\mu$ has no point mass at x. Hence

$$\begin{aligned}\frac{1}{|S(x,\rho)|}&\int_{S(x,\rho)} |u(y)-u(x)|\, dy \\ &= \frac{1}{|S(x,\rho)|}\int_{S(x,\rho)}\left|\int_{R^n}[G_\alpha(y-z)-G_\alpha(x-z)]\, d\mu(z)\right| dy \\ &\leqslant \int_{R^n}\left[\frac{1}{|S(x,\rho)|}\int_{S(x,\rho)}|G_\alpha(y-z)-G_\alpha(x-z)|\, dy\right]|d\mu|(z).\end{aligned}$$

For $x \neq z$, the function in square brackets converges pointwise to 0 as $\rho \downarrow 0$ by (4.1), II and is majorated by a constant times $G_\alpha(x, z)$ for ρ sufficiently small by Prop. 1), § 4, II. The result follows by the dominated convergence theorem.

6') *If $u \in P^\alpha$ then $u^L \in P^\alpha$ and $u(x) = u^L(x)$ exc. $\mathfrak{A}_{2\alpha}$.*

Proof. — If $u \in P^\alpha$ then exc. $\mathfrak{A}_{2\alpha}$ $u(x) = \int G_\alpha(x-y)f(y)\, dy$, $f \in L^2(R^n)$ and the result follows by setting $d\mu(y) = f(y)\, dy$ in Prop. 6).

The last proposition together with the previous ones leads immediately to the following theorems.

THEOREM I. — *If u is equivalent to a function in P^α (or $P^\alpha_{loc}(D)$) then for any φ, $u^\varphi \in P^\alpha$ (or $\in P^\alpha_{loc}(D)$).*

THEOREM II. — *If $u \in P^\alpha$ and j is any system of indices with $|j| \leqslant \alpha$, then for any φ*

$$D_j u(x) = \lim_{\rho \downarrow 0} \int i^{|j|}\xi^j e^{i(x,\xi)}\hat{u}(\xi)\, \hat{\varphi}(\rho\xi)\, d\xi \quad exc.\ \mathfrak{A}_{2\alpha-2|j|}.$$

Remark 1. — The corrected function u^φ of some function u can be characterized intrinsically by the fact that

$$(u^\varphi)^\varphi = u^\varphi.$$

Therefore when we say that v is a corrected function it means that $v^\varphi = v$ for some φ. Usually the particular choice of φ is immaterial and will not be mentioned.

1. The space $P^\alpha(D)$.

$P^\alpha(D)$ is the space (defined in Chap. I, § 5) of all restrictions to D of functions in P^α, with the norm

$$(1.1) \qquad ||u||_{\alpha,D} = \min ||\tilde{u}||_\alpha,$$

the minimum being taken over all $\tilde{u} \in P^\alpha$ such that $\tilde{u} = u$ on D except on a subset of D of 2α-capacity 0. By Prop. 1) of Chap. I, § 5, $P^\alpha(D)$ is a complete functional space relative to the class of subsets of D of 2α-capacity 0. Throughout the chapter it is assumed that D is an open set.

1) $P^\alpha(D)$ *is the perfect functional completion of the class of restrictions to* D *of functions in* $C_0^\infty(R^n)$.

Proof. — The proposition follows from the general properties of functional completions.

2) $P^\alpha(D) \subset P^\alpha_{loc}(D)$.

3) *If* $\beta < \alpha$, *then* $P^\beta(D) \supset P^\alpha(D)$ *and* $||u||_{\beta,D} \leqslant ||u||_{\alpha,D}$. *Moreover, if* D *is bounded then* $||u||_{\beta,D}$ *is completely continuous with respect to* $||u||_{\alpha,D}$.

Proof. — The first statement is obvious from the fact that $P^\beta \supset P^\alpha$ and $||u||_\beta \leqslant ||u||_\alpha$.

The statement about complete continuity is proved as follows. For each $u \in P^\alpha(D)$, let $\tilde{u} \in P^\alpha$ be the extension of u with $||\tilde{u}||_\alpha = ||u||_{\alpha,D}$. Let φ be a function of class C_0^∞ which is equal to 1 on D. By Prop. 6), § 2, II, there is a constant c such that

$$||\varphi\tilde{u}||_\alpha \leqslant c||\tilde{u}||_\alpha = c||u||_{\alpha,D}.$$

If $\{u_n\}$ is a bounded sequence in $P^\alpha(D)$, then $\{\varphi\tilde{u}_n\}$ is a bounded sequence in P^α, and each term vanishes outside of a fixed compact. Therefore, by Prop. 4) § 2, II, there is a subsequence $\{\varphi \tilde{u}_{n_k}\}$ which converges in P^β. The sequence $\{u_{n_k}\}$ must then converge in $P^\beta(D)$ since

$$||u_{n_k} - u_{n_l}||_{\beta,D} \leqslant ||\varphi \tilde{u}_{n_k} - \varphi \tilde{u}_{n_l}||_\beta.$$

φ will be called a *multiplier of order* k *on* D (*an open set*) *with Lipschitz constant* M if $\varphi \in C^{(k,1)}(D) \cap L^\infty(D)$ (if $k = -1$ we only require that $\varphi \in L^\infty(D)$) with common Lipschitz constant M for all derivatives of order $\leqslant k$ and $|\varphi|_{\infty,D} \leqslant M$.

4) *If* $u \in P^\alpha(D)$ *and* φ *is a multiplier of order* α^* *on* R^n *with Lipschitz constant* M *then* $\varphi u \in P^\alpha(D)$ *and* $||\varphi u||_{\alpha,D} \leqslant cM||u||_{\alpha,D}$ *where* c *depends only on* α^* *and* n.

Proof. — If $\tilde{u} \in P^\alpha$ is such that $\tilde{u} = u$ on D and $||\tilde{u}||_\alpha = ||u||_{\alpha,D}$ then by (2.4) of Chap. II, and by the definition (1.1) (4)

$$||\varphi u||_{\alpha,D} \leqslant ||\varphi u||_\alpha \leqslant cM||\tilde{u}||_\alpha = cM||u||_{\alpha,D}.$$

5) *If* $u \in P^\alpha(D)$ *and* m *is an integer* $\leqslant \alpha$, *then for* $|i| \leqslant m$, $D_i u \in P^{\alpha-m}(D)$ *and*

$$\sum_{k=0}^{m} \binom{m}{k} \sum_{|i|=k} ||D_i u||^2_{\alpha-m,D} \leqslant ||u||^2_{\alpha,D}. \tag{1.2}$$

Proof. — Take $\tilde{u} \in P^\alpha$ such that $\tilde{u} = u$ on D and $||\tilde{u}||_\alpha = ||u||_{\alpha,D}$ and use formula (7.1) of Chap. II (5).

The next proposition deals with restrictions of functions in $P^\alpha(D)$ to the intersection of D with a k-dimensional hyperplane R^k. In accordance with the conventions used earlier, quantities associated with R^k are primed.

6) *If* $D \cap R^k$ *is non-empty, and if* $2\alpha > n - k$, *then*

(*a*) *if* $u \in P^\alpha(D)$, *then* $u' \in P^{\alpha - \frac{n-k}{2}}(D \cap R^k)$ *and*

$$||u'||'^2_{\alpha-\frac{n-k}{2},\, D\cap R^k} \leqq \frac{\Gamma\left(\alpha - \frac{n-k}{2}\right)}{2^{n-k}\,\pi^{\frac{n-k}{2}}\,\Gamma(\alpha)} ||u||^2_{\alpha,D} \tag{1.3}$$

(4) The constant c is not the same as the one in (2.4) Chap. II, since we are using $\|\ \|_\alpha$ instead of $|\ |_\alpha$.

(5) If the boundary of D is irregular, the reverse inequality in (1.2) does not hold, even when the left side is multiplied by an arbitrarily large constant.

(*b*) *If* $u' \in P^{\alpha-\frac{n-k}{2}}(D \cap R^k)$ *then there exists* $u \in P^\alpha(D)$ *such that the restriction of* u *to* $D \cap R^k$ *is* u' *and such that equality holds in* (1.3).

Proof. — Theorems I*a* and I*b*, § 8, Chap. II.

7) *There is a constant c such that if, for*

$$q = 0,1, \ldots, r < \alpha - \frac{1}{2}, \qquad u'_q \in P^{\alpha-q-\frac{1}{2}}(D \cap R^{n-1})$$

then there is a function $u \in P^\alpha(D)$ *such that* $\left(\frac{\partial^q u}{\partial x_n^q}\right) = u'_q$ *except on a subset of* $D \cap R^{n-1}$ *of* (2α-$2q$-1)*-capacity* 0 *in* R^{n-1} *and*

$$(1.4) \qquad ||u||^2_{\alpha,D} \leqslant c \sum_{q=0}^{r} ||u'_q||'^2_{\alpha-q-\frac{1}{2}, D\cap R^{n-1}}.$$

Proof. — Theorem I*c*, § 8, II.

We will use the general notation: If U is an open set in R^n then $U^\delta = [x : \text{dist. } (x, R^n - U) > \delta]$, $\delta > 0$. It is easy to see that U^δ has the following properties: $|\partial(U^\delta)| = 0$ (6), $(U^{\delta_1})^{\delta_2} = U^{\delta_1+\delta_2}$ and if $U = R^n$, then $U^\delta = U$.

LEMMA 1. — *Let* U *be an open set in* R^n *and* A *a closed subset of* U^δ, *then there exists a function* $\varphi \in C^\infty(R^n)$ *such that*

$$0 \leqslant \varphi(x) \leqslant 1$$

everywhere, $\varphi(x) = 1$ *on* A, *and* $\varphi(x) = 0$ *outside of* U. *Furthermore,* $|D_i\varphi(x)| \leqslant C_{|i|}\delta^{-|i|}$ *where* C_m *is a constant depending only on m and n.*

Proof. — Consider the characteristic function $\chi(x)$ of the $\delta/3$ neighborhood of A. Then the regularization (7) $\chi_{\delta/3}(x)$ of χ by $e(x)$ satisfies all the requirements of the lemma and

$$C_m = \frac{\omega_n}{n} 3^m \sup_{|i|=m} |D_i e(x)|.$$

(6) If $x \in \partial(U^\delta)$ there exists a $y \in \partial U$ such that $x \in \partial S(y, \delta)$. Hence

$$\overline{\lim_{r \downarrow 0}} |S(x, r) \cap \partial(U^\delta)|/|S(x, r)| \leqslant \frac{1}{2}$$

for all $x \in \partial(U^\delta)$, i.e. $\partial(U^\delta)$ is at each point of upper density $\leqslant \frac{1}{2}$. By a classical theorem of Lebesgue if follows that $|\partial(U^\delta)| = 0$.

(7) Cf. § 2, II.

We recall that a transformation of D^* into D is of class $C^{(m,1)}(D^*)$ with Lipschitz constant M if each of its coordinate functions is of class $C^{(m,1)}(D^*)$ with common Lipschitz constant M for all derivatives of order $\leqslant m$. Finally, a homeomorphism T of D^* onto D is of class $C^{(m,1)}$ *with Lipschitz constant* M if T and T^{-1} are transformations of class $C^{(m,1)}(D^*)$ and $C^{(m,1)}(D)$ respectively with common Lipschitz constant M.

8) *Let* T *be a homeomorphism of class* $C^{(\alpha^*,1)}$ *with Lipschitz constant* M *of an open* U^* *on an open set* U *and let* $\overline{D} \subset U^\delta$. *Then if* $u \in P^\alpha(D)$,

$$T^*u(x^*) = u(Tx^*) \in P^\alpha(D^*) \quad (D^* = T^{-1}(D))$$

and $\|T^*u\|_{\alpha,D^*} \leqslant C\delta^{-\alpha} M^{\alpha+3n/2}\|u\|_{\alpha,D}$ *where* C *depends only on* α^* *and* n.

Proof. — It is clear that $\overline{D}^* \subset (U^*)^{\delta^*}$ where $\delta^* = M^{-1}\delta$. Let $\tilde{u} \in P^\alpha(R^n)$ be the extension of u to R^n with $\|\tilde{u}\|_{\alpha,R^n} = \|u\|_{\alpha,D}$ and for $x^* \in U^*$ let $\tilde{u}^*(x^*) = \tilde{u}(Tx^*)$. By Prop. 3, § 9, II, $\tilde{u}^* \in P^\alpha_{loc}(U^*)$.

Since $\overline{D}^* \subset ((U^*)^{\delta^*/2})^{\delta^*/2}$, by Lemma 1 there exists a function φ^* such that $\varphi^*(x^*) = 1$ on D^*, $\varphi^*(x^*) = 0$ outside of $(U^*)^{\delta^*/2}$, and its derivatives satisfy $|D_i^*\varphi^*(x^*)| \leqslant C_{|i|}(\delta^*)^{-|i|}$ where C_m depends only on m and n.

Therefore, if we extend $\varphi^*\tilde{u}^*$ by zero outside U^*,

$$\varphi^*\tilde{u}^* \in P^\alpha_{loc}(R^n);$$

and it is clear, that for an integer m:

$$\begin{aligned} d_m(\varphi^*\tilde{u}^*) &= \sum_{|i|=m} \int_{R^n} |D_i^*\varphi^*u^*(x^*)|^2\, dx^* \\ &\leqslant c\delta^{-2m} M^{n+2m} \sum_{l\leqslant m} d_l(\tilde{u}), \end{aligned}$$

c depending only on n and m.

For non-integral values of α an evaluation similar to that in the proof of Prop. 4), § 9,II, yields

$$d_\alpha(\varphi^*\tilde{u}^*) \leqslant c\delta^{-2\alpha} M^{2\alpha+3n}(d_\alpha(\tilde{u}) + \sum_{l\leqslant\alpha^*} d_l(\tilde{u}))$$

where c depends only on α^* and n, hence $\varphi^*\tilde{u}^* \in P^\alpha$.

Since $T^*u(x^*) = \varphi^* u^*(x^*)$ for $x^* \in D^*$, it follows that

$$\begin{aligned} ||T^*u||_{\alpha,D^*} &\leqslant ||\varphi^*\tilde{u}^*||_{\alpha,R^n} \\ &\leqslant c\delta^{-\alpha} M^{\alpha+\frac{3n}{2}} ||\tilde{u}||_{\alpha,R^n} \\ &= c\delta^{-\alpha} M^{\alpha+\frac{3n}{2}} ||u||_{\alpha,D}, \end{aligned}$$

where c depends only on n and α^*, from which the proposition follows. Next is a special result for $0 \leqslant \alpha \leqslant 1$.

9) *If u is of class $C^{(0,1)}$ on D and vanishes outside of a bounded subset of* D, *then* $u \in P^\alpha(D)$ *for* $0 \leqslant \alpha \leqslant 1$.

Proof. — Suppose u vanishes outside $S(0, r) \cap D$, and let φ be a function in $C^{(0,1)}(R^n)$ which is 1 on $S(0, r)$ and is 0 outside $S(0, r + 1)$. It is well known that each function u of class $C^{(0,1)}$ on a subset of R^n has an extension $\tilde{u}$ of class $C^{(0,1)}$ on R^n. By Prop. 6), § 2, II, $\varphi\tilde{u} \in P^\alpha$, and obviously u is the restriction to D of $\varphi\tilde{u}$, so $u \in P^\alpha(D)$.

Corollary. — *If* D *is bounded, then for* $0 \leqslant \alpha \leqslant 1$,

$$P^\alpha(D) \supset C^{(0,1)}(D).$$

2. The space $\check{P}^\alpha(D)$.

The *standard* α-norm over D, $|u|_{\alpha,D}$, is defined by direct formula in (2.1) below. The space $\check{P}^\alpha(D)$ is defined to be the subspace of $P^\alpha_{loc}(D)$ on which the α-norm is finite. The definition is such that $\check{P}^\alpha(D) \supset P^\alpha(D)$ and when the boundary of D satisfies suitable regularity conditions the two are equal ([8]). This will give the intrinsic characterization of functions in $P^\alpha(D)$.

Now for the definition. For $u \in P^\alpha_{loc}(D)$

(2.1*a*) $$|u|^2_{0,D} = \int_D |u(x)|^2\, dx.$$

(2.1*b*) For $0 < \alpha < 1$,

$$|u|^2_{\alpha,D} = |u|^2_{0,D} + \frac{1}{C(n,\alpha)G_{2n+2\alpha}(0)} \int_D \int_D \frac{G_{2n+2\alpha}(x-y)}{|x-y|^{n+2\alpha}} |u(x) - u(y)|^2\, dx\, dy$$

([8]) A large part of the rest of this paper is directed towards the proof of this fact which is not simple unless ∂D is of class $C^{(\alpha^*,1)}$.

where $C(n, \alpha)$ is defined by (1.3), II. For arbitrary $\alpha \geqslant 0$, let $m = [\alpha]$ [9] and $\beta = \alpha - m$, then

$$(2.1c) \qquad |u|^2_{\alpha,D} = \sum_{k=0}^{m} \binom{m}{k} \sum_{|i|=k} |D_i u|^2_{\beta,D}.$$

We also introduce a norm equivalent to $|u|_{\alpha,D}$, the *approximate* α-norm, $|u\rceil_{\alpha,D}$ (cf. Prop. 3) below). This norm is introduced because it is simpler and easier to handle in many of the proofs. The main distinction between the standard and approximate norms is their behavior as $\alpha \downarrow m$, m an integer. For more details see § 4. The motivation for calling $|u|_{\alpha,D}$ the standard α-norm is the fact that if $D = R^n$ then

$$|u|_{\alpha,D} = \|u\|_\alpha$$

(cf. Prop. 2) below).

For $u \in \check{P}^\beta(D)$, $0 \leqslant \beta \leqslant 1$, we define the Dirichlet integral of order β (cf. (1.2) and (1.4) of II) by

$$(2.2) \quad d_{0,D}(u)| = u|^2_{0,D}, \; d_{1,D}(u) = |u|^2_{1,D} - |u|^2_{0,D} = \sum_{l=1}^{n} \int_D \left|\frac{\partial u}{\partial x_l}\right|^2 dx$$

and for

$$0 < \beta < 1, \quad d_{\beta,D}(u) = \frac{1}{C(n, \beta)} \int_D \int_D \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\beta}} \, dx \, dy.$$

The approximate α-norm for $u \in \check{P}^\alpha(D)$ is

$$(2.3a) \qquad |u\rceil^2_{0,D} = d_{0,D}(u) = |u|^2_{0,D};$$

for

$$0 < \alpha < 1, \qquad |u\rceil^2_{\alpha,D} = |u\rceil^2_{0,D} + d_{\alpha,D}(u).$$

For arbitrary $\alpha \geqslant 0$ we let $m = [\alpha]$ and $\beta = \alpha - m$, then

$$(2.3b) \qquad |u\rceil^2_{\alpha,D} = \sum_{k=0}^{m} \binom{m}{k} \sum_{|i|=k} |D_i u\rceil^2_{\beta,D}.$$

Remark 1. — If $u \in P^\alpha_{loc}(D)$ then its distribution derivatives of order $\leqslant \alpha$ are equal to its ordinary derivatives. As a converse to this we have:

1) *If u is a distribution on D such that each distribution*

[9] $[\alpha]$ is the greatest integer $\leqslant \alpha$.

derivative $D_i u \in L^1_{loc}(D)$ *for* $|i| \leqslant m = [\alpha]$ *and*

$$\sum_{k=0}^{m} \binom{m}{k} \sum_{|i|=k} |D_i u|^2_{\beta,D} < \infty, \qquad \beta = \alpha - m,$$

then the correction of u *is in* $\check{P}^\alpha(D)$.

Proof. — By Prop. 7), § 2, II (localized) u is equal a.e. to a function v in $P^\alpha_{loc}(D)$. By Theorem 1, § 0, $u^L \in P^\alpha_{loc}(D)$ and the proposition is proved.

For the same reason we have:

1′) *If* u *is a corrected function in* $\check{P}^m(D)$, $m = [\alpha]$, *and* $|u|_{\alpha,D} < \infty$ *then* $u \in \check{P}^\alpha(D)$.

2) *If* $u \in P^\alpha$ *then* $|u|_{\alpha,R^n} = ||u||_\alpha$.

Proof. — By the last formula in (1.10), II it is sufficient to consider $0 \leqslant \alpha < 1$ and if $\alpha = 0$, the proof is trivial. Now suppose $0 < \alpha < 1$.

It can be shown by a simple rewriting of the second formula in (1.10), II (cf. the development in § 4, II) that

$$||u||^2_\alpha = \frac{1}{C(n+1,\alpha)} \times$$

$$\int_{-\infty}^{+\infty} \int_{R^n} \int_{R^n} \frac{|u(x)+u(y)|^2 \sin^2 \frac{1}{2} z_0 + |u(x)-u(y)|^2 \cos^2 \frac{1}{2} z_0}{[|x-y|^2 + z_0^2]^{\frac{n+1+2\alpha}{2}}} \, dx \, dy \, dz_0$$

$$= \frac{1}{C(n+1,\alpha)} \times$$

$$\int_{-\infty}^{+\infty} \int_{R^n} \int_{R^n} \frac{2(|u(x)|^2 + |u(y)|^2) \sin^2 \frac{1}{2} z_0 + |u(x)-u(y)|^2 \cos z_0}{[|x-y|^2 + z_0^2]^{\frac{n+1+2\alpha}{2}}} \, dx \, dy \, dz_0$$

$$= \int_{R^n} |u(x)|^2 \left[\frac{1}{C(n+1,\alpha)} \int_{-\infty}^{\infty} \int_{R^n} \frac{4 \sin^2 \frac{1}{2} z_0}{[|x-y|^2 + z_0^2]^{\frac{n+1+2\alpha}{2}}} \, dy \, dz_0 \right] dx$$

$$+ \int_{R^r} \int_{R^n} |u(x)-u(y)|^2 \left[\frac{1}{C(n+1,\alpha)} \int_{-\infty}^{\infty} \frac{\cos z_0}{[|x-y|^2 + z_0^2]^{\frac{n+1+2\alpha}{2}}} \, dz_0 \right] dx \, dy.$$

By the integral representation of $C(n+1,\ \alpha)$ preceding (1.3), II $\left(\text{and by writing } 4\sin^2 \frac{1}{2} z_0 = |e^{iz_0} - 1|^2\right)$, it is immediately seen that the expression in the square brackets in the first integral is $= 1$, and by (2.8), II,

$$\int_{-\infty}^{\infty} \frac{\cos z_0}{[|x-y|^2 + z_0^2]^{\frac{n+1+2\alpha}{2}}}\, dz_0 = \frac{2\pi}{|x-y|^{n+2\alpha}}\, G^{(1)}_{n+1+2\alpha}(|x-y|)$$

where $G^{(1)}_{n+1+2\alpha}$ is the kernel for R^1. Transforming this kernel by (4.1), II to the kernel for R^n we have finally,

$$||u||_\alpha^2 = \int_{R^n} |u|^2\, dx + \frac{\Gamma(n+\alpha)\pi^{\frac{n}{2}+\frac{1}{2}}\,\omega_n}{\Gamma\left(\alpha + \frac{n+1}{2}\right) C(n+1,\ \alpha)} \times \int_{R^n}\int_{R^n} \frac{G_{2n+2\alpha}(x-y)}{|x-y|^{n+2\alpha}}\, |u(x) - u(y)|^2\, dx\, dy.$$

By (1.3), II, and (4.2), II, it is checked immediately that the constant in front of the second integral is

$$[G_{2n+2\alpha}(0)\, C(n,\ \alpha)]^{-1}$$

which, after comparison with (2.1), completes the proof.

3) *If* $u \in \check{P}^m(D)$, $m = [\alpha]$, *then*

$$2^{-1/2} |u\rceil_{\alpha,D} \leqslant |u|_{\alpha,D} \leqslant |u\rceil_{\alpha,D} \tag{2.4}$$

and

$$0 \leqslant |u\rceil^2_{\alpha,D} - |u|^2_{\alpha,D} \leqslant |u|^2_{m,D}$$ [10].

Remark 2. — The second member in the inequality is to be interpreted as the integral of the difference of its integrands if the norms are infinite.

Proof. — By (2.1*c*) and (2.3*b*), it is clear that we may assume $0 \leqslant \alpha < 1$. Our hypothesis is now $u \in \check{P}^0(D)$. Put $\tilde{u}(x) = u(x)$ on D and $= 0$ on $R^n - D$, and let $u_n \in P^\alpha$ be such that

$$||u - u_n||_0 < \frac{1}{n} \ (P^\alpha \text{ is dense in } P^0).$$

[10] These inequalities are best possible for α not an integer, see example 1, § 13

Consider the function $F(\rho) = 1 + \rho^{2\alpha} - (1 + \rho^2)^\alpha$; it is an increasing function of ρ for $\rho \geqslant 0$ and $F(0) = 0$, $F(\infty) = 1$.

By (2.1), (2.2), and the fact that $G_{2n+2\alpha}(x)$ is a decreasing function of $|x|$ (see § 4, II), we have

$$0 \leqslant |u\rceil^2_{\alpha,D} - |u|^2_{\alpha,D} = \frac{1}{G_{2n+2\alpha}(0)C(n, \alpha)} \times$$
$$\int_D \int_D \frac{G_{2n+2\alpha}(0) - G_{2n+2\alpha}(x-y)}{|x-y|^{n+2\alpha}} |u(x) - u(y)|^2 \, dx \, dy$$
$$\leqslant \frac{1}{G_{2n+2\alpha}(0)\, C(n, \alpha)} \times$$
$$\int_{R^n} \int_{R^n} \frac{G_{2n+2\alpha}(0) - G_{2n+2\alpha}(x-y)}{|x-y|^{n+2\alpha}} |\tilde{u}(x) - \tilde{u}(y)|^2 \, dx \, dy.$$

Applying Fatou's lemma, (1.2) and (1.9) of II, and the remark concerning $F(\rho)$ we have from this inequality

$$\begin{aligned} 0 \leqslant |u\rceil^2_{\alpha,D} - |u|^2_{\alpha,D} &\leqslant \liminf_m \left[|u_m\rceil^2_{\alpha,R^n} - |u_m|^2_{\alpha,R^n}\right] \\ &= \liminf_m \int_{R^n} (1 + |\xi|^{2\alpha} - (1 + |\xi|^2)^\alpha) |\hat{u}_m(\xi)|^2 \, d\xi \\ &\leqslant \liminf_m \int_{R^n} |\hat{u}_m(\xi)|^2 \, d\xi \\ &= \liminf_m ||u_m||^2_0 = |\tilde{u}|^2_{0,R^n} = |u|^2_{0,D}. \end{aligned}$$

From this we also have that

$$|u\rceil^2_{\alpha,D} \leqslant |u|^2_{0,D} + |u|^2_{\alpha,D} \leqslant 2|u|^2_{\alpha,D}$$

which completes the proof.

Remark 3. — It is easily seen from the above proof that the difference of the two norms converges to 0 as $\alpha \uparrow m + 1$ for any $u \in \check{P}^m(D)$, e.g. by noting that $(1 + |\xi|^{2\beta} - (1 + |\xi|^2)^\beta) \downarrow 0$ as $\beta \uparrow 1$.

4) $P^\alpha(D) \subset \check{P}^\alpha(D)$ *and for* $u \in P^\alpha(D)$, $|u|_{\alpha,D} \leqslant ||u||_{\alpha,D}$.

Proof. — If $\tilde{u}$ is the extension of u in P^α such that

$$||\tilde{u}||_\alpha = ||u||_{\alpha,D}$$

then clearly $|u|_{\alpha,D} \leqslant |\tilde{u}|_{\alpha,R^n}$. The proposition then follows from Prop. 2).

5) *If* $\alpha \leqslant \gamma$ *and* $\alpha - [\alpha] \leqslant \gamma - [\gamma]$ *then* $\check{P}^\alpha(D) \supset \check{P}^\gamma(D)$

and $|u\rceil_{\alpha,D} \leqslant c|u\rceil_{\gamma,D}$ *for* $u \in \check{P}^{\gamma}(D)$ *where c depends only on n and* $\gamma - [\gamma]$ (11).

Proof. — Suppose at first that $0 < \alpha \leqslant \gamma < 1$. We have

$$d_{\alpha,D}(u) = \frac{1}{C(n,\alpha)}\left\{\int_D \int_{\substack{D \\ |x-y|<1}} + \int_D \int_{\substack{D \\ |x-y|>1}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy\right\}$$
$$\leqslant \frac{1}{C(n,\alpha)} \int_D \int_{\substack{D \\ |x-y|<1}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy$$
$$+ \frac{4}{C(n,\alpha)} \int_D |u(x)|^2 \int_{\substack{D \\ |x-y|>1}} \frac{dy}{|x-y|^{n+2\alpha}}\,dx$$
$$\leqslant \frac{C(n,\gamma)}{C(n,\alpha)} d_{\gamma,D}(u) + \frac{2\omega_n}{\alpha C(n,\alpha)} |u|^2_{0,D},$$

and the inequality is clear in this case from the properties of $C(n,\alpha)$. For arbitrary α and γ we have from the preceding that $|u\rceil^2_{\alpha,D} \leqslant c \sum_{k=0}^{[\alpha]} \binom{[\alpha]}{k} \sum_{|i|=k} |D_i u\rceil^2_{\gamma-[\gamma],D}, \leqslant c|u\rceil^2_{\gamma,D}$ which completes the proof.

We shall now introduce some special notation for indicial sets which are used to indicate partial derivatives. An *indicial set*, s, is a function defined on a finite well ordered set, the basis of s, into the integers. The number of elements in the basis is the length of s, written $|s|$, and the set of values is the range of s. (The empty set is considered an indicial set, e.g. $D_s u(x) \equiv u(x)$). Since there exists one and only one order preserving mapping of the basis into the integers, $1 \leqslant l \leqslant |s|$, we can represent the indicial set by a sequence of its values on the consecutive elements of its basis, e.g. $s = (s_1, \ldots, s_{|s|})$. An indicial set consisting of one element with value k will be written (k).

We will write $s' \subset s$ if the basis of s' is a subset of the basis of s with the induced order and the function s' is a restriction of the function s.

If s', $s^{(l)} \subset s$, $l = 1, \ldots, k$ and the basis of s' is a *disjoint*

(11) Example 2, § 13 shows that this is false in general for $\alpha \leqslant \gamma$ and

$$\gamma - [\gamma] < \alpha - [\alpha]$$

unless D is suitably restricted.

union of the bases of $s^{(l)}$ then we write $s' = \bigcup_l s^{(l)} \equiv \bigcup_{l=1}^{k} s^{(l)}$.

If $s' \subset s$ then there exists a unique s'', written $s - s'$, such that $s' \cup s'' = s$.

If $s = (s_1, \ldots, s_{|s|})$ is an indicial set with $1 \leqslant s_l \leqslant n$ then we shall write $D_s u(x) \equiv \dfrac{\partial^{|s|} u(x)}{\partial x_{s_{|s|}} \ldots \partial x_{s_1}}$ when the right side is defined.

An indicial set reprenseted by $\underbrace{(k, \ldots, k)}_{p\text{-times}}$ will be denoted by $\bigcup^{p} (k)$.

Examples of the above notation:

I) *Products*. Let $u_m \in C^p$ and $|i| \leqslant p$. Then

$$D_i\left(\prod_{m=1}^{l} u_m(x)\right) = \Sigma \prod_{m=1}^{l} D_{i^{(m)}} u_m(x), \tag{2.5}$$

where the summation is taken over all $i^{(m)}$ such that

$$\bigcup_{m=1}^{l} i^{(m)} = i.$$

(Note that some of the $i^{(m)}$ may be empty.) There are exactly $m^{|i|}$ terms in the summation. This formula is also valid when the u_m are not necessarily in C^p but where all the required derivatives exist at x.

II) *Composite functions*. Let $u, y_1, \ldots, y_n \in C^p$ and $|i| \leqslant p$. Let $\varphi(x) = u(y_1(x), \ldots, y_n(x))$ then

$$D_i \varphi(x) = \Sigma \frac{1}{|t|!} \prod_{m=1}^{|t|} D_{s^{(m)}} y_{t_m}(x) (D_t u)(y_1(x), \ldots, y_n(x)) \text{ [12]} \tag{2.6}$$

where the summation is taken over all $t = (t_1, \ldots, t_{|t|})$, $1 \leqslant |t| \leqslant |i|$, $1 \leqslant t_l \leqslant n$, and $s^{(m)}$ such that $\bigcup_{m=1}^{|t|} s^{(m)} = i$;

$$|s^{(m)}| \geqslant 1, \; m = 1, \ldots, |t|.$$

Again, the restriction to the class C^p is not necessary, the formula is valid when all the required derivatives exist at the point x and the corresponding point y.

[12] $(D_t u)(y_1(x), \ldots, y_n(x))$ means $D_t u(z)$ evaluated at $z = (y_1(x), \ldots, y_n(x))$.

We also note at this point two easily proved inequalities (essentially equivalent which will be used extensively:

For $0 < \beta < 1$,

$$(2.7) \quad \frac{\omega_n}{\beta(1-\beta)C(n,\beta)} \leqslant 2n \quad \textit{and} \quad \frac{2^{1-2\beta}\omega_n}{\beta(1-\beta)C(n,\beta)} \leqslant 2n \text{ [13]}.$$

The following propositions give some of the properties of multipliers.

6) *If* $u \in \check{P}^\alpha(D)$ *and* φ *is a multiplier of order* α^* *on* D *with Lipschitz constant* M *then* $\varphi u \in \check{P}^\alpha(D)$ *and* $|\varphi u\rceil_{\alpha,D} \leqslant cM|u\rceil_{\alpha,D}$ *where* $c = \sqrt{2}(1+2n)^{\frac{m+1}{2}}$ *if* $\beta > 0$ *and* $= (1+2n)^{m/2}$ *if* $\beta = 0$ *with* $m = [\alpha]$ *and* $\beta = \alpha - m$.

Proof. — From Prop. 1′, § 9, II, it follows that $\varphi u \in P^\alpha_{\text{loc}}(D)$. Following (2.5), we write for $|i| \leqslant m$

$$D_i\varphi u(x) = \sum_{j \cup k = i} D_j\varphi(x)\, D_k u(x)$$

where the summation is taken over j and k such that $j \cup k = i$, and we note that there are $2^{|i|}$ terms in the summation. The inequality for $\beta = 0$ follows from this and the properties of multipliers.

For $\beta > 0$,

$$(2.8) \quad d_{\beta,D}(D_i\varphi u)$$
$$\leqslant 2^{|i|+1} \sum_{j \cup k = i} \left\{ \frac{1}{C(n,\beta)} \int_D \int_D |D_j\varphi(x)|^2 \frac{|D_k u(x) - D_k u(y)|^2}{|x-y|^{n+2\beta}}\, dx\, dy \right.$$
$$\left. + \frac{1}{C(n,\beta)} \int_D |D_k u(y)|^2 \int_D \frac{|D_j\varphi(x) - D_j\varphi(y)|^2}{|x-y|^{n+2\beta}}\, dx\, dy \right\}.$$

By the hypotheses on φ the first term in the brackets is majorated by $M^2\, d_{\beta,D}(D_k u)$ and the inner integral in the second term is majorated by

$$M^2 \int_{\substack{D \\ |x-y|<2}} |x-y|^{2-n-2\beta}\, dx + 4\,M^2 \int_{\substack{D \\ |x-y|>2}} |x-y|^{-n-2\beta}\, dx \leqslant M^2 \frac{2^{1-2\beta}\omega_n}{\beta(1-\beta)}.$$

[13] It can be shown that $2n$ in the second inequality may be replaced by 2.038 and n for $n = 2$ and $n \geqslant 3$ respectively.

By (2.7) it follows that (2.8) is majorated by

$$2^{|i|+1} \sum_{k \subset i} \{ M^2 d_{\beta,D}(D_k u) + 2nM^2 |D_k u|^2_{0,D} \}$$

(the summation is taken over all $k \subset i$), hence

$$|D_i(\varphi u)\rceil^2_{\beta,D} \leqslant 2^{|i|}(1 + 4n)M^2 \sum_{k \subset i} |D_k u\rceil^2_{\beta,D}.$$

The proposition now follows by (2.3*b*).

7) *If* $u \in \check{P}^\alpha(D)$ *and* φ *is a multiplier of order* α^* *with Lipschitz constant* M *and has support in* D^δ *then* φu, *extended by* 0 *outside* D, *belongs to* P^α *and*

$$|\varphi u\rceil_{\alpha,R^n} \leqslant cM\{ |u\rceil_{\alpha,D} + (1 - \beta)^{1/2} \delta^{-\beta} |u\rceil_{m,D} \}$$

where $m = [\alpha]$, $\beta = \alpha - m$ *and* c *is the constant of Prop.* 6).

Remark 4. — We note that if $\beta = 0$ (i.e. α is an integer), Prop. 7) is valid when φ vanishes in some neighborhood of a D. In this case the second term in the brackets in the inequality is not needed.

Proof. — Clearly $\varphi u \in P^\alpha_{loc}(R^n)$ so we need only prove the inequality. If $\beta = 0$ then $|\varphi u\rceil_{\alpha,R^n} = |\varphi u\rceil_{\alpha,D}$ and the inequality follows from Prop. 6). So we assume $\beta > 0$. Then for $|i| \leqslant m$,

$$|D_i \varphi u\rceil^2_{\beta,R^n} = |D_i \varphi u\rceil^2_{\beta,D} + \int_{D^\delta} |D_i \varphi u(y)|^2 \left[\frac{2}{C(n, \beta)} \int_{R^n - D} \frac{dx}{|x - y|^{n+2\beta}} \right] dy.$$

By (2.7) it is clear that the square bracketed term is bounded by $\delta^{-2\beta}\omega_n / \beta C(n, \beta) \leqslant 2n(1 - \beta)\delta^{-2\beta}$. Hence by (2.3*b*),

$$|\varphi u\rceil^2_{\alpha,R^n} \leqslant |\varphi u\rceil^2_{\alpha,D} + 2n(1 - \beta)\delta^{-2\beta} |\varphi u\rceil^2_{m,D}$$

and the proof is completed by using Prop. 6).

8) *If* T *is a homeomorphism of class* $C^{(\alpha^*,1)}$, $\alpha^* \geqslant 0$, *with Lipschitz constant* $M \geqslant 1$ *of* D^* *onto* D *and* $u \in \check{P}^\alpha(D)$, *then* $u^*(x^*) = u(Tx^*) \in \check{P}^\alpha(D^*)$ *and* $|u^*\rceil_{\alpha,D^*} \leqslant cM^{\alpha+3n/2} |u\rceil_{\alpha,D}$ *where* c *depends only on* n *and* α^*.

Proof. — By Prop. 3), § 9, II, $u^* \in P^\alpha_{loc}(D^*)$.

Let $y_l(x^*)$, $l = 1, \ldots, n$, be the coordinate functions of T.

Then for $|i| \leqslant m$ and almost all $x^* \in D^*$,

$$D_i u^*(x^*) = D_i(u(Tx^*))$$
$$= \Sigma \frac{1}{|t|\,!} \prod_{m=1}^{|t|} D_{s^{(m)}} y_{t_m}(x^*)(D_t u)(y_1(x^*), \ldots, y_n(x^*)),$$

where the summation is taken over all $t = (t_1, \ldots, t_{|t|})$, $1 \leqslant |t| \leqslant |i|$, $1 \leqslant t_m \leqslant n$, and $s^{(m)}$ such that $\bigcup_m s^{(m)} = i$, $|s^{(m)}| \geqslant 1$ (cf. (2.6)). We define

$$\varphi(x^*) \equiv \varphi(x^*; s^{(1)}, \ldots, s^{(l)}, t) = \frac{1}{|t|\,!} \prod_{m=1}^{|t|} D_{s^{(m)}}\, y_{t_m}(x^*)$$

and it is easy to see that φ is a multiplier on D^* with Lipschitz constant $M^{|i|}/(|t| - 1)\,!$ and of at least order -1 when α is an integer and at least order 0 when α is not an integer. Hence by Prop. 6), the fact that T is Lipschitzian and the classical theorems on the transformation of integrals, we obtain

$$|D_i u^* \rceil_{\beta,D} \leqslant \Sigma|\varphi(x^*) D_t u(Tx^*)\rceil_{\beta,D^*} \leqslant cM^{|i|}\, \Sigma|D_t u(Tx^*)\rceil_{\beta,D^*}$$
$$\leqslant cM^{|i|+3n/2+\beta}\, \Sigma|D_t u\rceil_{\beta,D},$$

and the proposition follows by (2.3*b*) (if $\beta = 0$ the exponent of M may be reduced to $\alpha + \frac{n}{2}\Big)$.

Theorem I. — *$\check{P}^\alpha(D)$ is a complete functional space relative to the exceptional class $\mathfrak{A}_{2\alpha}(D)$ of subsets of* D *of 2α-capacity* 0. *It is the perfect functional completion of its subspace of infinitely differentiable functions.*

Proof. — Let $\{U_k\}$ be a locally finite covering of D by open relatively compact subsets of D, and let $\{\varphi_k\}$ be a corresponding partition of unity of class C^∞-that is: φ_k is of class C^∞ and vanishes outside a compact subset of U_k:

$$0 \leqslant \varphi_k(x) \leqslant 1 \textit{ for all } x, \textit{ and } \sum_{k=1}^{\infty} \varphi_k(x) = 1 \textit{ for all } x \in D.$$

If $\{u_n\}$ is a Cauchy sequence in $\check{P}^\alpha(D)$, then by Prop. 7) $\{\varphi_1 u_n\}$ is a Cauchy sequence in P^α. Therefore $\{\varphi_1 u_n\}$ contains a subsequence $\{\varphi_1 u_{1,n}\}$ which converges pointwise except on a subset of 2α-capacity 0. Similarly, $\{\varphi_2 u_{1,n}\}$ contains a subsequence $\{\varphi_2 u_{2,n}\}$ which converges pointwise except on a set of 2α-capacity 0. If we continue to extract subsequences and then take the diagonal sequence, we get a subsequence $\{v_n\}$ of $\{u_n\}$ such that for each k, $\{\varphi_k v_n\}$ converges pointwise except on a subset of D of 2α-capacity 0. This proves the functional space property. If v is the pointwise limit of $\{v_n\}$, then, since $\{\varphi_k v_n\}$ is Cauchy in P^α and converges pointwise to $\varphi_k v$ except on a set of 2α-capacity 0, it follows that $\varphi_k v \in P^\alpha$; hence that $v \in P^\alpha_{\mathrm{loc}}(D)$. By picking a further subsequence if necessary, it can be assumed that for $|i| \leqslant \alpha$, $D_i v_n$ converges pointwise except on a set of $(2\alpha - 2|i|)$-capacity 0. Then Fatou's lemma shows that

$$|v|_{\alpha,D} \leqslant \liminf_{n\to\infty} |v_n|_{\alpha,D}, \qquad |v - v_m|_{\alpha,D} \leqslant \liminf_{n\to\infty} |v_n - v_m|_{\alpha,D}$$

from which it follows that $v \in \check{P}^\alpha(D)$ and that $v_n \to v$ in $\check{P}^\alpha(D)$. This gives the completeness.

The argument to show that there cannot be a functional completion of the subspace of infinitely differentiable functions relative to a smaller exceptional class than the class $\mathfrak{A}_{2\alpha}(D)$ is familiar by now and will be omitted. What remains is to show that the infinitely differentiable functions are dense.

If $u \in \check{P}^\alpha(D)$, then since $\varphi_k u \in P^\alpha$ and vanishes outside a compact subset of U_k, there exists $w_k \in C_0^\infty(U_k)$ such that

$$(2.9) \qquad |\varphi_k u - w_k|_{\alpha,R^n} < \frac{\varepsilon}{2^k}.$$

Since the covering $\{U_k\}$ is locally finite, the sum

$$w(x) = \sum_{k=1}^{\infty} w_k(x)$$

is finite on a neighborhood of each point of D. Therefore w is of class C^∞ on D. From (2.9) and the completeness of $\check{P}^\alpha(D)$ it follows that $u - w = \Sigma(\varphi_k u - w_k) \in \check{P}^\alpha(D)$ — hence that $w \in \check{P}^\alpha(D)$ — and that $|u - w|_{\alpha,D} < \varepsilon$.

9) *Let* $\mathcal{B}_D$ *be the set of functions with bounded support in* D (*i.e.* $f \in \mathcal{B}_D$ *if* f *vanishes outside* $[|x| < R] \cap D$ *for some* R). *Then* $\check{P}^\alpha(D) \cap \mathcal{B}_D \cap C^\infty(D)$ *is dense in* $\check{P}^\alpha(D)$.

Proof. — Let $\varphi \in C_0^\infty(R^n)$ and $= 1$ on a neighborhood of 0 and $u \in \check{P}^\alpha(D) \cap C^\infty(D)$. Then the functions $u_{(\rho)}(x) = \varphi(\rho x)u(x)$ are bounded in norm in $\check{P}^\alpha(D)$. So, by a well known theorem from Hilbert space theory, there is a sequence $\rho_k \downarrow 0$ such that the arithmetic means of the sequence $\{u_{(\rho_k)}\}$ converge strongly in $\check{P}^\alpha(D)$. These arithmetic means must converge to u since they converge pointwise to $u(x)$. But the arithmetic means have bounded support since each $u_{(\rho_k)}$ has bounded support and are in $C^\infty(D)$. Hence the proposition follows by Theorem I.

10) *Let φ be a multiplier of order α^* on R^n and $= 1$ in a neighborhood of 0 and for $u \in \check{P}^\alpha(D)$ put $T_\rho u(x) = \varphi(\rho x)u(x)$. Then $T_\rho u \to u$ in $\check{P}^\alpha(D)$ as $\rho \downarrow 0$.*

Proof. — By Prop. 6) $\{T_\rho\}$ is a uniformly bounded family of linear transformations of $\check{P}^\alpha(D)$ into $\check{P}^\alpha(D)$. If $u \in \check{P}^\alpha(D)$ and has bounded support the proposition is trivial, but since such functions are dense in $\check{P}^\alpha(D)$ by Prop. 9), the proof is complete.

Remarks about the spaces $\check{P}^{\alpha,p}(D)$, $1 < p < \infty$. The spaces $\check{P}^{\alpha,p}$ were introduced in [2]. These spaces reduce to $P^\alpha(R^n)$ when $p = 2$. The definition of the norms in these spaces lends itself to a suitable definition of norm in an open set $D \subset R^n$. This allows us to define the spaces $\check{P}^{\alpha,p}(D)$. We introduce two norms, the *standard* and *approximate norm* which will be denoted by $|u|_{\alpha,p,D}$ and $|u\urcorner_{\alpha,p,D}$ respectively. However, for $p = 2$, $|u|_{\alpha,2,D} = |u\urcorner_{\alpha,2,D} = |u|_{\alpha,D}$ for all α and D. Therefore to have a norm analogous to the approximate norm which was introduced in this section for $\check{P}^\alpha(D)$ we will introduce the *second approximate norm* $|u]_{\alpha,p,D}$.

We shall define these norms formally after we introduce the following notation. Let $A^{(l)}(\nu_1, \ldots, \nu_l)$ be an l-linear complex valued functional on $\underbrace{R^n \times \ldots \times R^n}_{l\text{-times}}$ with $\nu_j \in R^n$ (i.e. $A^{(l)} = A^{(l)}_{re} + iA^{(l)}_{im}$ where $A^{(l)}_{re}$ and $A^{(l)}_{im}$ are l-covariant tensors). We note that in terms of any orthonormal basis of R^n we may write $A^{(l)}(\nu_1, \ldots, \nu_l) = \sum_{|i|=l} A^{(l)}_i \nu_1^{i_1} \ldots \nu_l^{i_l}$ where

$v_k = (v_k^1, \ldots, v_k^n)$. Note that the $A_i^{(l)}$ depend on the choice of the basis. Let

$$\Sigma^{(l)} = \underbrace{\partial S(0,1) \times \cdots \times \partial S(0,1)}_{l\text{-times}}, \quad \theta_{(l)} = (\theta_1, \ldots, \theta_l) \in \Sigma^{(l)}$$

and $d\theta_{(l)}$ be the element of volume in $\Sigma^{(l)}$. Then we define

$$|A^{(l)}|_p^p = \left(\frac{n}{\omega_n}\right)^l \int_{\Sigma^{(l)}} |A^{(l)}(\theta_{(l)})|^p \, d\theta_l \qquad \text{and} \qquad |A^{(l)}\rceil_p^p = \sum_{|i|=l} |A_i^{(l)}|^p.$$

$|A^{(l)}|_p$ is independent of the basis chosen in R^n whereas $|A^{(l)}\rceil_p$ depends on the basis except for $p = 2$ when $|A^{(l)}|_2 = |A^{(l)}\rceil_2$. For $p \neq 2$, one shows that

$$n^{-l/\min(2,p')}|A^{(l)}\rceil_p \leqslant |A^{(l)}|_p \leqslant n^{l/\min(2,p)}|A^{(l)}\rceil_p$$

where $\dfrac{1}{p'} + \dfrac{1}{p} = 1$.

Let $m = [\alpha]$, $\beta = \alpha - m$ and for $\beta > 0$ let

$$d\mu_\beta(x, y) = \frac{G_{2n+2\beta}(x - y)|x - y|^{-n}}{C(n, \beta)G_{2n+2\beta}(0)} \, dx \, dy.$$

With the above we define formally for $\beta > 0$,

$$|u|_{\alpha,p,D}^p = \sum_{l=0}^m \binom{m}{l} \left(\frac{2}{p}\right)^l \times \left[\int_D |\nabla^l u(x)|_p^p \, dx + \int_D \int_D \left|\frac{\nabla^l u(x) - \nabla^l u(y)}{|x - y|^\beta}\right|_p^p d\mu_\beta(x, y)\right]$$

where $\nabla^l u(x)$ is the l-th gradient of $u(x)$, i.e. the tensor $D_i u(x)$ with $|i| = l$. If $\beta = 0$, i.e. $m = \alpha$, then

$$|u|_{m,p,D}^p = \sum_{l=0}^m \binom{m}{l} \left(\frac{2}{p}\right)^l \int_D |\nabla^l u(x)|_p^p \, dx.$$

Similarly,

$$|u\rceil_{\alpha,p,D}^p = \sum_{l=0}^m \binom{l}{m} \left(\frac{2}{p}\right)^l \times \left[\int_D |\nabla^l u(x)\rceil_p^p \, dx + \int_D \int_D \left|\frac{\nabla^l u(x) - \nabla^l u(y)}{|x - y|^\beta}\right|_p^p d\mu_\beta(x, y)\right]$$

and if $\beta = 0$ we omit the double integral as in $|u|_{\alpha,p,D}$. To define the *second approximate norm* we merely replace

$$d\mu_\beta(x, y) \quad \text{by} \quad \frac{dx\,dy}{C(n, \beta)|x - y|^n} \quad \text{in } |u\rceil^p_{\alpha,p,D}.$$

It is clear from what has been mentioned above that

$$n^{-m/\min(2,p')}|u\rceil_{\alpha,p,D} \leqslant |u|_{\alpha,p,D} \leqslant n^{m/\min(2,p)}|u\rceil_{\alpha,p,D}.$$

The spaces $\check{P}^{\alpha,p}(R^n)$ are well determined as the perfect functional completion of $C_0^\infty(R^n)$ with respect to the norm $|u|_{\alpha,p,R^n}$. This was proved in [2]. The more familiar spaces W_p^α (introduced in [7] and [12]) were also considered in [2] and were defined as imperfect functional completions of $C_0^\infty(R^n)$ with the norm $|u|_{\alpha,p,R^n}$ relative to the class of sets with Lebesgue measure 0.

Having $\check{P}^{\alpha,p}(R^n)$ defined, by the usual localization we obtain the definition of $\check{P}^{\alpha,p}_{\text{loc}}(D)$ (and also $W^\alpha_{p,\text{loc}}(D)$). By the results of [2], therefore, for $u \in \check{P}^{\alpha,p}_{\text{loc}}(D)$ (or $\in W^\alpha_{p,\text{loc}}(D)$) all derivatives (or distribution derivatives) $D_i u$, $|i| \leqslant \alpha$, will be functions in $L^p_{\text{loc}}(D)$. Hence for these functions it is meaningful to consider $|u|_{\alpha,p,D}$, $|u\rceil_{\alpha,p,D}$ and $|u]_{\alpha,p,D}$. The space $\check{P}^{\alpha,p}(D)$ (or $W^\alpha_p(D)$) is defined as the subspace of $\check{P}^{\alpha,p}_{\text{loc}}(D)$ (or $W^\alpha_{p,\text{loc}}(D)$) on which $|u|_{\alpha,p,D} < \infty$.

Essentially all the results of the present paper concerning $\check{P}^\alpha(D)$ have exact analogues for $\check{P}^{\alpha,p}(D)$. In particular, Theorem I, § 0, is valid for $\check{P}^{\alpha,p}(R^n)$ (or $\check{P}^{\alpha,p}_{\text{loc}}(D)$) so that if $u \in W^\alpha_p(D)$ then the correction of u is in $\check{P}^{\alpha,p}(D)$ (cf. [2]).

To obtain the analogues to the results of § 1, we remark that $\check{P}^{\alpha,p}(R^n)$ corresponds to $P^\alpha = \check{P}^\alpha(R^n)$ and hence the analogue to $P^\alpha(D)$ is the class of restrictions to D of functions in $\check{P}^{\alpha,p}(R^n)$ with the norm $\|u\|_{\alpha,p,D} = \inf_{\tilde{u}} |\tilde{u}|_{\alpha,p,R^n}$ for all $\tilde{u} \in \check{P}^{\alpha,p}(R^n)$ with $\tilde{u}(x) = u(x)$ in D.

All the propositions in § 1 are either valid verbatim or have obvious analogues for $p \neq 2$ except for those pertaining to restrictions to hyperplanes (Props. 6) and 7)). These are still essentially valid if $\alpha - \frac{n-k}{2}$ is replaced by $\alpha - \frac{n-k}{p}$ and this number is not an integer.

The propositions and theorem of the present section are all valid, with a suitable change in constant, when $|u|_{\alpha,\mathrm{D}}$ is replaced by $|u\rceil_{\alpha,p,\mathrm{D}}$ and $|u\rceil_{\alpha,\mathrm{D}}$ is replaced by $|u]_{\alpha,p,\mathrm{D}}$ (and $\check{\mathrm{P}}^{\alpha}(\mathrm{D})$ by $\check{\mathrm{P}}^{\alpha,p}(\mathrm{D})$). However, some of the proofs have to be changed especially when we use Fourier transforms (these cannot be used for $p \neq 2$).

The spaces $\check{\mathrm{P}}^{\alpha,p}(\mathrm{R}^n)$ for $p = 1$ or ∞ were introduced in [2]; however this theory is complicated even on R^n, in particular, when $p = 1$ and α is an integer > 1. It has not been proved as yet that there exists a perfect functional completion of $\mathrm{C}_0^\infty(\mathrm{R}^n)$ in the last mentioned cases. They have many other exceptional features — for instance, they are not reflexive. We can introduce for $p = 1$ or $p = \infty$ the corresponding classes $\check{\mathrm{P}}^{\alpha,p}(\mathrm{D})$. If we consider the results of the present paper and attempt to extend them to similar results for these classes it turns out that there are many which can be extended, but also several which cannot be extended.

To avoid undue length in the present paper we shall restrict our remarks in the following sections to $1 < p < \infty$ (unless otherwise stated).

3. Inessential singularities of functions in $\check{\mathrm{P}}^{\alpha}(\mathrm{D})$ and Lichtenstein extensions.

In this section we give some results on the possibility of extending a function in $\check{\mathrm{P}}^{\alpha}(\mathrm{D})$ to a larger open set D_1. These results will be used later in this section when we introduce the Lichtenstein extension.

THEOREM I. — *Let* D *and* D_1 *be open sets such that* $\mathrm{D} \subset \mathrm{D}_1$ *and* $|\mathrm{D}_1 - \mathrm{D}| = 0$ *and let* $m = [\alpha]$. *The function* $u \in \check{\mathrm{P}}^{\alpha}(\mathrm{D})$ *has an extension in* $\check{\mathrm{P}}^{\alpha}(\mathrm{D}_1)$ *if and only if for every* $k = 1, \ldots, n$, *and for almost every line* l *parallel to the* x_k*-axis, each derivative* $\frac{\partial^j u}{\partial x_k^j}$ $j \leqslant m - 1$, *has an absolutely continuous extension to* $l \cap \mathrm{D}_1$. *If this condition is satisfied, the extension is unique.*

Remark 1. — It is assumed that the condition is satisfied for each k. However, it is not assumed that there is any relation between the extensions in different directions. That there is such a relation results from the proof.

Proof. — The necessity of the condition is obvious from the results of Chapter II, specifically Theorem 1, § 7, localized.

The proof of the sufficiency relies on Prop. 2′), § 9, chap. II. This proposition shows that on D_1, u is equal a.e. to a function in $P^{\alpha}_{loc}(D_1)$. Obviously we may assume that u is a corrected function in D (relative, say, to spherical means). Since $u \in L^2(D_1)$ we can introduce the corrected function u_1 of u in D_1 which is an extension of u and which is in $P^{\alpha}_{loc}(D_1)$ (see Theorem I, § 0). Clearly $|u_1|_{\alpha,D_1} = |u|_{\alpha,D} < \infty$, hence

$$u_1 \in \check{P}^{\alpha}(D_1).$$

The uniqueness of u_1 (up to sets of 2α-capacity 0) comes immediately from the Frostman mean value theorem (Prop. 1, § 4), Chap. II).

If it happens that $D_1 - D$ has $(n-1)$-dimensional measure 0, then for almost all lines l in any given direction, $D_1 \cap l = D \cap l$. Hence we have the following corollary.

THEOREM I′. — *If* $D_1 - D$ *has* $(n-1)$*-dimensional measure* 0, *then every function* $u \in \check{P}^{\alpha}(D)$ *has a unique extension* $u_1 \in \check{P}^{\alpha}(D_1)$.

Let h_μ, $\mu = 0,1,\ldots, q$ be a strictly increasing sequence of positive numbers and consider the $q+1$ linear equations for a_μ,

$$(3.1) \qquad \sum_{\mu=0}^{q} a_\mu(-h_\mu)^p = 1, \qquad p = 0,1,\ldots,q.$$

It is easy to see that

$$(3.2) \qquad a_\mu = \prod_{\substack{j=0 \\ j\neq\mu}}^{q} \frac{h_j+1}{h_j-h_\mu}.$$

Let D be an open set in R^n such that

$$D \subset [x_n < 0], \qquad D_+ = \bigcap_{\mu=0}^{q} [(x', x_n) : (x', -h_\mu x_n) \in D]$$

and $\tilde{D} = (D \cup D_+ \cup [x_n = 0])^o$. Let $\mathfrak{M}(D)$ be the class of functions which are measurable and finite a.e. on D. Then for $u \in \mathfrak{M}(\mathrm{D})$ we define the *Lichtenstein extension of order* q (relative to $\{h_\mu\}$) *of* u by

$$(3.3a)\qquad \tilde{u}(x) = u(x) \qquad \textit{for} \qquad x \in \mathrm{D},$$

$$(3.3b)\qquad \tilde{u}(x) = \sum_{\mu=0}^{q} a_\mu u(x', -h_\mu x_n) \qquad \textit{for} \qquad x \in \mathrm{D}_+,$$

whenever all terms in the sum are defined and finite.

(3.3c) $\tilde{u}(x) =$ *the correction of* $\tilde{u}$ *as defined above for* $x \in \tilde{\mathrm{D}} \cap [x_n = 0]$ *if it exists, otherwise not defined* (14).

The restriction of $\tilde{u}$ to D_+, as defined by (3.3*b*), is the *reflected* function and the operation leading from u in D to the reflected function is called Lichtenstein reflection of order q (rel. to $\{h_\mu\}$).

Remark 2. — If $u \in \mathrm{C}^p(\overline{\mathrm{D}})$, $p \leqslant q$, then it is easy to see that $\tilde{u} \in \mathrm{C}^p(\tilde{\mathrm{D}})$ (cf. (3.4) below) and u may be determined on

$$\tilde{\mathrm{D}} \cap [x_n = 0]$$

by continuity instead of (3.3*c*). The first idea of such an extension was due to L. Lichtenstein [9] who introduced it for $q = 1$ and who did it not only for hyperplanes but also for hypersurfaces of class C^1. The idea of Lichtenstein was applied by M. R. Hestenes [8] to define the extension of order q across hypersurfaces of class C^q. Quite recently this idea was further extended by R. T. Seeley [11] to define an extension of order ∞ across hyperplanes. We will use Seeley's idea in § 11 to define an extension of order ∞ across any hypersurface which is the graph of a Lipschitzian function.

1) *If* $u \in \check{\mathrm{P}}^\alpha(\mathrm{D})$, $\alpha^* \leqslant q$, *then* $\tilde{u}$ — *the Licthenstein extension of order* q *of* u — *belongs to* $\check{\mathrm{P}}^\alpha(\tilde{\mathrm{D}})$ *and* $|\tilde{u}\urcorner_{\alpha,\tilde{\mathrm{D}}} \leqslant c|u\urcorner_{\alpha,\mathrm{D}}$ *where* c *depends only on* n *and* q (*and the choice of* h_μ).

Proof. — Clearly $\tilde{u} \in \mathrm{P}^\alpha_{\mathrm{loc}}(\mathrm{D} \cup \mathrm{D}_+)$. We shall show in part *a*), below, that the inequality is valid when $\tilde{\mathrm{D}}$ is replaced by $\mathrm{D} \cup \mathrm{D}_+$. In part *b*), we shall show that $\tilde{u}$ as defined in (3.3*a*) and (3.3*b*) (i.e $\tilde{u}$ is restricted to $\mathrm{D} \cup \mathrm{D}_+$) satisfies the hypotheses of Theorem I with respect to $\mathrm{D} \cup \mathrm{D}_+$ and $\tilde{\mathrm{D}}$ which implies

(14) If $\tilde{u}$ is not integrable in a neighborhood of x the correction does not exist at x.

that this restriction of $\tilde{u}$ has an extension $\tilde{u}_1 \in \check{\mathrm{P}}^{\alpha}(\tilde{\mathrm{D}})$. By Theorem I, § 0, we have $\tilde{u} = \tilde{u}_1$ exc. $\mathfrak{A}_{2\alpha}$ on $\tilde{\mathrm{D}}$ which will complete the proof of the proposition.

a) Suppose i, $|i| \leqslant \alpha^* + 1$, contains k indices n. Then for $x \in \mathrm{D}_+$

$$(3.4) \quad \mathrm{D}_i\tilde{u}(x) = \sum_{\mu=0}^{q} a_\mu(-h_\mu)^k(\mathrm{D}_iu)(x', -h_\mu x_n) \text{ (15)}.$$

Therefore for $|i| \leqslant \alpha^* + 1$,

$$\begin{aligned}\lceil \mathrm{D}_i u\rceil_{0,\mathrm{D}_+} &\leqslant \sum_{\mu=0}^{q} |a_\mu| h_\mu^k \lceil (\mathrm{D}_iu)(x', -h_\mu x_n)\rceil_{0,\mathrm{D}_+} \\ &\leqslant \left(\sum_{\mu=0}^{q} |a_\mu h_\mu^{k-1/2}|\right) \lceil \mathrm{D}_i u\rceil_{0,\mathrm{D}},\end{aligned}$$

which completes the proof of part *a*) if α is an integer.

Suppose $\beta = \alpha - m > 0$, $m = [\alpha]$, then for $|i| \leqslant \alpha^* = m$

$$(3.5) \quad \begin{aligned} d_{\beta,\mathrm{D}\cup\mathrm{D}_+}(\mathrm{D}_iu) &= d_{\beta,\mathrm{D}}(\mathrm{D}_iu) + d_{\beta,\mathrm{D}_+}(\mathrm{D}_iu) \\ &+ \frac{2}{\mathrm{C}(n,\beta)}\int_{\mathrm{D}}\int_{\mathrm{D}_+} \frac{|\mathrm{D}_i\tilde{u}(x) - \mathrm{D}_iu(y)|^2}{|x-y|^{n+2\beta}}\,dx\,dy.\end{aligned}$$

By using (3.4) in the last two terms of (3.5) and then applying the Cauchy-Schwarz inequality we have for a bound of the last two terms,

$$\begin{aligned}&\left(\sum_{\mu=0}^{q} |a_\mu|^2 h_\mu^{2k}\right) \\ &\times \sum_{\mu=0}^{q}\left(\frac{1}{\mathrm{C}(n,\beta)}\int_{\mathrm{D}_+}\int_{\mathrm{D}_+} \frac{|(\mathrm{D}_iu)(x', -h_\mu x_n) - (\mathrm{D}_iu)(y', -h_\mu y_n)|^2}{|x-y|^{n+2\beta}}\,dx\,dy\right. \\ &\quad\left. + \frac{2}{\mathrm{C}(n,\beta)}\int_{\mathrm{D}}\int_{\mathrm{D}_+} \frac{|(\mathrm{D}_iu)(x', x_n) - (\mathrm{D}_iu)(y', -h_\mu y_n)|^2}{|x-y|^{n+2\beta}}\,dx\,dy\right) \\ &\leqslant \left(\sum_{\mu=0}^{q} |a_\mu|^2 h_\mu^{2k}\right)\left(\sum_{\mu=0}^{q} (h_\mu^{-2} + 2h_\mu^{-1})(\mathrm{Max}(h_\mu, 1))^{n+2\beta}\right) d_{\beta,\mathrm{D}}(\mathrm{D}_iu).\end{aligned}$$

This completes the proof of part *a*).

b) If l is a line parallel to the x_k-axis, $k = 1, \ldots, n$, then by Theorem I, § 7, II (localized) $\dfrac{\partial^j \tilde{u}}{\partial x_k^j}$, $j = 0, \ldots, m-1$, is abso-

(15) We remind the reader that $(\mathrm{D}_iu)(x', -h_\mu x_n)$ means $(\mathrm{D}_iu)(y)$ evaluated at $y = (x', -h_\mu x_n)$.

lutely continuous on $l \cap (\mathrm{D} \cup \mathrm{D}_+)$ for almost all l. From this it is clear we need only consider lines parallel to the x_n-axis which satisfy (i) $l \cap \tilde{\mathrm{D}} \cap [x_n = 0] \neq 0$. For $j \leqslant m - 1$ let $\varphi(x', x_n) = \dfrac{\partial^j \tilde{u}}{\partial x_n^j}(x', x_n)$ and consider lines

$$l_{x'} = [(x', t) : -\infty < t < \infty]$$

which satisfy (i) and in addition satisfy (ii) $\varphi(x', x_n)$ is absolutely continuous on $l_{x'} \cap (\mathrm{D} \cup \mathrm{D}_+)$ and (iii) $|\varphi|_{1, l_{x'} \cap (\mathrm{D} \cup \mathrm{D}_+)} < \infty$. It is clear by (iii) that both the left and right hand limits as $t \to 0$ of $\varphi(x', t)$ exist and also, by (3.1) and (3.4), that these limits are equal. Hence by taking this limit as the value of $\varphi(x', 0)$ we make φ absolutely continuous on $l_{x'} \cap \tilde{\mathrm{D}}$ [16]. Since (ii) and (iii) are valid for almost all $l_{x'}$ the hypotheses of Theorem I are satisfied, which completes the proof of the proposition.

Remark 3. — When we use the Lichtenstein extension for a function in $\check{\mathrm{P}}^{\alpha}(\mathrm{D})$ we will assume that the order $q \geqslant \alpha^*$, so that $\tilde{u}$ will be in $\check{\mathrm{P}}^{\alpha}(\tilde{\mathrm{D}})$. It should be kept in mind that in general the Lichtenstein extension of order q of a function will not belong to a class higher than $\check{\mathrm{P}}^{q+1}(\tilde{\mathrm{D}})$ no matter what class the initial function belongs to (a similar remark is valid for classes $\mathrm{C}^p(\overline{\mathrm{D}})$).

Remark 4. — Obviously we can consider the Lichtenstein extension with respect to any hyperplane; it is enough to change the coordinate axis suitably. As an example take the rectangle $\mathrm{D} = [c_i < x_i < d_i;\ i = 1, \ldots, n]$ (c_i or d_i possibly infinite). We can consider the Lichtenstein extension of order q with respect to the hyperplane $[x_l = c_l]$ then $\tilde{\mathrm{D}} = [c_i < x_i < d_i$ for $i \neq l$ and $c_l - h_q^{-1}(d_l - c_l) < x_l < d_l]$. A similar formula is valid for the extension across $[x_l = d_l]$.

We can also consider oblique Lichtenstein extensions which are extensions across an $(n - 1)$-dimensional hyperplane π in the direction of a unit vector θ which is not necessarily

[16] We use here the fact that if a function of one variable is absolutely continuous on $-b < t < 0$ and on $0 < t < b$ and is continuous on $-b < t < b$, and if its derivative is integrable on $-b < t < b$, then the function is absolutely continuous on $-b < t < b$.

orthogonal to π (but not parallel). An application of Prop. 1) and Prop. 5), § 2 to the affine transformation which carries θ into the unit vector orthogonal to π and which leaves π invariant would suffice to give a result similar to Prop. 1) except that the constant c would also depend on the angle between θ and π.

2) *If* $D = [c_i < x_i < d_i;\ i = 1, \ldots, n]$ *(c_i or d_i possibly infinite) there is a linear mapping,* $\tilde{u} = T_q u$, *of* $\mathfrak{M}(D)$ *into* $\mathfrak{M}(R^n)$ *such that if* $\alpha^* \leqslant q$ *and* $u \in \check{P}^\alpha(D)$ *then* $\tilde{u} = T_q u \in P^\alpha$ *and* $|\tilde{u}\urcorner_{\alpha,R^n} \leqslant C W^{-\alpha^*-1} |u\urcorner_{\alpha,D}$ *where* $W = \min_i (1, d_i - c_i)$ *and* C *depends only on* q, n *and* $\{h_\mu\}$.

Proof. — Let

$$D_l = [c_i^{(l)} < x_i < d_i^{(l)};\ i = 1, \ldots, n], \qquad 0 \leqslant l \leqslant 2n,$$

where $c_i^{(l)} = c_i$ for $i > l$ and $c_i^{(l)} = c_i - h_q^{-1}(d_i - c_i)$ for $i \leqslant l$; $d_i^{(l)} = d_i$ for $i > l - n$ and $d_i^{(l)} = d_i + h_q^{-1}(d_i + c_i^{(l)})$ for $i \leqslant l - n$ and h_q is the last term in the sequence $\{h_\mu\}$ which defines the Lichtenstein extension of order q.

We proceed by induction. Suppose $u \in \mathfrak{M}(D)$, let $D_0 = D$ and $\tilde{u}_0 = u$. Trivially $\tilde{u}_0 \in \mathfrak{M}(D_0)$. Suppose $\tilde{u}_l \in \mathfrak{M}(D_l)$, consider the Lichtenstein extension of order q of $\tilde{u}_l$ across the hyperplane $[x_l = c_l]$ for $l \leqslant n$ and across $[x_{l-n} = d_{l-n}]$ for $l > n$ (cf. the example of Remark 4). It is easy to see that $\tilde{D}_l = D_{l+1}$ and from (3.3) that $\tilde{u}_{l+1} \in \mathfrak{M}(D_{l+1})$. This completes the induction.

Hence $\tilde{u}_{2n} \in \mathfrak{M}(D_{2n})$ and $u \to \tilde{u}_{2n}$ is a linear mapping. If $D_{2n} = R^n$ the desired mapping is $\tilde{u} = \tilde{u}_{2n} = T_q u$. If $D_{2n} \neq R^n$ then dist $(\partial D_{2n}, D) \geqslant h_q^{-1} W$. Let $\varphi(x)$ be the function given by Lemma 1, § 1, which vanishes outside D_{2n}^δ where

$$\delta = \frac{1}{2} h_q^{-1} W,$$

and $= 1$ on $\bar{D}$. Then the desired mapping is $u = \varphi u_{2n} = T_q u$ (extended by 0 outside D_{2n}).

If $u \in \check{P}^\alpha(D)$, $\alpha^* \leqslant q$, then by Prop. 1) at each step of the induction

$$\tilde{u}_{l+1} \in \check{P}^\alpha(D_{l+1}) \qquad \text{and} \qquad |\tilde{u}_{l+1}\urcorner_{\alpha,D_{l+1}} \leqslant c_0 |\tilde{u}_l\urcorner_{\alpha,D_l} \leqslant c_0^l |\tilde{u}\urcorner_{\alpha,D}$$

where c_0 is the constant of Prop. 1) which depends only on q, n, and $\{h_\mu\}$. If $D_{2n} = R^n$ this completes the proof; if

$$D_{2n} \neq R^n,$$

an application of Lemma 1, § 1 and Prop. 6), § 2 completes the proof.

2′) *If* D *is a rectangle then* $\check{P}^\alpha(D) = P^\alpha(D)$.

Proof. — Proposition 2).

Remark 5. — We could use the oblique Lichtenstein extension to show that Props. 2) and 2′) are valid for a convex polyhedron (17). However, a much more general result will be obtained later from a general theorem concerning Lipschitzian graph domains (Section 11).

Remarks about the spaces $\check{P}^{\alpha,p}(D)$, $1 < p < \infty$. All the statements of the present section are true for the classes $\check{P}^{\alpha,p}(D)$. In the proof of Theorem I we would refer to results and methods in [2] instead of Chapter II (18). All the other statements are obtained by proofs similar to those given in this section.

4. Behavior of $d_{\alpha,D}(u)$, $|u\rceil_{\alpha,D}$ and $|u|_{\alpha,D}$ for varying α.

We will study $d_{\alpha,D}(u)$, $|u\rceil_{\alpha,D}$ and $|u|_{\alpha,D}$ for α varying between two consecutive integers $m \leqslant \alpha \leqslant m + 1$, $m = 0, 1, \ldots$, especially when $\alpha \uparrow m + 1$ or $\alpha \downarrow m$.

Throughout this section $u(x)$ will be assumed to be a corrected function in D (cf. § 0). Hence, if $u \in \check{P}^m(D)$, $m = [\alpha]$, and $|u|_{\alpha,D} < \infty$, then $u \in \check{P}^\alpha(D)$ by Prop. 1′), § 2.

For $\theta = (\theta_1, \ldots, \theta_n) \in \partial S = \partial S(0,1)$, denote by E_θ the orthogonal projection on the $(n - 1)$-dimensional subspace orthogonal to θ. For $x \in D$ let $z' = E_\theta(x) \in E_\theta(D)$ and

$$I(\theta, z') = [t : z' + t\theta \in D].$$

(17) This was actually done in the first version of this paper.

(18) However, one result which was not given in [2] would have to be used, viz. that the local finiteness of the L^p-norms of the pure derivatives of order k in each direction implies the local finiteness of the L^p-norms of all mixed derivatives of order k.

It is then easy to verify that for $f(x, y) \geqslant 0$ (and measurable on $D \times D$)

$$(4.1) \quad \int_D \int_D f(x, y)\, dx\, dy$$
$$= \frac{1}{2} \int_{\partial S} \int_{E_\theta(D)} \int_{I(\theta,z')} \int_{I(\theta,z')} f(z' + t\theta, z' + s\theta)|s - t|^{n-1}\, ds\, dt\, dz'\, d\theta.$$

We now list some useful transformations of $d_{\alpha,D}(u)$:

$$(4.1a) \quad |u|^2_{0,D} = \int_D |u(x)|^2\, dx$$
$$= \frac{1}{\omega_n} \int_{\partial S} \int_{E_\theta(D)} |u(z' + s\theta)|^2_{0,I(\theta,z')}\, dz'\, d\theta,$$

applying (4.1) to $d_{\alpha,D}(u)$ for $0 < \alpha < 1$ we have

$$(4.1b) \quad d_{\alpha,D}(u) = \frac{C(1, \alpha)}{2C(n, \alpha)} \int_{\partial S} \int_{E_\theta(D)} d_{\alpha,I(\theta,z')}(u(z' + s\theta))\, dz'\, d\theta$$

and by noting that

$$\int_{\partial S} \theta_i \theta_j\, d\theta = \frac{\omega_n}{n} \delta_{ij} \qquad \text{and} \qquad \sum_{l=1}^{n} \theta_l \frac{\partial u}{\partial x_l} = \frac{d}{ds} u(z' + s\theta)$$

we have

$$(4.1c) \quad d_{1,D}(u) = \sum_{l=1}^{n} \left|\frac{\partial u}{\partial x_l}\right|^2_{0,D} = \frac{n}{\omega_n} \int_D \int_{\partial S} \left|\sum_{l=1}^{n} \theta_l \frac{\partial u}{\partial x_l}\right|^2 d\theta\, dx$$
$$= \frac{n}{\omega_n} \int_{\partial S} \int_{E_\theta(D)} d_{1,I(\theta,z')}\,(u(z' + s\theta))\, dz'\, d\theta.$$

For later reference, we note from the formulae for the Gamma function that

$$(4.2) \qquad \frac{C(1, \alpha)}{2C(n, \alpha)} = \frac{\Gamma(\alpha + n/2)}{2\pi^{\frac{n-1}{2}}\, \Gamma\left(\alpha + \frac{1}{2}\right)}$$

is an increasing function of α, $0 \leqslant \alpha \leqslant 1$, which varies from $\frac{1}{\omega_n}$ to $\frac{n}{\omega_n}$.

Lemma 1. — *Let $D \subset R^n$ be convex and $u \in \check{P}^0(D)$. Define*

$$I(\alpha) = \begin{cases} 0 \quad \textit{for} \quad \alpha = 0, \\ \dfrac{1}{C(n, \alpha)} \displaystyle\int_D \int_{\substack{D \\ |x-y|<1}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}}\, dx\, dy \quad \textit{for} \quad 0 < \alpha < 1, \\ d_{1,D}(u) \quad \textit{if} \quad u \in \check{P}^1(D), \quad \textit{otherwise} \quad = +\infty \quad \textit{for} \quad \alpha = 1. \end{cases}$$

Then $I(\alpha)$ *is an increasing function of* α *which is continuous on any subinterval of* $[0,1]$ *where it is finite. Furthermore,* $\lim_{\alpha \uparrow 1} I(\alpha)$ *is finite if and only if* $u \in \check{P}^1(D)$.

Remark 1. — It is clear by considering

$$d_{\alpha,R^n}(u) = \int_{R^n} |\xi|^{2\alpha}|\hat{u}(\xi)|^2 \, d\xi$$

that in general $d_{\alpha,D}(u)$ is not an increasing function.

Proof. — *i*) We consider first $0 \leqslant \alpha < 1$. By (4.1) and (4.2) it is clear that we may restrict ourselves to considering

$$D = (a, b) \subset R^1.$$

If $I(\alpha) \equiv \infty$ for $0 < \alpha < 1$, $I(\alpha)$ is obviously increasing. So we suppose $I(\alpha) < \infty$ for some α and write

$$(4.3) \quad I(\alpha) = \frac{2}{C(1, \alpha)} \int_a^b \int_a^b_{0<s-t<1} \frac{|u(s) - u(t)|^2}{|s - t|^{1+2\alpha}} \, ds \, dt = \frac{2}{C(1, \alpha)} \int_0^1 \frac{1}{r^{2\alpha-1}} \varphi(r) \, dr$$

where $\varphi(r) = \int_{I_r} \left| \frac{u(r + t) - u(t)}{r} \right|^2 dt \leqslant \frac{4}{r^2} |u|^2_{0,(a,b)}$ and

$$I_r = (a, \max(a, b - r)).$$

By the Minkowski inequality we have

$$0 \leqslant \varphi(r)^{\frac{1}{2}} \leqslant \frac{1}{2} \left[\int_{I_r} \left| \frac{u(r + t) - u\left(\frac{r}{2} + t\right)}{r/2} \right|^2 dt \right]^{\frac{1}{2}} + \frac{1}{2} \left[\int_{I_r} \left| \frac{u\left(\frac{r}{2} + t\right) - u(t)}{r/2} \right|^2 dt \right]^{\frac{1}{2}} \leqslant \varphi\left(\frac{r}{2}\right)^{\frac{1}{2}} < \infty.$$

(This inequality was suggested by E. Gagliardo.) Therefore for fixed r, $\varphi(2^{-k} r)$ is an increasing function of k. Applying

Abel's summation formula ([19]) we have

$$(4.4)\quad I(\alpha) = \frac{2}{C(1, \alpha)} \sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} \frac{1}{r^{2\alpha-1}} \varphi(r)\, dr$$
$$= \frac{2}{C(1, \alpha)} \sum_{k=0}^{\infty} \int_{\frac{1}{2}}^{1} \frac{2^{2k(\alpha-1)}}{r^{2\alpha-1}} \varphi(2^{-k} r)\, dr$$
$$= \frac{2}{C(1, \alpha)(1 - 2^{2(\alpha-1)})} \int_{\frac{1}{2}}^{1} \frac{1}{r^{2\alpha-1}}$$
$$\times \left\{ \varphi(r) + \sum_{k=1}^{\infty} 2^{2k(\alpha-1)} (\varphi(2^{-k} r) - \varphi(2^{-k+1} r)) \right\} dr.$$

It is not difficult to show that $2/C(1, \alpha)(1 - 2^{2(\alpha-1)})$ is an increasing function of α which converges to $1/\log 2$ as $\alpha \uparrow 1$, and to 0 as $\alpha \downarrow 0$. Since the integrand in the last formula in (4.4) is a positive increasing function of α this completes the proof for $0 \leqslant \alpha < 1$.

ii) We now show that $\lim_{\alpha \uparrow 1} I(\alpha)$ is finite if and only if $u \in \check{P}^1(D)$ and that $I(\alpha)$ is continuous at $\alpha = 1$.

We assume at first, as in *i*), that $D = (a, b)$ and use (4.3). Suppose, in this case that $\lim_{\alpha \uparrow 1} I(\alpha) < \infty$. Then from (4.4) we see that

$$\lim_{\alpha \uparrow 1} I(\alpha) = \frac{1}{\log 2} \int_{\frac{1}{2}}^{1} \frac{1}{r} \lim_{k \uparrow \infty} \varphi(2^{-k} r)\, dr < \infty$$

and

$$\lim_{k \uparrow \infty} \varphi(2^{-k} r) < \infty$$

for almost every r. But this implies from a well-known Hilbert space theorem that for every such r there is a subsequence $(u(2^{-k_l} r + t) - u(t))/2^{-k_l} r$ which converges weakly to

$$v(t) \in L^2(a, b),$$

([19]) If $b_\nu \geqslant 0$ is an increasing sequence and $a_\nu \geqslant 0$, then

$$\sum_{\nu=0}^{\infty} a_\nu b_\nu = s_0 b_0 + \sum_{\nu=1}^{\infty} s_\nu (b_\nu - b_{\nu-1})$$

where $s_\nu = \sum_{l=\nu}^{\infty} a_l$.

the distribution derivative of $u(t)$. Hence by Prop. 1), § 2, $u \in \check{P}^1(a, b)$. By Prop. 2), § 3. there exists $\tilde{u} \in P^1(R^1)$ which is an extension of u. By using Fourier transforms and cominated convergence on proves immediately that

$$r^{-1}[\tilde{u}(x+r) - \tilde{u}(x)]$$

converges strongly to $\dfrac{\partial \tilde{u}}{\partial x}$ in $L^2(R^1)$ for $r \searrow 0$. Hence

$$\varphi(r) = |r^{-1}[u(x+r) - u(x)]|^2_{0,I_r} \to \left|\frac{\partial u}{\partial x}\right|^2_{0,D} \quad \text{for} \quad r \searrow 0,$$

$$\lim_{\alpha \nearrow 1} I(\alpha) = \frac{1}{\log 2}\int_{1/2}^{1} \frac{1}{r} \lim_{k \nearrow \infty} \varphi(2^{-k} r)\, dr = \left|\frac{\partial u}{\partial x}\right|^2_{0,D}.$$

which completes the proof in the special case $D = (a, b)$.

Now if D is a convex set in R^n and $\lim\limits_{\alpha \uparrow 1} I(\alpha) < \infty$ this implies from the preceding and (4.1*b*) that $d_{1,I(\theta,z')}(u(z' + s\theta)) < \infty$ for almost all θ and z' in particular for at least n linearly independent $\theta^{(m)}$. Hence from Prop. 1), § 2, $u \in \check{P}^1(D)$ and the rest of the proof follows.

1) *If* $D \subset R^n$ *is convex, then a) for* $\alpha \leqslant \gamma$, $\check{P}^\alpha(D) \supset \check{P}^\gamma(D)$ *and* $|u|_{\alpha,D} \leqslant \sqrt{10n}|u|_{\gamma,D}$ *for* $u \in \check{P}^\alpha(D)$, *b) if*

$$u \in \check{P}^\alpha(D), \quad 0 \leqslant \alpha < \alpha_0,$$

then $|u|_{\alpha,D}$ *is continuous for* $0 \leqslant \alpha < \alpha_0$ *and* $\lim\limits_{\alpha \uparrow \alpha_0} |u|_{\alpha,D}$ *always exists (possibly* $= +\infty$*). The limit is finite if and only if* $u \in \check{P}^{\alpha_0}(D)$ *and in this case the limit is equal to* $|u|_{\alpha_0,D}$.

Proof of a). — By Prop. 3, § 1 (localized), it is sufficient to prove the inequality and from (2.1) it is clear that we may restrict ourselves to the case $0 < \alpha \leqslant \gamma \leqslant 1$. By Lemma 1 and (2.7) we have that

$$\begin{aligned} |u\rceil^2_{\alpha,D} &= |u\rceil^2_{0,D} + d_{\alpha,D}(u) \\ &\leqslant |u|^2_{0,D} + I(\alpha) + 4\int |u(x)|^2 \left[\frac{1}{C(n,\alpha)} \int_{\substack{D \\ |x-y|>1}} \frac{dy}{|x-y|^{n+2\alpha}}\right] dx \\ &\leqslant I(\gamma) + [4n(1-\alpha) + 1]|u|^2_{0,D} \leqslant 5n|u\rceil^2_{\gamma,D} \end{aligned}$$

and part *a*) follows by Prop. 3), § 2.

Proof of b). — Suppose part *b*) is true for $0 \leqslant \alpha_0 \leqslant 1$. Let $m+1$ be an integer $\leqslant \alpha_0$. From (2.1) and our assumption

we have

$$\lim_{\alpha \uparrow m+1} |u|^2_{\alpha,D} = \sum_{k=0}^{m} \binom{m}{k} \sum_{|i|=k} \lim_{\beta \uparrow 1} |D_i u|^2_{\beta,D}$$
$$= \sum_{k=0}^{m} \binom{m}{k} \sum_{|i|=k} \left(|D_i u|^2_{0,D} + \sum_{j=1}^{n} \left| \frac{\partial}{\partial x_j} D_i u \right|^2_{0,D} \right)$$
$$= \sum_{k=0}^{m+1} \left(\binom{m}{k} + \binom{m}{k-1}\right) \sum_{|i|=k} |D_i u|^2_{0,D} = |u|^2_{m+1,D}.$$

The rest of *b*) follows easily from our assumption so it is clear that we need only consider $0 \leqslant \alpha_0 \leqslant 1$.

For $0 < \alpha < 1$, we have by § 4, II that $\frac{G_{2n+2\alpha}(x)}{G_{2n+2\alpha}(0)}$, $x \neq 0$, is an analytic function of α, which is majorated by 1 and by (1.3), II, that $\frac{1}{C(n, \alpha)}$ is analytic which completes the proof in the case $0 < \alpha < 1$.

By (4.4), II we have for $0 \leqslant \alpha \leqslant 1$ and $0 < |x| \leqslant 1$ that $1 \geqslant \frac{G_{2n+2\alpha}(x)}{G_{2n+2\alpha}(0)} > 0$ and by (2.10), (3.6) and (4.2) of II

(4.5) *If* $|x| \geqslant 1$ *and* $\alpha \geqslant 0$ *then* $\frac{G_{2n+2\alpha}(x)}{G_{2n+2\alpha}(0)} \leqslant |x|^{\alpha+\frac{n}{2}-\frac{1}{2}} e^{1-|x|}$.

Hence, by Lemma 1 and (2.7) we have for $u \in \check{P}^{\alpha_0}(D)$, $\alpha_0 > 0$, that

$$0 \leqslant |u|^2_{\alpha,D} - |u|^2_{0,D}$$
$$\leqslant I(\alpha) + \frac{4}{C(n,\alpha)} \int_D |u(x)|^2 \int_{|x-y|>1} \frac{G_{2n+2\alpha}(x-y)}{G_{2n+2\alpha}(0)} |x-y|^{-n-2\alpha} dy\, dx$$
$$\leqslant I(\alpha) + 8n\alpha(1-\alpha) |u|^2_{0,D}\, e\Gamma\left(\frac{n}{2}+1\right) \downarrow 0 \text{ as } \alpha \downarrow 0.$$

We now consider $\alpha \uparrow 1$. By Remark 3, § 2,

$$|u\rceil^2_{\alpha,D} - |u|^2_{\alpha,D} \downarrow 0$$

as $\alpha \uparrow 1$ for $u \in \check{P}^0(D)$. By (2.7) it is clear that

$$|I(\alpha) - d_{\alpha,D}(u)| \leqslant 4n(1-\alpha)|u|^2_{0,D}.$$

Hence by Lemma 1, if $u \in \check{P}^{\alpha}(D)$, $0 \leqslant \alpha < 1$, then $\lim_{\alpha \uparrow 1} |u|^2_{\alpha,D}$ exists and is finite if and only if $u \in \check{P}^1(D)$. Furthermore, if $u \in \check{P}^1(D)$, then $\lim_{\alpha \uparrow 1} |u|^2_{\alpha,D} = |u|^2_{1,D}$. This completes the proof of the proposition.

Remark 2. — Although $|u|^2_{\alpha,R^n} = \int_{R^n} (1 + |\xi|^2)^\alpha |\hat{u}(\xi)|^2 \, d\xi$ is an increasing function of α, it is not known whether $|u|^2_{\alpha,D}$, D convex, is increasing or not.

In the remaining part of this section we shall generally consider $0 \leqslant \alpha \leqslant 1$. The propositions and remarks have obvious analogues for $m \leqslant \alpha \leqslant m + 1$, $m = 1,2, \ldots$.

Remark 3. — For a function $u \in \check{P}^0(D)$ the behavior of $d_{\alpha,D}(u)$ (or $|u|^2_{\alpha,D} - |u|^2_{0,D}$) inside the interval $0 < \alpha < 1$ is easily analyzed by decomposing

$$d_{\alpha,D}(u) = \frac{1}{C(n, \alpha)} \left[\int_D \int_D_{|x-y|<1} + \int_D \int_D_{|x-y|>1} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}} \, dx \, dy \right]$$

and noting that the second term is bounded by

$$4n(1 - \alpha)|u|^2_{0,D}.$$

The following cases are possible and exhaust all possibilities: 1° $d_{\alpha,D}(u) \equiv \infty$ for all $\alpha > 0$; 2° $d_{\alpha,D}(u)$ is finite and continuous in $0 < \alpha < \alpha_0 \leqslant 1$ and infinite elsewhere, or 3° $d_{\alpha,D}(u)$ is finite and continuous in $0 < \alpha \leqslant \alpha_0 \leqslant 1$ and infinite elsewhere.

We next give some properties of $|u|_{\alpha,D}$ in an arbitrary domain $D \subset R^n$ as $\alpha \uparrow 1$. We remind the reader that by Remark 3, § 2, $|u|^2_{\alpha,D} - |u\rceil^2_{\alpha,D} \downarrow 0$ as $\alpha \uparrow 1$; hence $|u|_{\alpha,D}$ may be replaced in any of the propositions by $|u\rceil_{\alpha,D}$.

2) *If* $u \in \check{P}^0(D)$ *and* $\liminf_{\alpha \uparrow 1} |u|_{\alpha,D} < \infty$, *then* $u \in \check{P}^1(D)$ *and* $\liminf_{\alpha \uparrow 1} |u|_{\alpha,D} \geqslant |u|_{1,D}$ [20].

Proof. — For $x \in D$ let $S = S(x, \rho) \subset D$ then $|u|_{\alpha,S} \leqslant |u|_{\alpha,D}$, and it follows from Prop. 1) that $u \in \check{P}^1(S)$; a fortiori $u \in P^1_{loc}(D)$. Now let $\{S_k\}$, $k = 1,2, \ldots$ be a disjoint sequence of spheres such that $S_k \subset D$ and $\left| D - \bigcup_{k=1}^{\infty} S_k \right| = 0$. Then

$$\sum_{k=1}^{\infty} |u|^2_{\alpha,S_k} \leqslant |u|^2_{\alpha,D}$$

[20] By considering $S(0,1) - [x_1 = 0]$ in R^2 and setting $u(x_1, x_2) = 0$ for $x_1 > 0$ and $= 1$ for $x_1 < 0$ we see that the converse of this proposition is false.

and by Prop. 1),

$$\infty > \liminf_{\alpha\uparrow 1} |u|^2_{\alpha,D} \geqslant \sum_{k=1}^{\infty} \lim_{\alpha\uparrow 1} |u|^2_{\alpha,S_k} = \sum_{k=1}^{\infty} |u|^2_{1,S_k} = |u|^2_{1,D}.$$

As an immediate application of Prop. 2) we can give another criterion for the extension of functions (cf. Theorem I, § 3) from a given domain to a larger domain.

3) *Let* $D \subset D_1$ *and* $|D_1 - D| = 0$. *If* $u \in \check{P}^\alpha(D)$ *and*

$$\liminf_{\alpha'\uparrow m} |u|_{\alpha',D} < \infty, \qquad m = [\alpha],$$

then u *has a unique extension* $\tilde{u} \in \check{P}^\alpha(D_1)$.

Proof. — Take as $\tilde{u}$ the correction of u in D_1. Since

$$|\tilde{u}|_{\alpha,D_1} = |u|_{\alpha,D},$$

we need only prove that $\tilde{u} \in P^\alpha_{loc}(D_1)$.

Let v_i be the correction of $D_i u$ in D_1 for $0 < |i| \leqslant m$. Then by Prop. 2), $v_i \in \check{P}^1(D_1)$. The proposition now follows by Prop. 2'), § 9, II and Theorem 1, § 0.

Remark 4. — In connection with Prop. 3), it might be of interest to introduce the class $\tilde{P}^\alpha(D)$ of all functions which belong to $\check{P}^\alpha(D)$ and such that $|u|_{\beta,D}$ is uniformly bounded for all $\beta \leqslant \alpha$. Then, Prop. 3) might be restated « if $D \subset D_1$ and $|D_1 - D| = 0$, each function $u \in \tilde{P}^\alpha(D)$ has a unique extension to $\tilde{P}^\alpha(D_1)$ ». It is clear that $P^\alpha(D) \subset \tilde{P}^\alpha(D) \subset \check{P}^\alpha(D)$, and examples might be given to show that $\tilde{P}^\alpha(D)$ is not always equal to $P^\alpha(D)$ (cf. example 3, § 13).

4) *If* $u \in \check{P}^1(D)$ *and* $\lim_{\alpha\uparrow 1} |u|_{\alpha,D} = |u|_{1,D}$ *then for every* $D' \subset D$ *such that* $|D \cap \partial D'| = 0$ *we have* $\lim_{\alpha\uparrow 1} |u|_{\alpha,D'} = |u|_{1,D'}$.

Proof. — Let $D'' = D - \overline{D'}$ then $|D - (D' \cup D'')| = 0$, $u \in \check{P}^1(D')$ and $u \in \check{P}^1(D'')$. By Prop. 2), $|u|^2_{1,D'} \leqslant \liminf_{\alpha\uparrow 1} |u|^2_{\alpha,D}$ and $|u|^2_{1,D''} \leqslant \liminf_{\alpha\uparrow 1} |u|^2_{\alpha,D''}$. From $|u|^2_{\alpha,D'} + |u|^2_{\alpha,D''} \leqslant |u|^2_{\alpha,D}$ it follows that

$$|u|^2_{1,D} = |u|^2_{1,D'} + |u|^2_{1,D''} \leqslant \limsup_{\alpha\uparrow 1} |u|^2_{\alpha,D'} + \liminf_{\alpha\uparrow 1} |u|^2_{\alpha,D''} \leqslant \lim_{\alpha\uparrow 1} |u|^2_{\alpha,D} = |u|^2_{1,D}.$$

Hence, $\limsup_{\alpha \uparrow 1} |u|^2_{\alpha,D'} = |u|^2_{1,D'} \leqslant \liminf_{\alpha \uparrow 1} |u|^2_{\alpha,D'}$ and

$$|u|^2_{1,D'} = \lim_{\alpha \uparrow 1} |u|^2_{\alpha,D'}.$$

5) *If* $u \in \check{P}^1(D)$ *then* $\lim_{\alpha \uparrow 1} |u|_{\alpha,D} = |u|_{1,D}$ *if and only if for every* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that* $\limsup_{\alpha \uparrow 1} |u|_{\alpha,D-\overline{D^\delta}} \leqslant \varepsilon$.

Proof. — Assume that $\lim_{\alpha \uparrow 1} |u|_{\alpha,D} = |u|_{1,D}$, then by noting that $|D \cap \partial(D - \overline{D}^\delta)| = 0$ it follows immediately by Prop. 4) that $\lim_{\alpha \uparrow 1} |u|_{\alpha,D-\overline{D^\delta}} = |u|_{1,D-\overline{D^\delta}} \leqslant \varepsilon$ for δ sufficiently small.

Conversely, suppose $\limsup_{\alpha \uparrow 1} |u|_{\alpha,D-\overline{D^\delta}} \leqslant \varepsilon$. It is easy to see by Lemma 1, § 1, and Prop. 7) § 2 that there is an extension $\tilde{u}$ of u restricted to $D^{\delta/2}$ which belongs to P^α. Hence by applying Prop. 4) to $D^{\delta/2} \subset R^n$ we have that $\lim_{\alpha \uparrow 1} |u|_{\alpha,D^{\delta/2}} = |u|_{1,D^{\delta/2}}$. Since

$$\operatorname{dist}(D - D^{\delta/2}, D^\delta) = \delta/2$$

it is clear from the properties of $G_{2n+2\alpha}(x)$ (21) and (2.7) that

$$\begin{aligned} |u|^2_{\alpha,D} &\leqslant |u|^2_{\alpha,D^{\delta/2}} + |u|^2_{\alpha,D-\overline{D^\delta}} + \frac{2}{C(n,\alpha)G_{2n+2\alpha}(0)} \\ &\quad \times \int_{D^\delta} \int_{D-\overline{D^{\delta/2}}} G_{2n+2\alpha}(x-y) \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\, dx\, dy \\ &\leqslant |u|^2_{\alpha,D^{\delta/2}} + |u|^2_{\alpha,D-\overline{D^\delta}} + \left(\frac{\delta}{2}\right)^{-2\alpha} \frac{8\omega_n}{2\alpha C(n,\alpha)} |u|^2_{0,D} \\ &\leqslant |u|^2_{\alpha,D^{\delta/2}} + |u|^2_{\alpha,D-\overline{D^\delta}} + 32n\,\delta^{-2}(1-\alpha)|u|^2_{0,D}. \end{aligned}$$

Hence, $\limsup_{\alpha \uparrow 1} |u|^2_{\alpha,D} \leqslant |u|^2_{1,D} + \varepsilon^2$. Therefore, since ε is arbitrary and by Prop. 2), $\liminf_{\alpha \uparrow 1} |u|_{\alpha,D} \geqslant |u|_{1,D}$, we have

$$\lim_{\alpha \uparrow 1} |u|_{\alpha,D} = |u|_{1,D}.$$

6) *If* T *is a* $C^{(0,1)}$ *homeomorphism of* D^* *onto* D *then* $u \in \check{P}^1(D)$ *and* $\lim_{\alpha \uparrow 1} |u|_{\alpha,D} = |u|_{1,D}$ *implies* $u^* \in \check{P}^1(D^*)$ *and*

$$\lim_{\alpha \uparrow 1} |u^*|_{\alpha,D^*} = |u^*|_{1,D^*}$$

where $u^*(x^*) = u(Tx^*)$.

(21) In particular $G_{2n+2\alpha}(x)/G_{2n+2\alpha}(0) \leqslant 1$.

Proof. — By Prop. 8), § 2, $u^* \in \check{P}^1(D^*)$. If M is the common Lipschitz constant of T and T^{-1}, $T(D^* - \overline{D^{*\delta/M}}) \subset D - \overline{D^\delta}$. Then by Prop. 8), § 2, $|u|^2_{\alpha, D^* - \overline{D^{*\delta/M}}} \leqslant CM^{\alpha + \frac{3}{2}n} |u|_{\alpha, D - \overline{D^\delta}}$. An application of the preceding proposition completes the proof.

In Remark 3, § 2, we noted that $|u\rceil^2_{\alpha,D} - |u|^2_{\alpha,D} \downarrow 0$ as $\alpha \uparrow 1$. It can be shown that

$$(4.6) \quad 0 \leqslant |u\rceil^2_{\alpha,D} - |u|^2_{\alpha,D} \leqslant 4\left(n + \log \frac{e}{1 - \alpha_0}\right)\left(\frac{\alpha}{\alpha_0}\right)(1 - \alpha)|u|^2_{\alpha_0,D},$$

for $0 < \alpha_0 \leqslant \alpha < 1$ and $u \in \check{P}^\alpha(D)$. If we interpret the difference in the inequality as the integral of the difference of the integrands the inequality is valid for all $u \in \check{P}^{\alpha_0}(D)$.

To characterize the behavior of $d_{\alpha,D}(u)$ as $\alpha \downarrow 0$ we introduce the function

$$(4.7) \quad \delta(\alpha) \equiv \delta(\alpha, D) = \frac{2}{C(n, \alpha)} \int_{D \cap [|y| > 1]} \frac{1}{|y|^{n+2\alpha}} \, dy.$$

7) *If* $u \in \check{P}^\alpha(D)$ *for some* α, $0 < \alpha < 1$, *then*

$$\lim_{\alpha' \downarrow 0} (d_{\alpha',D}(u) - \delta(\alpha')|u|^2_{0,D}) = 0.$$

To prove this we need the following lemma.

Lemma 2. — *For* $0 < \alpha < 1$, *a)* $\delta(\alpha, D) \leqslant 2n$ *for all* D, $\lim_{\alpha \downarrow 0} \delta(\alpha, R^n) = 1$ *and if* D *is bounded,* $\lim_{\alpha \downarrow 0} \delta(\alpha, D) = 0$.

b) Let D *be a fixed domain. Put*

$$\delta(x, R, \alpha) = \frac{2}{C(n, \alpha)} \int_{\substack{D \\ |x-y| > 2R}} \frac{1}{|x - y|^{n+2\alpha}} \, dy$$

for $R > 1$ *and* $x \in D \cap [|x| < R]$. *Then* $\delta(x, R, \alpha) \leqslant 2n$ *and* $\limsup_{\alpha \downarrow 0} |\delta(\alpha) - \delta(x, R, \alpha)| = 0$ *uniformly in x for*

$$x \in D \cap [|x| < R].$$

Proof. — By (2.7),

$$\delta(\alpha, D) \leqslant \frac{2}{C(n, \alpha)} \int_{|y| > 1} |y|^{-n-2\alpha} \, dy = \frac{\omega_n}{\alpha C(n, \alpha)} \leqslant 2n.$$

It is clear from (1.3), II, that $\delta(\alpha, R^n) = \frac{\omega_n}{\alpha C(n, \alpha)} \to 1$ as

$\alpha \downarrow 0$. If $D \subset S(0, R)$ then $\delta(\alpha, D) \leqslant 2n(1 - R^{-2\alpha}) \downarrow 0$ as $\alpha \downarrow 0$ which completes the proof of part *a*).

For $R > 1$ and $x \in D \cap [|x| < R]$, $|y| > 3R$ implies

$$|x - y| > 2R,$$

hence

$$\delta(\alpha) - \delta(x, R, \alpha) = \frac{2}{C(n, \alpha)} \int_{\substack{D \\ 1<|y|<3R}} |y|^{-n-2\alpha}\, dy - \frac{2}{C(n, \alpha)} \int_{\substack{D \\ R<|y|<3R \\ |x-y|>2R}} |y - x|^{-n-2\alpha}\, dy + \frac{2}{C(n, \alpha)} \int_{\substack{D \\ |y|>3R}} (|y|^{-n-2\alpha} - |x - y|^{-n-2\alpha})\, dy.$$

The integrand of the third term is majorated by

$$(n + 2\alpha)|x|(\min\ [|y|, |x - y|])^{-n-2\alpha-1} \leqslant (n + 2\alpha)R \cdot \left(\frac{2}{3}|y|\right)^{-n-2\alpha-1}.$$

Therefore by (2.7)

$$|\delta(\alpha) - \delta(x, R, \alpha)| \leqslant 2n(1 - (3R)^{-2\alpha}) + 2n((2R)^{-2\alpha} - (4R)^{-2\alpha}) + \frac{(n + 2\alpha)}{(1 + 2\alpha)} 2n\alpha(1 - \alpha)\left(\frac{2}{3}\right)^{-n-2\alpha} (3R)^{-2\alpha},$$

which establishes the convergence in part *b*). The bound for $\delta(x, R, \alpha)$ is obtained the same way as for $\delta(\alpha)$.

Proof of Prop. 7). — Let $u \in \check{P}^{\alpha_0}(D)$ and $\alpha < \alpha_0 < 1$. Given $\varepsilon > 0$, choose R sufficiently large ($R > 1$) so that

$$|u|^2_{0, D\cap[|x|>R]} < \varepsilon.$$

Then

$$d_{\alpha, D}(u) = \frac{1}{C(n, \alpha)} \left\{ \int_D \int_{\substack{D \\ |x-y|<2R}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}}\, dx\, dy + \int_{\substack{D \\ |x-y|>2R \\ |x|>R, |y|>R}} \int_D \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}}\, dx\, dy + 2 \int_{\substack{D \\ |x|<R, |y|>R \\ |x-y|>2R}} \int_D \frac{|u(x)|^2 + |u(y)|^2 - 2\mathrm{Re}\, u(x)\overline{u(y)}}{|x - y|^{n+2\alpha}}\, dx\, dy \right\}.$$

The first and second terms are bounded by

$$\alpha \frac{C(n, \alpha_0)(2R)^{2\alpha_0} 2n}{\omega_n} d_{\alpha_0, D}(u)$$

and $4n\varepsilon$ respectively. Since

$$\left| \frac{2}{C(n, \alpha)} \int_D \int_D_{\substack{|x|<R, |y|>R \\ |x-y|>2R}} \frac{|u(x)|^2 + |u(y)|^2 - 2\mathrm{Re}\, u(x)\overline{u(y)}}{|x-y|^{n+2\alpha}} dx\, dy - \int_{D \atop |x|<R} \delta(x, R, \alpha)|u(x)|^2 dx \right| \leqslant 2n\varepsilon + 4n\varepsilon^{\frac{1}{2}} |u|_{0,D}$$

it follows by Lemma 2 that

$$\limsup_{\alpha \downarrow 0} |d_{\alpha,D}(u) - \delta(\alpha)|u|^2_{0,D}| \leqslant 8n\varepsilon + 4n\varepsilon^{\frac{1}{2}} |u|_{0,D}$$

and the proposition follows.

It is clear from the above that for bounded domains,

$$\lim_{\alpha \downarrow 0} \delta(\alpha) = 0;$$

consequently, $d_{\alpha,D}(u)$ as $\alpha \downarrow 0$ depends only the the behavior of D in a neighborhood of ∞. Furthermore, if

$$D = R^n, \qquad \lim_{\alpha \downarrow 0} \delta(\alpha) = 1,$$

and examples can be constructed (cf. example 5, § 13) so that $\liminf_{\alpha \downarrow 0} \delta(\alpha) = a$, $\limsup_{\alpha \downarrow 0} \delta(\alpha) = b$ for arbitrary a, b; $0 \leqslant a \leqslant b \leqslant 1$.

In contrast with $|u\rceil_{\alpha,D}$, $|u|_{\alpha,D}$ is continuous from the right at $\alpha = 0$ and the next proposition gives an estimate for the modulus of continuity.

8) *If* $u \in \check{P}^{\alpha_0}(D)$, $0 < \alpha_0 < 1$, *then for* $\alpha \leqslant \alpha_0(1-\alpha_0)/e$,

$$0 \leqslant |u|^2_{\alpha,D} - |u|^2_{0,D} \leqslant \frac{2\alpha}{\alpha_0(1-\alpha_0)} \left(\mathrm{Log}\, \frac{\alpha_0(1-\alpha_0)}{\alpha} \right)^{\gamma} \left(\frac{\sqrt{\pi}}{\Gamma_0} + 4\, en \right) |u|^2_{\alpha_0,D}$$

where $\Gamma_0 = \min_{1<\beta<2} \Gamma(\beta)$ (*approximately*.88560) *and*

$$\gamma = \max\left(2\alpha_0, \frac{n}{2} - \frac{1}{2}\right).$$

Proof. — By (1.3), II, it is easy to check that for

$$0 < \alpha < \alpha_0 < 1,$$

$\dfrac{C(n, \alpha_0)}{C(n, \alpha)} \leqslant \dfrac{\alpha(1-\alpha)}{\alpha_0(1-\alpha_0)} \cdot \dfrac{\sqrt{\pi}}{\Gamma_0}$. Hence by (4.5), (2.7), and (2.4), we have for $r \geqslant 1$,

$$0 \leqslant |u|^2_{\alpha,D} - |u|^2_{0,D} = \frac{1}{C(n, \alpha)G_{2n+2\alpha}(0)} \left\{ \int_D \int_D_{|x-y|<r} + \int_D \int_D_{|x-y|>r} \frac{G_{2n+2\alpha}(x-y)}{|x-y|^{n+2\alpha}} |u(x) - u(y)|^2 \, dx \, dy \right\}$$

$$\leqslant r^{2\alpha_0 - 2\alpha} \frac{C(n, \alpha_0)}{C(n, \alpha)} d_{\alpha_0, D}(u) + 4n(1-\alpha) \frac{G_{2n+2\alpha}(r) r^{-2\alpha}}{G_{2n+2\alpha}(0)} |u|^2_{0,D} \text{ (22)}$$

$$\leqslant \left\{ \frac{\alpha r^{2\alpha_0}}{\alpha_0(1-\alpha_0)} \frac{\sqrt{\pi}}{\Gamma_0} + 4nr^{-\alpha + \frac{n}{2} - \frac{1}{2}} e^{1-r} \right\} 2|u|^2_{\alpha_0, D}.$$

The proof is completed by setting $r = \operatorname{Log} \dfrac{\alpha_0(1-\alpha_0)}{\alpha} \geqslant 1.$

Remarks on the spaces $\check{P}^{\alpha,p}(D)$. All the results of this section remain true for the spaces $\check{P}^{\alpha,p}(D)$, $1 < p < \infty$, with analogous proofs. Some of the proofs were already given in [2] for $D = R^n$.

5. Localization and L-convex domains.

In this section we make some rather general definitions which will be used in subsequent sections to localize various properties of domains. We will be especially interested in *boundary properties*, i.e. properties of ∂D relative to D. (If ∂D is the boundary of another domain D_1, ∂D relative to D_1 will not necessarily satisfy the same properties as ∂D relative to D.) As an immediate application we define the L-convex domains and establish some of their more useful properties.

A δ-loose covering of an arbitrary set $A \subset R^n$ is an open covering $\{U_k\}$ of A such that $A \subset \bigcup_k U_k^\delta$. In cases where it is not necessary to make δ explicit we shall call it a *loose covering*.

(22) Since $G_{2n+2\alpha}(x)$ is a function of $|x|$ alone we write r for $|x|$ to make the notation simpler.

A *covering with rank p* (p a positive integer) of an arbitrary set $A \subset R^n$ is an open covering $\{U_k\}$ such that for each k, $U_k \cap U_l \neq 0$ for at most p indices l. In cases where it is not necessary to make p explicit, we shall call it a *covering with finite rank.*

Consider a boundary property (P); we say the open set D satisfies the *weakly localized boundary property* (P), and for brevity we write D satisfies (P_{wl}), if for each $x \in \partial D$ there exists a neighborhood N_x of x such that $N_x \cap D$ satisfies (P).

Remark 1. — This definition is weaker than the usual definition of localization in that the usual definition requires that (P) be satisfied by $N_x \cap D$ with N_x as small as we please.

For our needs we must introduce a much more restrictive definition which we call strong localization.

Consider the boundary property (P). We say that D satisfies the *strongly localized boundary property* (P) *with constants* δ *and* p; for brevity we write D satisfies $(P_{sl,\delta,p})$, if there exists a δ-loose covering with rank p, $\{U_k\}$ of ∂D such that for each k, $U_k \cap D$ satisfies (P).

We say D satisfies the *strongly localized boundary property* (P); for brevity, D satisfies (P_{sl}) if there exists a δ and p such that D satisfies $(P_{sl,\delta,p})$.

Remark 2. — For ∂D compact, (P_{wl}) is equivalent to (P_{sl}).

If property (P) is stronger than property (P'), i.e.D satisfies (P) implies D satisfies (P') for all D, we shall write $(P) \succ (P')$ (or $(P') \prec (P)$). If $(P) \prec (P')$ and $(P') \prec (P)$, then we write $(P) \equiv (P')$.

With this symbolism we list the following without proof.

(5.1*a*) $(P) \succ (P_{sl,\delta,p}) \succ (P_{sl}) \succ (P_{wl})$.

(5.1*b*) $(P) \succ (P')$ *implies* $(P_{sl,\delta,p}) \succ (P'_{sl,\delta,p})$, $(P_{sl}) \succ (P'_{sl})$ *and* $(P_{wl}) \succ (P'_{wl})$.

(5.1*c*) $(P_{sl,\delta,p}) \succ ((P_{sl,\delta',p'})_{sl,\delta,p}) \succ (P_{wl})$.

(5.1*d*) *If* $\delta \geqslant \delta'$ *and* $p \leqslant p'$ *then* $(P_{sl,\delta,p}) \succ (P_{sl,\delta',p'})$.

(5.1*e*) *If* $(P) \succ (P')$, $\delta \geqslant \delta'$ *and* $p \leqslant p'$, *then* $(P_{sl,\delta,p}) \succ (P'_{sl,\delta',p'})$.

Lemma 1. — *If* $\{U_k\}$ *is a covering with rank* $p < \infty$ *then the sequence* $\{k\} = \{1, 2, \dots\}$ *can be divided into at most* p

mutually disjoint subsequences $\{k_l^{(1)}\}$, $\{k_l^{(2)}\}, \ldots$ *such that each of the corresponding sequences* $\{U_{k_l^{(i)}}\}$ *is composed of mutually disjoint sets.*

Proof. — Denote by $\{U_{k_l^{(1)}}\}$ a maximal subsequence of mutually disjoint sets. If $\{U_{k_l^{(i-1)}}\}$ is already defined, choose a maximal subsequence of mutually disjoint sets among the U_k's not belonging to $\{U_{k_l^{(1)}}\}, \ldots, \{U_{k_l^{(i-1)}}\}$. If the first p such sequences did not exhaust $\{U_k\}$, the remaining U_k's would satisfy $U_k \cap U_h \neq 0$ for some index h in each of the sequences $k_l^{(i)}$, $i = 1, \ldots, p$.

LEMMA 2. — *If* U_k *is a* δ*-loose covering with rank* p *of an arbitrary set* $A \subset R^n$, *then there exists a partition of unity corresponding to* U_k, $\psi_k(x)$, *such that*

i) $0 \leqslant \psi_k(x) \leqslant 1$, $\psi_k \in C^\infty(R^n)$, $\psi_k(x) = 0$ *outside* $U_k^{\delta/3}$ *and* $\sum_k \psi_k(x) = 1$ *for* $x \in S(A, \delta/8)$, *the* $\delta/8$ *neighborhood of* A.

ii) $|D_i\psi_k(x)| \leqslant \delta^{-|i|} C_{|i|}$, C_m *depending only on* p, n, *and* m.

Proof. — By Lemma 1, § 1, there exists

$$\varphi_k(x), \qquad 0 \leqslant \varphi_k(x) \leqslant 1, \qquad \varphi_k \in C^\infty(R^n),$$

such that $\varphi_0(x) = 1$ on $\overline{S(A, \delta/8)} \subset (S(A, \delta/4))^{\delta/9}$ and $= 0$ outside $S(A, \delta/4)$; for $k \geqslant 1$, $\varphi_k(x) = 1$ on $\overline{U_k^{2\delta/3}}$ and $\varphi_k(x) = 0$ outside $U_k^{\delta/3}$; and $|D_i\varphi_k(x)| \leqslant \delta^{-|i|} C'_{|i|}$, and $C'_{|i|}$ depends only on n and $|i|$.

Since $1 \leqslant \sum_{k \geqslant 1} \varphi_k(x) \leqslant p$ on $S(A, \delta/4)$ for $k \geqslant 1$ we define

$$\psi_k(x) = \begin{cases} \dfrac{\varphi_0(x)\varphi_k(x)}{\sum_{l \geqslant 1} \varphi_l(x)} & \textit{for} \quad x \in S(A, \delta/4) \\ 0 & \textit{for} \quad x \notin S(A, \delta/4) \end{cases}$$

and $\psi_k(x)$ satisfies the requirements of the Lemma.

A domain $D \subset R^n$ is $C^{(0,1)}$-*convex with bound* M if there is a $C^{(0,1)}$-homeomorphism T defined on D such that $T(D) \subset R^n$ is convex and the Lipschitz constants of T and T^{-1} are $\leqslant M$ (note that $M \geqslant 1$).

A domain $D \subset R^n$ *is an* L-*convex domain with constants* δ, p *and* M (we shall usually suppress « with constants δ, p, and M » unless the constants are explicity needed) if D satisfies $(P_{sl,\delta,p})$ where property (P) means « $C^{(0,1)}$-convex with bound M », i.e. there is a δ-loose covering with rank p of ∂D, say, $\{U_k\}$, and $C^{(0,1)}$-homeomorphisms T_k defined on $U_k \cap D$, $k = 1, 2, \ldots$, such that $T_k(U_k \cap D) \subset R^n$ is convex and the Lipschitz constants of T_k and T_k^{-1} are bounded by M.

L-convex domains will be studied extensively throughout the rest of this paper because many of the properties of convex domains with regard to Bessel potentials carry over to L-convex domains as we shall illustrate in the next proposition (cf. Prop. 1), § 4).

1) *If* D *is an* L-*convex domain with constants* δ, p *and* M (*and we assume for simplicity that* $\delta \leqslant 1$) *then*

a) *for* $\alpha \leqslant \gamma$, $\check{P}^{\alpha}(D) \supset \check{P}^{\gamma}(D)$ *and* $|u|_{\alpha,D} \leqslant C|u|_{\gamma,D}$ *where* $C = (\delta^{-1} + p^{1/2}M^{2+3n})\,C'$ *and* C' *depends only on* n.

b) *If* u *is a corrected function in* $\check{P}^{\alpha}(D)$, $0 \leqslant \alpha < \alpha_0$ *then* $|u|_{\alpha,D}$ *is a continuous function of* α *for* $0 \leqslant \alpha < \alpha_0$ *and*

$$\lim_{\alpha \uparrow \alpha_0} |u|_{\alpha,D}$$

always exists (*possibly* $= +\infty$). *The limit is finite if and only if* $u \in \check{P}^{\alpha_0}(D)$ *and in this case the limit is equal to* $|u|_{\alpha_0,D}$.

Proof of part a). — By Prop. 3, § 1 (localized) it is sufficient to prove the inequality and by the definition of $|u|^2_{\alpha,D}$ we may restrict ourselves to $0 \leqslant \alpha < \gamma \leqslant 1$. We may also assume that $u \in \check{P}^{\gamma}(D)$.

If $\{U_k\}$ is the covering of ∂D, we set

$$(5.2) \qquad V_1 = \bigcup_{k=1}^{\infty} U_k^{\delta/4}, \qquad V_2 = D - \overline{\bigcup_{k=1}^{\infty} U_k^{(7/8)\delta}}.$$

One checks easily that $\{V_1, V_2\}$ is a $\delta/4$ loose covering of D.

By Prop. 1) and 8) of § 2, and Prop. 1), § 4, there is a constant C_1 depending only on n such that

$$|u|_{\alpha,D\cap U_k} \leqslant C_1 M^{\alpha+\gamma+3n} |u|_{\gamma,D\cap U_k}, \quad k = 1, 2, \ldots .$$

From the fact that $\mathrm{dist}(V_2, R^n - D) \geqslant \delta/8$, we see by Lemma 1, § 1 and Prop. 7, § 2 that there is a $\varphi \in C^{\infty}(R^n)$ such that $\varphi u \in \check{P}^{\gamma}$ (extended by 0 outside D) and

$$|u|^2_{\alpha,D\cap V_2} \leqslant |\varphi u|^2_{\alpha,R^n} \leqslant |\varphi u|^2_{\gamma,R^n} \leqslant C_2 \delta^{-2\gamma} |u|^2_{\gamma,D}$$

where C_2 depends only on n.

Hence by (2.7) (and the fact that $G_{2n+2\alpha}(x)$ is a decreasing function of $|x|$) we have

$$\begin{aligned} |u|^2_{\alpha,D} &\leqslant |u|^2_{\alpha,D\cap V_2} \\ &\quad + \sum_k |u|^2_{\alpha,D\cap U_k} + \frac{1}{C(n,\alpha)} \int_D \int_{D \atop |x-y|>\delta/4} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}} dx\, dy \\ &\leqslant C_2\delta^{-2\gamma}|u|^2_{\gamma,D} + C_1^2 M^{2(\alpha+\gamma+3n)} \sum |u|^2_{\gamma,D\cap U_k} + 4n\delta^{-2\alpha}|u|^2_{0,D} \\ &\leqslant (C_2\delta^{-2\gamma} + pC_1^2 M^{2(\alpha+\gamma+3n)} + 4n\delta^{-2\alpha})|u|^2_{\gamma,D}, \end{aligned}$$

which completes the proof of this part.

The proof of part *b*) is based on the following lemma :

LEMMA 3. — *Let* $\{U_k\}$ *be a loose covering of finite rank of* D *and let* $u \in P^1_{loc}(D)$ *satisfy*

i) $\lim\limits_{\alpha\uparrow 1} |u|_{\alpha,D\cap U_k} = |u|_{1,D\cap U_k} < \infty$, $k = 1,2,\ldots;$

ii) *if* $\{U_k\}$ *is infinite there is a constant* C *such that for each* k *and* $\alpha \leqslant 1$, $|u|_{\alpha,D\cap U_k} \leqslant C|u|_{1,D\cap U_k}$.

Then $\lim\limits_{\alpha\uparrow 1} |u|_{\alpha,D}$ *exists and is finite if and only if* $u \in \check{P}^1(D)$. *If* $u \in \check{P}^1(D)$, *the limit is equal to* $|u|_{1,D}$.

Proof. — If the covering is finite clearly $u \in \check{P}^1(D)$ and we need only prove that $\lim\limits_{\alpha\uparrow 1} |u|_{\alpha,D} = |u|_{1,D}$ which we now prove by induction. The statement is obvious if the covering consists of one set. Suppose next that the covering consists of two sets, U_1 and U_2. Given $\varepsilon > 0$, choose δ', $0 < \delta' \leqslant \delta$, sufficiently small so that $|u|_{1,(U_1-\overline{U_1^{\delta'}})\cap D} < \varepsilon$ and $D_2 = D - \overline{U_1^{\delta'}} \subset U_2$. Then by Prop. 4, § 4 (with $D_2 \subset D \cap U_2$), $\lim\limits_{\alpha\uparrow 1} |u|_{\alpha,D_2} = |u|_{1,D_2}$. Now

$$\operatorname{dist}(D - \overline{U}_1, D \cap U_1^{\delta'}) \geqslant \delta'.$$

Therefore by (2.7) we see easily that

$$\begin{aligned} \limsup_{\alpha\uparrow 1} |u|^2_{\alpha,D} &\leqslant \limsup_{\alpha\uparrow 1} \Big(|u|^2_{\alpha,D\cap U_1} + |u|^2_{\alpha,D_2} \\ &\quad + \frac{4}{C(n,\alpha)} \int_D |u(x)|^2 \int_{|x-y|>\delta'} |x-y|^{-n-2\alpha} dx\, dy\Big) \\ &= |u|^2_{1,D\cap U_1} + |u|^2_{1,D_2} \leqslant |u|^2_{1,D} + \varepsilon^2. \end{aligned}$$

Since ε is arbitrary the proof in this case follows by Prop. 2), § 4.

Suppose now the proposition is true for coverings with m sets, $m \geqslant 2$, and the covering of D contains $m+1$ sets. Then $D' = D \cap \bigcup_{k=1}^{m} U_k^{\delta/2}$ has a finite loose covering of m sets, viz., $U_1, \ldots, U_m$. By Prop. 4), § 4,

$$\lim_{\alpha \uparrow 1} |u|_{\alpha, D' \cap U_k} = |u|_{1, D' \cap U_k}, \quad k = 1, \ldots, m.$$

Thus by the inductive hypotheses, $\lim_{\alpha \uparrow 1} |u|_{\alpha, D'} = |u|_{1, D'}$. Since $\bigcup_{k=1}^{m} U_k^{\delta/2}$ and U_{m+1} is a loose covering of D, it follows that the proposition is true for all finite coverings.

Now suppose the covering is infinite. By Prop. 2), § 4, if $|u|_{1,D} = \infty$, i.e. $u \notin \check{P}^1(D)$, then $\lim_{\alpha \uparrow 1} |u|_{\alpha, D} = \infty$; hence to prove the lemma we may assume that $|u|_{1,D} < \infty$. By Lemma 1 we can decompose the covering of D, $\{U_k\} = \bigcup_{i=1}^{q} \{U_{k_l^{(i)}}\}$, $q \leqslant p$, where for each i $\{U_{k_l^{(i)}}\}$ is disjoint. Let

$$U_{(i)} = \bigcup_{l} U_{k_l^{(i)}}^{\delta/2};$$

then $\{U_{k_l^{(i)}}\}$ is a $(\delta/2)$-loose disjoint covering of $D_{(i)} = D \cap U_{(i)}$. By Prop. 4), § 4, $\lim_{\alpha \uparrow 1} |u|_{\alpha, D_{(i)} \cap U_{k_l^{(i)}}} = |u|_{1, D_{(i)} \cap U_{k_l^{(i)}}}$ for $i = 1, \ldots, q$ and all l. Then since

$$\operatorname{dist}\left(U_{k_l^{(i)}}^{\delta/2} \cap D_{(i)},\ D_{(i)} - U_{k_l^{(i)}}^{\delta/2}\right) \geqslant \delta/2, \quad l = 1, 2, \ldots,$$

we have that

$$\limsup_{\alpha \uparrow 1} |u|^2_{\alpha, D_{(i)}} \leqslant \limsup_{\alpha \uparrow 1} \left(\sum_l |u|^2_{\alpha, D_{(i)} \cap U_{k_l^{(i)}}} + 4n(1-\alpha)\left(\frac{\delta}{2}\right)^{-2\alpha} |u|^2_{0, D_{(i)}} \right) = \sum_l |u|^2_{1, D_{(i)} \cap U_{k_l^{(i)}}} = |u|^2_{1, D_{(i)}}.$$

since the infinite sum $\sum_l |u|^2_{\alpha, D_{(i)} \cap U_{k_l^{(i)}}}$ is dominated by

$$\sum_l c|u|^2_{1, D \cap U_{k_l^{(i)}}} \leqslant c|u|^2_{1,D} < \infty.$$

Hence by Prop. 2), § 4,

$$\lim_{\alpha \uparrow 1} |u|_{\alpha, D \cap U_{(i)}} = \lim_{\alpha \uparrow 1} |u|_{\alpha, D_{(i)}} = |u|_{1, D_{(i)}} = |u|_{1, D \cap U_{(i)}}.$$

Since $\{U_{(i)}\}$, $i = 1, \ldots, q$ is a finite loose covering of D which satisfies the hypotheses of the lemma the conclusion for infinite coverings follows from the previously considered finite case.

Proof of part b) of Prop. 1). — By the definition of the norm $|u|^2_{\alpha,D}$ it is clearly sufficient to assume that $\alpha_0 = 1$. By Remark 3, § 4, Prop. 1), § 2 and Prop. 8), § 4 it is easy to see that we need only consider $\lim\limits_{\alpha \uparrow 1} |u|_{\alpha,D}$. By Prop. 2), § 4, we may also assume that $u \in \check{P}^1(D)$.

Since U_k is a loose covering of $V_1^{\delta/8} \cap D$ and

$$|U_k \cap D \cap (\partial V_1^{\delta/8} \cap D))| = 0,$$

we have by Props. 1), 4), and 6) of § 4 and Prop. 8) of § 2 that the hypotheses of Lemma 3 are satisfied with respect to $V_1^{\delta/8} \cap D$. Hence

$$\lim_{\alpha \uparrow 1} |u|_{\alpha, D\cap V_1^{\delta/8}} = |u|_{1, D\cap V_1^{\delta/8}} < \infty.$$

Since $\operatorname{dist}(V_2, R^n - D) \geqslant \delta/8$ and $|\partial(V_2^{\delta/8} \cap D)| = 0$, we have, by Lemma 1, § 1, Prop. 7), § 2, the continuity of $|\ \ |_{\alpha,R^n}$ at $\alpha = 1$, and by Prop. 4, § 4 that

$$\lim_{\alpha \uparrow 1} |u|_{\alpha, D\cap V_2^{\delta/8}} = |u|_{1, D\cap V_2^{\delta/8}} < \infty.$$

Since $\{V_1^{\delta/8}, V_2^{\delta/8}\}$ is a loose covering of D the proof of the proposition is completed by another application of Lemma 3.

Remarks about the spaces $\check{P}^{\alpha,p}(D)$. — All the results of this section extend to the spaces $\check{P}^{\alpha,p}(D)$ (since all the statements of the preceding sections on which these results are based extend to $\check{P}^{\alpha,p}(D)$).

6. Density of $P^\alpha(D)$ in $\check{P}^\alpha(D)$ and graph type domains.

For an arbitrary domain D, $P^\alpha(D) \subset \check{P}^\alpha(D)$ and it is not always true that $P^\alpha(D) = \check{P}^\alpha(D)$ (23). In the subsequent sec-

(23) As a simple example consider the unit circle in R^2 less the x_1-axis. Define $u(x) = 0$ for $x_2 < 0$ and $= 1$ for $x_2 > 0$. Then $u \in \check{P}^1(D)$, but since $d_{\beta,D}(u) = \infty$ for $\frac{1}{2} \leqslant \beta < 1$, $u \notin P^1(D)$.

tions we shall prove $P^\alpha(D) = \check{P}^\alpha(D)$ for a rather general class of domains which appears to give a very nearly complete description of domains where equality holds.

Although the question of equality is certainly the most important, a pertinent question is whether, in general, $P^\alpha(D)$ is dense in $\check{P}^\alpha(D)$. This may be answered in the negative. The most general class known for which $\check{P}^\alpha(D) = \overline{P^\alpha(D)}$, the *α-density property*, are the graph domains defined below. But first we show that density is a weakly localized boundary property.

Remark 1. — Since $C_0^\infty(R^n)$ is dense in P^α, it follows by Theorem 1, § 2, that the α-density property is equivalent to the fact that the restrictions to D of functions in $C_0^\infty(R^n)$ are dense in $\check{P}^\alpha(D) \cap C^\infty(D) \cap \mathcal{B}(D)$ in the norm of $\check{P}^\alpha(D)$ ($\mathcal{B}(D)$ is the class of functions in D with bounded support).

1) *If* (P) *is the « α-density property » then* $(P) \equiv (P_{wl})$.

Proof. — Since $(P) \succ (P_{wl})$ we need only show $(P_{wl}) \succ (P)$. It is enough by Prop. 9), § 2, to prove that if D satisfies (P_{wl}) then $P^\alpha(D)$ is dense in $\mathcal{B}(D) \cap \check{P}^\alpha(D)$.

Let $u \in \check{P}^\alpha(D) \cap \mathcal{B}(D)$ and have support in $S_R = [|x| < R]$. Since $\partial D \cap \overline{S_{R+2}}$ is compact and D satisfies (P_{wl}) there is a covering of $\partial D \cap \overline{S_{R+2}}$, say $\{\tilde{U}_k\}, k = 1, \ldots, N-1$, such that $\overline{P^\alpha(D \cap \tilde{U}_k)} = \check{P}^\alpha(D \cap \tilde{U}_k)$. Let

$$U_k = \tilde{U}_k \cap S_{R+2},\ 2\delta_1 = \operatorname{dist}\left(\partial D \cap \overline{S_{R+1}},\ R^n - \bigcup_{k=1}^{N-1} U_k\right)$$

and define $U_N = (R^n - (\partial D \cap S_{R+1}))^{\delta_1}$. Then $\{U_k\}, k = 1, \ldots, N$, is a δ-loose covering of $\overline{D}$, for some δ with $0 < \delta \leqslant \delta_1$.

Let $\{\varphi_k\}$ be the partition of unity given in Lemma 2, § 5, corresponding to the covering $\{U_k\}$. Then $\varphi_k u$ has support in $U_k^{\delta/3} \cap D$, $k = 1, \ldots, N$, and by Prop. 6), § 2, $\varphi_k u \in \check{P}^\alpha(D)$ a fortiori $\varphi_k u \in \check{P}^\alpha(D \cap \tilde{U}_k)$.

Now, given $\varepsilon > 0$, let $u_k \in P^\alpha$, $k = 1, \ldots, N-1$, be such that

$$|u_k - \varphi_k u|_{\alpha, D\cap U_k} \leqslant |u_k - \varphi_k u|_{\alpha, D\cap\tilde{U}_k} < \varepsilon/(N-1)\left(1 + 2n\left(\frac{\delta}{6}\right)^{-2\beta}\right)^{\frac{1}{2}}$$

where $m = [\alpha]$ and $\beta = \alpha - m$. We may assume u_k has support in $U_k^{\delta/6}$ [24]. Therefore it is easy to see by (2.7) that

$$(6.1)\quad |u_k - \varphi_k u|^2_{\alpha,D} \leqslant |u_k - \varphi_k u|^2_{\alpha,D\cap U_k} + \frac{2}{C(n,\beta)} \sum_{l=0}^{m} \binom{m}{l} \sum_{|i|=l} \int_{D\cap U_k} |D_i(u_k(x) - \varphi_k u(x))|^2 \int_{|x-y|>\delta/6} |x-y|^{-n-2\beta}\, dx\, dy$$
$$\leqslant \left(1 + 2n\left(\frac{\delta}{6}\right)^{-2\beta}\right) |u_k - \varphi_k u|^2_{\alpha,D\cap U_k} < \varepsilon^2/(N-1)^2$$

(the second term in each inequality is omitted if $\beta = 0$).

Since $\operatorname{dist}(R^n - D, S_R \cap U_N^{\delta/3}) \geqslant \delta_1$, $\varphi_N u$ has support in D^{δ_1}. Hence $\varphi_N u$, when extended by 0 outside D, is in P^α (cf. Prop. 7), § 2).

Thus $u' = \varphi_N u + \sum_{k=1}^{N-1} u_k$ is in P^α and

$$|u - u'|_{\alpha,D} = \left| \sum_{k=1}^{N} (\varphi_k u) - \varphi_N u - \sum_{k=1}^{N-1} u_k \right|_{\alpha,D} \leqslant \sum_{k=1}^{N-1} |\varphi_k u - u_k|_{\alpha,D} < \varepsilon$$

which completes the proof.

2) *If* D *satisfies the* α*-density property and* T^{-1} *is a* $C^{(k,1)}$*-homeomorphism,* $k \geqslant \alpha^*$*, defined on the open set* U, $U \supset \overline{D}$, *then* $D^* = T^{-1}(D)$ *satisfies the* α*-density property.*

Proof. — By Prop. 9), § 2, we need only prove that $P^\alpha(D^*)$ is dense in $\check{P}^\alpha(D^*) \cap \mathcal{B}(D^*)$.

Let $u^* \in \check{P}^\alpha(D^*) \cap \mathcal{B}(D^*)$ and $\varepsilon > 0$ be given. By Prop. 8), § 2, $u(x) = u^*(T^{-1}x) \in \check{P}^\alpha(D)$ and clearly u has bounded support. By hypothesis there is a $\nu \in P^\alpha$ such that

$$|u - \nu|_{\alpha,D} < \varepsilon / CM^{\alpha+3n/2}$$

where C is the constant given in Prop. 8), § 2, and M is the common Lipschitz constant of T and T^{-1}. Since u has bounded support it is clear that we may suppose that the support S ($S = [x : \nu(x) \neq 0]$) of ν is bounded. Since $\overline{D^* \cap T^{-1}(S)}$ is compact, $\operatorname{dist}(R^n - T^{-1}(U), D^* \cap T^{-1}(S)) = 2\delta > 0$. Let φ be

[24] Let $\varphi \in C^\infty(R^n) = 1$ on $U_k^{\delta/3}$ and $= 0$ outside $U_k^{\delta/6}$ (cf. Lemma 1, § 1); then φu_k has support in $U_k^{\delta/6}$ and we may replace u_k by φu_k by applying Prop. 6), § 2, and noting that $\varphi u_k - \varphi_k u = \varphi(u_k - \varphi_k u)$.

the function given in Lemma 1, § 1, which vanishes outside $(T^{-1}(U))^{\delta/2}$ and $= 1$ on $\overline{D \cap T(S)}$. Then

$$v^*(x^*) = \varphi(x^*)v(Tx^*) = v(Tx^*) \qquad \text{for} \qquad x \in D^*$$

and by Prop. 7), § 2, v^* when extended by 0 outside $(T^{-1}(U))^{\delta/2}$ is in P^α. Furthermore, by Prop. 6), § 2,

$$|u^* - v^*|_{\alpha,D^*} \leqslant c|u - v|_{\alpha,D} < \varepsilon.$$

D is a *simple graph domain*, for brevity an SG-*domain*, if $D = J(\tilde{D})$ where J is an isometry of R^n and $\tilde{D}$ is of the following type. Let $\tilde{B} \subset R^{n-1}$ be a rectangle and $\tilde{f}$ a continuous function on $\overline{\tilde{B}}$ with a positive lower bound, then

$$\tilde{D} = [(x', x_n) : x' \in \tilde{B},\ 0 < x_n < \tilde{f}(x')].$$

$B = J(\tilde{B})$ is the basis of D and $f(x) = \tilde{f}(J^{-1}(x))$ for $x \in \overline{B}$ is the graph function of D.

D is a *graph domain*, for brevity a G-*domain*, if D satisfies the property (D is an SG-domain)$_{wl}$.

3) *If* D *is an* SG-*domain then* D *satisfies the* α-*density property for all* α.

Proof. — By Prop. 9), § 2 and Theorem I, § 2, it is sufficient to show that $P^\alpha(D)$ is dense in $\check{P}^\alpha(D) \cap \mathcal{B}(D) \cap C^\infty(D)$ and clearly we may assume that $D = [(x', x_n) : 0 < x_n < f(x'),\ x' \in B]$.

Let $u \in \check{P}^\alpha(D) \cap \mathcal{B}(D) \cap C^\infty(D)$ and $2\delta = \min_{x' \in \overline{B}} f(x') (> 0)$. For $t \geqslant 0$ let $D_t = [(x', x_n) : 0 < x_n < f(x') + t,\ x' \in B]$, $\{h_\mu\}$, $\mu = 0, 1, \ldots, \alpha^*$, a strictly increasing sequence of positive numbers and $\tilde{u}$ the Lichtenstein extension of order α^* of u given in (3.3). By Prop. 3), § 3, $\tilde{u} \in \check{P}^\alpha(\tilde{D})$ and $|\tilde{u}\rceil_{\alpha,\tilde{D}} \leqslant c|u\rceil_{\alpha,D}$ where c depends only on n and α^*. For $0 \leqslant t < \delta h_{\alpha^*}^{-1}$ define

$$u_t(x) = \tilde{u}(x', x_n - t), \quad x \in D_t.$$

Hence $u_t \in \check{P}^\alpha(D_t)$ and $|u\rceil_{\alpha,D} \leqslant |u_t\rceil_{\alpha,D_t} \leqslant |\tilde{u}\rceil_{\alpha,\tilde{D}} \leqslant c|u\rceil_{\alpha,D}$.

For $t \downarrow 0$ the functions u_t are uniformly bounded in norm in $\check{P}^\alpha(D)$. Therefore by a well known theorem from Hilbert space theory there is a sequence $\{u_{t_k}\}$ converging weakly in $\check{P}^\alpha(D)$ and a subsequence of $\{u_{t_k}\}$ whose arithmetic means converge strongly in $\check{P}^{\alpha}(D)$. These aritmetic means must converge to u since they converge pointwise to $u(u \in C^{\infty}(D))$.

Therefore given $\varepsilon > 0$ there exists $v \in \check{P}^{\alpha}(D_{\bar{t}})$, $\bar{t} > 0$, such that $|u - v\urcorner_{\alpha,D} < \varepsilon$. Let $C = [(x', x_n) : x' \in B, x_n > 0]$, a rectangle, and $V = [x \in D_t : v(x) \neq 0]$. Since u has a bounded support the same is true for v (by the above construction). Hence $\bar{V}$ is compact and $2\delta_1 = \text{dist}(\bar{V} \cap \bar{D}, \bar{C} - \bar{D}_t) > 0$. Let φ be the function in $C^{\infty}(R^n)$ given in Lemma 1, § 1, such that $\varphi = 1$ on $\bar{D} \cap \bar{V}$ and $= 0$ outside a δ_1-neighborhood of $\bar{D} \cap \bar{V}$. It is easy to see that $\varphi v \in P^{\alpha}_{\text{loc}}(C)$ (extended by 0 in $C - D_t$) and by a calculation similar to (6.1) and Prop. 6), § 2, that

$$|\varphi v\urcorner^2_{\alpha,C} \leqslant (1 + 4n\delta_1^{-2\beta})|\varphi v\urcorner^2_{\alpha,D_t} \leqslant c(1 + 4n\delta_1^{-2\beta})|v|^2_{\alpha,D_t}$$

where $\beta = \alpha - [\alpha]$ and c depends only on δ_1, α^*, and n. Hence $\varphi v \in \check{P}^{\alpha}(C)$. By Prop. 2'), § 3, $\varphi v \in P^{\alpha}(C)$ and it is clear that $|\varphi v - u\urcorner_{\alpha,D} = |v - u\urcorner_{\alpha,D} < \varepsilon$ which completes the proof.

4) *If* D *is a* G-*domain then* D *satisfies the* α-*density property for any* α.

Proof. — Propositions 1) and 3).

5) *If* D *is* L-*convex then* D *satisfies the* α-*density property for* $0 \leqslant \alpha \leqslant 1$.

The proof follows from Propositions 1), 2) and 4) and the fact that a convex set is a G-domain.

Remarks on spaces $\check{P}^{\alpha,p}(D)$. — For $1 < p < \infty$ all the theorems of this section can be extended with some slight changes in proofs (where we used the Hilbert space structure of $\check{P}^{\alpha}(D)$). The problem of density of $P^{\alpha}(D)$ in $\check{P}^{\alpha}(D)$ is replaced here by the one of density of $\check{P}^{\alpha,p}(R^n|_D)$ (25) in $\check{P}^{\alpha,p}(D)$. As concerns the spaces $\check{P}^{\alpha,1}(D)$ and $\check{P}^{\alpha,\infty}(D)$ the following remarks can be made.

The spaces $\check{P}^{\alpha,1}(D)$ are defined for all $\alpha \geqslant 0$ except for the integers $\alpha \geqslant 2$. In the exceptional cases we can replace $\check{P}^{\alpha,1}(D)$ by $W_1^{\alpha}(D)$ the imperfect completions rel. $\mathfrak{A}_0$ (or by the « almost » perfect completions as in [2]). The theorems of

(25) $\check{P}^{\alpha,p}(R^n|_D)$ is the class of restrictions of functions in $\check{P}^{\alpha,p}(R^n)$ to D.

the present section can still be extended with more drastic changes in proofs due to the fact that $\check{P}^{\alpha,1}(D)$ are not reflexive.

The spaces $\check{P}^{\alpha,\infty}(D)$ are proper functional spaces for

$$\alpha > 0 \ (\check{P}^{0,\infty}(D) = L^{\infty}(D)).$$

Their norms are obtained by the usual limit procedure from the norms in $\check{P}^{\alpha,p}(D)$ for $p \uparrow \infty$. We have (putting as usual $m = [\alpha]$, $\beta = \alpha - m$)

$$(6.2) \quad |u|_{\alpha,\infty,D} = \max_{|i| \leqslant m} \left[\sup_{x \in D} |D_i u(x)|, \sup_{x,y \in D} \frac{|D_i u(x) - D_i u(y)|}{|x - y|^{\beta}} \right].$$

(When α is an integer we omit the second sup in the square bracket).

For $\check{P}^{\alpha,\infty}(D)$ all the propositions of the present section are in general false. Some of them become true under suitable restrictions on D. For instance Propositions 1) and 2) are true for D bounded. Proposition 2) is true even for D unbounded as long as $D \subset U^{\delta}$ for some $\delta > 0$.

7. Localization of extension theorems.

The statement $P^{\alpha}(D) = \check{P}^{\alpha}(D)$ is equivalent to the assertion that for every $u \in \check{P}^{\alpha}(D)$ there exists a $\tilde{u} \in P^{\alpha}$ such that $u(x) = \tilde{u}(x)$ in D. Since P^{α}, as well as $\check{P}^{\alpha}(D)$ are Hilbert spaces the preceding statement is equivalent to the fact that there exists a linear bounded extension mapping of $\check{P}^{\alpha}(D)$ into P^{α}. In Prop. 2), § 3, we have given such a mapping when D is a rectangle.

This linear mapping is clearly not unique but there does exist a distinguished mapping, viz. the mapping with minimum bound. This mapping is not in general easy to construct explicitly.

We now introduce the following notation. Let $I \subset [0, \infty)$ be an interval and $\Gamma \equiv \Gamma(I') \geqslant 1$ be a finite valued increasing function defined on the compact subintervals I' of I. Then we write $D \in \mathcal{E}(I, \Gamma)$ if there is a linear mapping E with domain $\mathfrak{D}_E \subset \mathfrak{M}(D)$ and range $\mathfrak{M}(R^n)$ such that

(7.1a) $\quad$ Eu *is an extension of* u *for* $u \in \mathfrak{D}_E$

and

$$(7.1b) \qquad \textit{if } \alpha \in \mathrm{I} \textit{ then } \check{\mathrm{P}}^\alpha(\mathrm{D}) \subset \mathfrak{D}_\mathrm{E}, \quad \mathrm{E}(\check{\mathrm{P}}^\alpha(\mathrm{D})) \subset \mathrm{P}^\alpha$$

and for any compact subinterval I′ of I and $\alpha \in \mathrm{I}'$,

$$(7.1c) \qquad |\mathrm{E}|_{\mathrm{P}^\alpha, \check{\mathrm{P}}^\alpha(\mathrm{D})} \leqslant \Gamma(\mathrm{I}').$$

We shall write $\mathrm{D} \in \mathcal{E}(\mathrm{I})$ [26] if $\mathrm{D} \in \mathcal{E}(\mathrm{I}, \Gamma)$ for some Γ.

We note that if $\mathrm{D} \in \mathcal{E}(\mathrm{I})$ then $\check{\mathrm{P}}^\alpha(\mathrm{D}) = \mathrm{P}^\alpha(\mathrm{D})$ for $\alpha \in \mathrm{I}$. In Prop. 2), § 3 we have proved that if D is a rectangle then $\mathrm{D} \in \mathcal{E}([0, q])$ for any positive integer q.

THEOREM I. — *Let* T *be a homeomorphism of class* $\mathrm{C}^{(q,1)}$ *with Lipschitz constant* M *of the open set* U *onto* U* *and let* $\mathrm{D} \subset \mathrm{U}^\delta$. *If* $\mathrm{D} \in \mathcal{E}(\mathrm{I}, \Gamma)$, $\mathrm{I} \subset [0, q+1]$ *then* $\mathrm{D}^* = \mathrm{T}(\mathrm{D}) \in \mathcal{E}(\mathrm{I}, \Gamma^*)$ *where* $\Gamma^*(\mathrm{I}') = c\mathrm{M}^{2q+3(n+1)}\delta^{-1}\,\Gamma(\mathrm{I}')$ *and* c *depends only on* n *and* q.

Proof. — Let E be the extension mapping which satisfies (7.1). If $u^* \in \check{\mathrm{P}}^\alpha(\mathrm{D}^*)$, $\alpha \in \mathrm{I}$, then $u(x) = u^*(\mathrm{T}x) \in \check{\mathrm{P}}^\alpha(\mathrm{D})$ by Prop. 8), § 2 and $\tilde{u}(x) = \mathrm{E}u(x) \in \mathrm{P}^\alpha$. Let $\varphi \in \mathrm{C}^\infty(\mathrm{R}^n)$ be the function which $= 1$ on D and $= 0$ outside $\mathrm{U}^{\delta/2}$ given in Lemma 1, § 1. Then by Prop. 7), § 2, $\varphi(x)u(x) \in \mathrm{P}^\alpha(= \mathrm{P}^\alpha(\mathrm{R}^n))$ and by Prop. 8), § 2, $\tilde{u}^*(x^*) = \varphi(\mathrm{T}^{-1}x^*)\tilde{u}(\mathrm{T}^{-1}x^*) \in \check{\mathrm{P}}^\alpha(\mathrm{U}^*)$. Since $\tilde{u}^*(x^*) = u^*(x^*)$ for $x^* \in \mathrm{D}^*$ and $\tilde{u}^*$ vanishes outside $(\mathrm{U}^*)^{\delta/2\mathrm{M}}$, it is easily seen (cf. Prop. 7, § 2) that $\tilde{u}^* \in \mathrm{P}^\alpha$ (extended by 0 outside U*) and from the quoted propositions that

$$\mathrm{E}^*u^* = \tilde{u}^*$$

satisfies (7.1) with Γ replaced by Γ^*.

THEOREM II. — *Let property* (P) *be* « $\mathrm{D} \in \mathcal{E}(\mathrm{I}, \Gamma)$ ». *If* D *satisfies* $(\mathrm{P}_{sl,\delta,p})$ (*with* $\delta \leqslant 1$) *then* $\mathrm{D} \in \mathcal{E}(\mathrm{I}, \Gamma^*)$ *where*

$$\Gamma^*(\mathrm{I}') = \Gamma^*([\alpha_1, \alpha_2]) = c_0\delta^{-\alpha_2}\Gamma(\mathrm{I}')$$

and c_0 *depends only on* p, n, *and* α_2^*.

Proof. — Let $\{\mathrm{U}_k\}$ be the δ-loose covering with rank p of $\partial\mathrm{D}$ such that $\mathrm{D} \cap \mathrm{U}_k \in \mathcal{E}(\mathrm{I}, \Gamma)$ and E_k the corresponding extension mapping which satisfies (7.1).

[26] $\mathcal{E}(\alpha)$ is the extension class corresponding to the degenerate interval formed by a single point α.

Let $\{\psi_k\}$ be the partition of unity corresponding to $\{U_k\}$ and ∂D given in Lemma 2, § 5. Then $\left(1 - \sum_{k=1}^{\infty} \psi_k(x)\right) \in C^\infty$, $= 0$ in $S(\partial D, \delta/8)$ and $= 1$ outside of $S(\partial D, \delta/4)$ and

$$\left|D_i\left(1 - \sum_{k=1}^{\infty} \psi_k(x)\right)\right| \leqslant C_{|i|}\delta^{-|i|}$$

where $C_{|i|}$ depends only on p, $|i|$ and n. We construct now an extension mapping E for D. We define

$$\mathfrak{D}_E = \{u:\ u \in \mathfrak{M}(D) \quad \text{and} \quad u|_{U_k \cap D} \in \mathfrak{D}_{E_k}\}\ (^{27});$$

it is obviously a linear class of functions. We put then for $u \in \mathfrak{D}_E$

$$\tilde{u} = Eu = \sum_{k=1}^{\infty} \psi_k E_k u_k(x) + E_0 u(x) \tag{7.2}$$

where $E_0 u(x) = \left(1 - \sum_{k=1}^{\infty} \psi_k\right) u(x)$ for $x \in D$ and $= 0$ for $x \notin D$. Obviously condition (7.1*a*) is satisfied. Suppose now, $u \in \check{P}^{\alpha}(D)$, $\alpha \in I' \subset I$, $I' = [\alpha_1, \alpha_2]$. Then by Prop. 6), § 2,

$$|\psi_k E_k u_k|_{\alpha, R^n} \leqslant c\delta^{-[\alpha_2]}\Gamma(I')|u|_{\alpha, D \cap U_k}$$

and by Prop. 7), § 2, $|E_0 u|_{\alpha, R^n} \leqslant c\delta^{-[\alpha_2]}|u|_{\alpha, D}$ where c depends only on p, n, and α_2^*, and clearly can be chosen to depend increasingly on $[\alpha_2]$. Therefore

$$\begin{aligned}|\tilde{u}|^2_{\alpha, R^n} &\leqslant 2\left|\sum_{k=1}^{\infty} \psi_k E_k u_k\right|^2_{\alpha, R^n} + 2|E_0 u|^2_{\alpha, R^n} \\ &\leqslant 4p \sum_{k=1} |\psi_k E_k u_k|^2_{\alpha, R^n} + 2|E_0 u|^2_{\alpha, R^n} \\ &\leqslant 4c^2 p\delta^{-2[\alpha_2]}\Gamma(I')^2 \sum_{k=1}^{\infty} |u|^2_{\alpha, D \cap U_k} + 2c^2\delta^{-2[\alpha_2]}|u|^2_{\alpha, D} \\ &\leqslant c^2\delta^{-2[\alpha_2]}(4p^2\Gamma(I')^2 + 2)|u|^2_{\alpha, D},\end{aligned}$$

which completes the proof (we put $c_0 = 3pc$).

Remarks about the spaces $\check{P}^{\alpha,p}(D)$. — For $p \neq 2$ the extension mappings should transform $\check{P}^{\alpha,p}(D)$ into $\check{P}^{\alpha,p}(R^n)$. Since now we deal with Banach spaces which are not Hilbert spaces, the fact that each $u \in \check{P}^{\alpha,p}(D)$ can be extended to $\tilde{u} \in \check{P}^{\alpha,p}(R^n)$

(27) The symbol $u|_F$ means the restriction of u to F.

does not automatically imply that there exists a bounded linear extension mapping.

We introduce the classes $\mathcal{E}^{(p)}(\mathrm{I}, \Gamma)$ in a similar way as $\mathcal{E}(\mathrm{I}, \Gamma)$ by replacing $\check{\mathrm{P}}^{\alpha}(\mathrm{D})$ and P^{α} by $\check{\mathrm{P}}^{\alpha,p}(\mathrm{D})$ and $\check{\mathrm{P}}^{\alpha,p}(\mathrm{R}^n)$ in the definition. The theorems of this section are then immediately extended to $p \neq 2$ since all the results on which they are based have corresponding extensions. The preceding statement holds even for $p = 1$ and $p = \infty$.

8. Regularized distance, simple Lipschitzian graph domains and singular multipliers.

In this section we shall develop some notions for later use. We start by a well-known theorem of H. Whitney. Our proof is much shorter than other published proofs. Let $\mathrm{F} \subset \mathrm{R}^n$ be an arbitrary non-empty set and

$$r_{\mathrm{F}}(x) = \operatorname{dist}(x, \mathrm{F}).$$

THEOREM I. — *For arbitrary* ε, $0 < \varepsilon < 1$, *there exists a function* $\rho(x) \equiv \rho_{\mathrm{F},\varepsilon}(x)$ *defined on* R^n *such that:*

i) $(1 - \varepsilon)r_{\mathrm{F}}(x) \leqslant \rho(x) \leqslant r_{\mathrm{F}}(x)$, $x \in \mathrm{R}^n$,

ii) $\rho \in \mathrm{C}^{\infty}(\mathrm{R}^n - \overline{\mathrm{F}})$ *and*

$$|\mathrm{D}_i\rho(x)| \leqslant r_{\mathrm{F}}(x)^{1-|i|}\varepsilon^{-|i|}\mathrm{B}_{|i|}, \quad x \in \mathrm{R}^n - \overline{\mathrm{F}},$$

where $\mathrm{B}_{|i|}$ *depends only on* $|i|$ *and* n.

Proof. — Let $e(x) \in \mathrm{C}^{\infty}(\mathrm{R}^n)$ be a fixed decreasing function of $|x|$ which vanishes for $|x| \geqslant 1$ and such that

$$\int e(x)\,dx = 1.$$

Then we define

$$\rho(x) = \begin{cases} c^{-n}\dfrac{(1-c)^n}{(1+c)^{n-1}}\displaystyle\int r_{\mathrm{F}}(y)^{1-n}\, e\left(\frac{x-y}{c r_{\mathrm{F}}(y)}\right) dy, & x \in \mathrm{R}^n - \overline{\mathrm{F}}, \\ 0, & x \in \overline{\mathrm{F}}, \end{cases} \tag{8.1}$$

where $c = \varepsilon/(4n - 2)$.

For $x \in \overline{F}$, part i) is trivial. With $r(y) \equiv r_F(y)$ we have for x and y satisfying $|x - y| < cr(y)$:

$$(8.1a) \quad r(x) \leqslant r(y) + |x - y| < (1 + c)r(y),$$
$$r(x) \geqslant r(y) - |x - y| > (1 - c)\, r(y),$$

$$(8.1b) \quad \text{for } x \in S\left(x_0, \frac{1}{2} r(x_0)\right) (x_0 \notin \overline{F}),$$
$$r(y) > (1 + c)^{-1} r(x) > \frac{1}{2(1 + c)} r(x_0) > 0.$$

From the monotonicity of $e(x)$ in $|x|$ and (7.1a) we obtain for $x \in R^n - \overline{F}$

$$\rho(x) \leqslant r(x) \left(\frac{cr(x)}{1 - c}\right)^{-n} \int e\left(\frac{x - y}{\dfrac{c}{1 - c} r(x)}\right) dy = r(x),$$

and

$$\rho(x) \geqslant r(x) \left(\frac{cr(x)}{1 + c}\right)^{-n} \left(\frac{1 - c}{1 + c}\right)^{2n-1} \int e\left(\frac{x - y}{\dfrac{c}{1 + c} r(x)}\right) dy$$
$$= r(x) \left(\frac{1 - c}{1 + c}\right)^{2n-1}$$

and i) is clear from the choice of c.

That $\rho \in C^\infty(R^n - \overline{F})$ is clear from (8.1b) and from (8.1a),

$$|D_i \rho(x)| \leqslant c^{-n-|i|} \frac{(1 - c)^n}{(1 + c)^{n-1}} \int r(y)^{1-|i|-n} \left|(D_i e)\left(\frac{x - y}{cr(y)}\right)\right| dy$$
$$\leqslant r(x)^{1-|i|} \left(\frac{c}{1 + c}\right)^{-|i|} \frac{\omega_n}{n} \sup_{z \in R^n} |D_i e(z)|$$

and ii) follows.

A domain D is a *simple Lipschitzian graph domain with constant* M, for brevity, an SLG-*domain*, if D is an SG-domain and the graph function f is Lipschitzian with Lipschitz constant M.

THEOREM II. — *Let* $D = [(x', x_n) : 0 < x_n < f(x'),\ x' \in R^{n-1}]$ *be an* SLG-*domain with constant* M *and basis* $B = R^{n-1}$. *Put* $D_+ = [(x', x_n) : f(x') < x_n,\ x' \in R^{n-1}]$. *Then for arbitrary* ε, $0 < \varepsilon < 1$, *there is a function* $\rho(x) \equiv \rho_{D,\varepsilon}(x)$ *defined on* D_+ *which satisfies:*

i) $(1 - \varepsilon)(x_n - f(x')) \leqslant \rho(x) \leqslant (x_n - f(x'))$,

ii) $\rho \in C^\infty(D_+)$ *and* $|D_i\rho(x)| \leqslant (x_n - f(x'))^{1-|i|}M_0^{|i|}\varepsilon^{-|i|}B_{|i|}$ *where* $M_0 = \max(M, 1)$ *and* $B_{|i|}$ *depends only on* $|i|$ *and* n,

iii) $\dfrac{\partial\rho}{\partial x_n}(x) > (1 - \varepsilon)$.

Remark 1. — Since for $x \in D_+$,

$$r_D(x) \leqslant x_n - f(x') \leqslant (1 + M^2)^{1/2} r_D(x)$$

we can replace $x_n - f(x')$ in the lower bound of i) by $r_D(x)$ and in the upper bound by $2M_0 r_D(x)$. In ii) we may replace $(x_n - f(x'))^{1-|i|}$ by $\rho(x)^{1-|i|}$ or $r_D(x)^{1-|i|}$.

Proof of Theorem II. — Without loss of generality we will assume that $M \geqslant 1$, hence $M_0 = M$.

Let $e \in C_0^\infty[0, \infty)$ be a decreasing function which is constant in a neighborhood of 0, vanishes for $r \geqslant 1$ and such that

$$\int_{R^{n-1}} e(|z'|)\, dz' = \omega_{n-1} \int_0^\infty e(r) r^{n-2}\, dr = 1.$$

For $z' \in R^{n-1}$ we let $\hat{e}(z') = e(|z'|) \in C_0^\infty(R^{n-1})$. Then we define

$$\text{(8.2}a\text{)} \qquad \rho(x) = k \int_{R^{n-1}} (x_n - f(y'))^{2-n}\, \hat{e}\left(\frac{y' - x'}{c(x_n - f(y'))}\right) dy'$$

where

$$\text{(8.2}b\text{)} \qquad c = \max\left[t : \left(\frac{1 - tM}{1 + tM}\right)^{2n-3} \geqslant 1 - \varepsilon \text{ and } (1 - (2n - 3)tM)\frac{(1 - tM)^{n-2}}{(1 + tM)^n} \geqslant 1 - \varepsilon\right]$$

and

$$\text{(8.2}c\text{)} \qquad k = (1 + cM)\left(\frac{c(1 + cM)}{1 - cM}\right)^{1-n}.$$

We note that with this choice of c, $cM < 1$ and $c < \dfrac{1}{M} \leqslant 1$. Furthermore, it follows easily from (8.2b) that there exists a positive constant a (depending only on n) such that

$$\text{(8.2}d\text{)} \qquad cM > a\varepsilon.$$

For $x \in D_+$ (i.e. $x_n > f(x')$) and y satisfying

$$|x' - y'| \leqslant c|x_n - f(y')|$$

we have

$$x_n - f(y') \geqslant x_n - f(x') - |f(x') - f(y')|$$
$$\geqslant x_n - f(x') - \mathrm{M}|x' - y'| \geqslant x_n - f(x') - c\mathrm{M}|x_n - f(y')|.$$

Since $c\mathrm{M} < 1$ we have $x_n > f(y')$. Similarly,

$$x_n - f(y') \leqslant x_n - f(x') + c\mathrm{M}|x_n - f(y')|.$$

From these inequalities we conclude

(8.3) $x_n > f(x')$ *and* $|x' - y'| \leqslant c|x_n - f(y')|$ *imply*

$$0 < \frac{1}{1 + c\mathrm{M}}(x_n - f(x')) \leqslant x_n - f(y') \leqslant \frac{1}{1 - c\mathrm{M}}(x_n - f(x')).$$

From (8.3) we get $\rho \in \mathrm{C}^\infty(\mathrm{D}_+)$ and in analogy with the proof of Theorem I,

$$\left(\frac{1 - c\mathrm{M}}{1 + c\mathrm{M}}\right)^{2n-3}(x_n - f(x')) \leqslant \rho(x) \leqslant x_n - f(x')$$

which by the choice of c proves i).

It can be checked that

$$\mathrm{D}_i\rho(x) = kc^{-|i|}\int (x_n - f(y'))^{2-n-|i|}\, h\left(c, \frac{x' - y'}{c(x_n - f(y'))}\right) dy'$$

where $h(c, z')$ is a polynomial in c with coefficients which are C_0^∞ functions in z' vanishing for $|z'| \geqslant 1$. Furthermore, $h(c, z')$ is completely determined by $\hat{e}(z')$, n, and the indicial system i.

Since $c \leqslant 1$ we have $|h(c, z')| \leqslant \mathrm{A}_{|i|,n}$, where $\mathrm{A}_{|i|,n}$ depends only on $|i|$ and n.

To complete the proof of part ii) we use the inequalities (8.3) and the formulas (8.2c) and (8.2d). The constant $\mathrm{B}_{|i|}$ is then given by $\dfrac{\omega_{n-1}}{n - 1}\left(\dfrac{2}{a}\right)^{|i|}\mathrm{A}_{|i|,n}$.

For a fixed $x \in \mathrm{D}_+$ consider the transformation

$$z' = \frac{y' - x'}{c(x_n - f(y'))} \tag{8.4}$$

of points y' in the domain $[y' : |y' - x'| < c(x_n - f(y'))]$ into points z'. It is obviously a Lipschitzian transformation. What is more, the inverse exists and is also Lipschitzian.

In fact (8.3) gives

$$\left|\frac{\overline{y}' - x'}{c(x_n - f(\overline{y}'))} - \frac{y' - x'}{c(x_n - f(y'))}\right|$$
$$= \frac{|(\overline{y}' - y')(x_n - f(y')) + (y' - x')(f(\overline{y}') - f(y'))|}{c(x_n - f(\overline{y}'))(x_n - f(y'))}$$
$$\geqslant \frac{(1 - cM)^2}{c(x_n - f(x'))^2}\,[|\overline{y}' - y'|(x_n - f(y')) - |y' - x'|M|\overline{y}' - y'|]$$
$$\geqslant \frac{(1 - cM)^3|\overline{y}' - y'|}{c(1 + cM)(x_n - f(x'))}.$$

It follows that the transformation (8.4) can be applied in the usual way to change variables in an integral. Its gradient and Jacobian may only exist almost everywhere. The Jacobian, wherever it exists, can be evaluated as follows:

$$\frac{\partial z_k}{\partial y_j} = \left(\frac{\delta_{jk}}{c(x_n - f(y'))} + \frac{y_k - x_k}{c(x_n - f(y'))^2}\cdot\frac{\partial f}{\partial y_j}(y')\right)$$
$$= (x_n - f(y'))^{-1}\left(\frac{\delta_{jk}}{c} + z_k\frac{\partial f}{\partial y_j}\right).$$

It is easy to verify that

$$\det\left(\frac{\partial z_k}{\partial y_j}\right) = [c(x_n - f(y'))]^{1-n}\left[1 + c\sum_{l=1}^{n-1} z_l\frac{\partial f}{\partial y_l}\right].$$

Furthermore since $|\nabla f(y')| \leqslant M$ and $cM < 1$ we have for $|z'| \leqslant 1$

$$(8.5)\quad 0 < \frac{c^{n-1}}{1 + cM} \leqslant \frac{(x_n - f(y'))^{1-n}}{\det\left(\frac{\partial z_k}{\partial y_j}\right)} \leqslant \frac{c^{n-1}}{1 - cM} < \infty.$$

We pass now to the proof of iii). From (8.2) we have (with $e'(r) = \frac{de}{dr}(r)$),

$$\frac{\partial \rho}{\partial x_n} = k\int (x_n - f(y'))^{1-n}$$
$$\left[(2 - n)e\left(\frac{|x' - y'|}{c(x_n - f(y'))}\right) - \frac{|x' - y'|}{c(x_n - f(y'))}\,e'\left(\frac{|x' - y'|}{c(x_n - f(y'))}\right)\right]dy'$$
$$= k\int (x_n - f(y'))^{1-n}\,[(2 - n)e(|z'|) - |z'|e'(|z'|)]\left|\det\left(\frac{\partial z_k}{\partial y_j}\right)\right|^{-1} dz'.$$

From the properties of e we note that $e'(|z'|) \leqslant 0$ for all z' and that $e'(|z'|)$ and $e(|z'|) = 0$ for $|z'| \geqslant 1$. Thus from (8.5) we have

$$\frac{\partial \rho}{\partial x_n} \geqslant k\omega_{n-1}$$
$$\left[-\frac{c^{n-1}}{1-c\mathrm{M}}(n-2)\int_0^1 e(r)r^{n-2}\,dr - \frac{c^{n-1}}{1+c\mathrm{M}}\int_0^1 e'(r)r^{n-1}\,dr\right]$$
$$= kc^{n-1}\left[-\frac{n-2}{1-c\mathrm{M}} + \frac{n-1}{1+c\mathrm{M}}\right]$$
$$= kc^{n-1}(1-(2n-3)c\mathrm{M})/(1-c\mathrm{M})(1+c\mathrm{M}) \geqslant 1 - \varepsilon$$

by (8.2b) and (8.2c). This completes the proof of Theorem II.

Let F and F_1 be non-empty closed sets in R^n, $\mathrm{F} \neq \mathrm{F}_1$, and $\mathrm{H} = \mathrm{F} \cap \mathrm{F}_1$. For $\varepsilon > 0$ we define

$$\mathrm{U}_\varepsilon \equiv \mathrm{U}_\varepsilon(\mathrm{F}, \mathrm{F}_1) = [x : r_{\mathrm{F}}(x) < \varepsilon r_{\mathrm{F}_1}(x)],$$

an open set. Clearly, $\mathrm{F} - \mathrm{H} \subset \mathrm{U}_\varepsilon \subset \mathrm{R}^n - \mathrm{F}_1$.

For $0 < \varepsilon < 1$ we define the *singular multiplier for the triple* $\{\mathrm{F}, \mathrm{F}_1, \varepsilon\}$ by

$$(8.6) \quad \varphi(x) = \begin{cases} (\theta\rho(x))^{-n} \displaystyle\int_{\mathrm{R}^n-\mathrm{F}_1} \chi(y) e\left(\frac{x-y}{\theta\rho(x)}\right) dy, & x \in \mathrm{R}^n - \mathrm{F}_1 \\ 0, & x \in \mathrm{F}_1 \end{cases}$$

where $\rho(x) \equiv \rho_{\mathrm{F}_1,\varepsilon}(x)$ is the regularized distance given in Theorem I, $\chi(y)$ is the characteristic function of $\mathrm{U}_{\varepsilon/2}(\mathrm{F}, \mathrm{F}_1)$, $\theta = \varepsilon/(2+\varepsilon)$ and $e(z)$ is a fixed regularizing function with support in $|z| < 1$.

Theorem III. — *The singular multiplier* $\varphi(x)$ *for the triple* $\{\mathrm{F}, \mathrm{F}_1, \varepsilon\}$ *satisfies:*

i) $0 \leqslant \varphi(x) \leqslant 1$, $\varphi(x) = 1$ *on* $\mathrm{F} - \mathrm{H}$ *and* $= 0$ *on* $\mathrm{R}^n - \mathrm{U}_\varepsilon$,

ii) $\varphi \in \mathrm{C}^\infty(\mathrm{R}^n - [(\overline{\mathrm{F} - \mathrm{F}_1}) \cap \mathrm{F}_1])$ *and*

$$|\mathrm{D}_i\varphi(x)| \leqslant \mathrm{B}^1_{|i|}(\mathrm{Max}[\varepsilon r_{\mathrm{F}_1}(x), r_{\mathrm{F}-\mathrm{F}_1}(x)])^{-|i|}$$

where $\mathrm{B}^1_{|i|}$ *depends only on* $|i|$ *and* n.

Proof. — The first part of i) follows from the properties of a regularizing function. If $x \in \mathrm{F} - \mathrm{H}$ and $|x - y| < \theta\rho(x)$, then by Theorem I, $r_{\mathrm{F}}(y) \leqslant |x-y| < \theta r_{\mathrm{F}_1}(x)$ and

$$r_{\mathrm{F}_1}(y) \geqslant r_{\mathrm{F}_1}(x) - |x-y| > (1-\theta) r_{\mathrm{F}_1}(x)$$

which implies $y \in U_{\varepsilon/2}$ and by (8.7), that $\varphi(x) = 1$. If $x \in R^n - U_\varepsilon$ and $|x - y| < \theta\rho(x)$ then

$$r_{F_1}(y) \leqslant r_{F_1}(x) + |x - y| \leqslant (1 + \theta) r_{F_1}(x)$$

and $r_F(y) \geqslant r_F(x) - |x - y| \geqslant (\varepsilon - \theta) r_{F_1}(x)$ which implies $y \notin U_{\varepsilon/2}$ and by (8.6) that $\varphi(x) = 0$, which completes the proof of part i).

To prove the first part of ii) it is clear by part i) that we may restrict our considerations to $\overline{U}_\varepsilon$. If $x \in \overline{U}_\varepsilon - F_1$, then we know that $\varphi(x)$ has infinitely many derivatives. If $x \in \overline{U}_\varepsilon \cap F_1$ then $0 = r_F(x) = r_{F_1}(x)$, but since $U_\varepsilon \cap F_1 = 0$, this implies that $x \in \overline{(F - F_1)} \cap F_1$ (28). Hence

$$\varphi \in C^\infty(R^n - [\overline{(F - F_1)} \cap F_1]).$$

To prove the inequality in ii) it is enough to consider $x \in \overline{U}_\varepsilon - [\overline{(F - F_1)} \cap F_2] = \overline{U}_\varepsilon - F_1$.

By (2.6) and Theorem I we have for p a positive integer (the indices in the summation below satisfy: $1 \leq l \leq |j|$, $|s^{(m)}| \geq 1$, $\bigcup_1^l s^{(m)} = j$)

$$\text{(8.7)} \quad |D_j(\theta\rho(x))^{-p}| = \left| \sum \theta^{-p} \frac{(-1)^l}{l!} \frac{(p + l - 1)!}{(p-1)!} \rho(x)^{-p-l} \prod_{m=1}^{l} D_{s^{(m)}}\rho(x) \right| \leqslant M^{(1)}_{|j|} \varepsilon^{-p-|j|} r_{F_1}(x)^{-p-|j|},$$

where $M^{(1)}_{|j|}$ depends only on p, $|j|$, and n.

For $|x_l - y_l| < \theta\rho(x)$ we have by (2.5) and (8.7),

$$\left| D_s^{(x)} \left(\frac{x_l - y_l}{\theta\rho(x)} \right) \right| = \left| (x_l - y_l) D_s(\theta^{-1}\rho(x)^{-1}) + \sum_{j \cup (l) = s} D_j(\theta^{-1}\rho(x)^{-1}) \right| \text{(29)}$$
$$\leqslant \theta\rho(x) M^{(1)}_{|s|} \varepsilon^{-1-|s|} r_{F_1}(x)^{-1-|s|} + \sum_{j \cup (l) = s} M^{(1)}_{|j|} \varepsilon^{-1-|j|} r_{F_1}(x)^{-1-|j|} \leqslant M^{(2)}_{|s|} \varepsilon^{-|s|} r_{F_1}(x)^{-|s|}$$

where $M^{(2)}_{|s|}$ depends only on n and $|s|$.

(28) In fact $x = \lim x^{(k)}$, $r_F(x^{(k)}) = |x^{(k)} - y^{(k)}| < \varepsilon r_{F_1}(x^{(k)})$, $y^{(k)} \in F - F_1$, $y^{(k)} \to x$.

(29) $D_s^{(x)} f(x, y)$ means differentiate $f(x, y)$ with respect to the x variables.

Since $e(z)$ vanishes for $|z| \geqslant 1$ we have by the above and (2.6)

$$(8.8)\quad \left| D_k^{(x)} e\left(\frac{x-y}{\theta\rho(x)}\right)\right|$$
$$= \left| \sum_{\substack{\bigcup_{m=1}^{|t|} s^{(m)} = k \\ |s^{(m)}| \geqslant 1}} \frac{1}{|t|\,!} (D_t e)\left(\frac{x-y}{\theta\rho(x)}\right) \prod_{m=1}^{|t|} D_{s^{(m)}}^{(x)} \left(\frac{x_{t_m} - y_{t_m}}{\theta\rho(x)}\right)\right|$$
$$\leqslant \Sigma \frac{1}{|t|\,!} \left| (D_t e)\left(\frac{x-y}{\theta\rho(x)}\right) \sum_{m=1}^{|t|} M_{|s^{(m)}|}^{(2)} \varepsilon^{-|s^{(m)}|} r_{F_1}(x)^{-|s^{(m)}|}\right|$$
$$\leqslant M_{|k|}^{(3)} \varepsilon^{-|k|} r_{F_1}(x)^{-|k|}$$

where $M_{|k|}^{(3)}$ depends only on k and n. Hence by (8.7), (8.8), and (2.5),

$$D_i^{(x)} \left[(\theta\rho(x))^{-n} e\left(\frac{x-y}{\theta\rho(x)}\right)\right]\Bigg|$$
$$= \left| \sum_{j \cup k = i} D_j(\theta^{-n}\rho(x)^{-n}) D_k^{(x)} \left(e\left(\frac{x-y}{\theta\rho(x)}\right)\right)\right| \leqslant M_{|i|}^{(4)} (\varepsilon r_{F_1}(x))^{-n-|i|}$$

where $M_{|i|}^{(4)}$ depends only on $|i|$ and n. Therefore we have by (8.6), $|D_i\varphi(x)| \leqslant B_{|i|}^1 (\varepsilon r_{F_1}(x))^{-|i|}$. To replace $\varepsilon r_{F_1}(x)$ here by $\max\,[\varepsilon r_{F_1}(x), r_{F-F_1}(x)]$ it is enough to notice that for $x \in \overline{U}_\varepsilon - F_1$ there is a $y \in F$ such that $|x - y| = r_F(x) \leqslant \varepsilon r_{F_1}(x) < r_{F_1}(x)$, i.e. $y \notin F_1$. This implies $r_{\overline{F-F_1}}(x) = r_F(x) \leqslant \varepsilon r_{F_1}(x)$ which completes the proof of part ii).

Remark 2. — The singular multiplier will be used for sets F and F_1 which are the closures of open sets, i.e. $F = \overline{D}$, $F_1 = \overline{D}_1$, D and D_1 open sets. In these cases we shall write that « φ is the singular multiplier corresponding to the triple $\{D, D_1, \varepsilon\}$ » instead of « $\{F, F_1, \varepsilon\}$ ».

9. Vanishing of potentials.

In this section we prove three theorems in connection with the vanishing of potentials. These theorems will be applied in § 11.

We begin with a few definitions. Let $x(t)$, $0 \leqslant t \leqslant 1$ be a simple arc in R^n. For any $\varkappa$, $0 < \varkappa < 1$, we call the open set $\bigcup_t S(x(t), \varkappa|x(t) - x(0)|)$ a *conoid with vertex* $x(0)$, *opening* $\varkappa$, *axial arc* $x(t)$ and *radius* $|x(1) - x(0)|$. We shall consider the following property of an open set D:

(C) *There exists* r *and* $\varkappa$, $r > 0$, $0 < \varkappa < 1$ *such that for every* $x \in \partial D$ *there is a conoid lying in* D *with vertex* x, *opening* $\varkappa$ *and radius* r.

For brevity we will call a domain with the above property a (C)-domain with constants $\varkappa$ and r.

Remark 1. — Two immediate properties of condition (C) are:

a) If D is a (C)-domain then $|\partial D| = 0$ (similar proof to the one given in footnote (6), p. 16).

b) The (C) condition is invariant under a $C^{(0,1)}$-homeomorphism. In particular if C^* is the image of a conoid C with constants $\varkappa$ and r and vertex x under a $C^{(0,1)}$-homeomorphism T, and M is the Lipschitz constant of T and T^{-1} then C^* contains a conoid with constants $\varkappa/M^2$ and r/M and vertex $T(x)$.

Let l^- be the half-line from x in the direction $-\theta$ (θ a unit vector) and D an open set. We define

$$r_{D,\theta}(x) = \begin{cases} r_{l^- \cap D}(x) & \text{if} \quad l^- \cap D \neq 0 \\ \infty & \text{if} \quad l^- \cap D = 0. \end{cases}$$

It is easy to prove that $r_{D,\theta}(x)$ is an upper semi-continuous function of θ and x.

LEMMA 1. — *For* $\alpha \geqslant 0$, *a*) *If* F *is a measurable subset of* R^n *and* $x \in R^n - \overline{F}$ *then*

$$\int_F \frac{dy}{|x-y|^{n+2\alpha}} \leqslant \frac{\omega_n}{2\alpha} r_F(x)^{-2\alpha}. \tag{9.1}$$

b) *If* D *is an open set and* $x \in R^n - \overline{D}$, *and at a point* $x_0 \in \partial D$ *where* $r_D(x) = |x - x_0|$ *there exists a conoid lying in* D *with vertex* x_0, *radius* r *and opening* $\varkappa$, *then*

$$\int_D \frac{dy}{|x-y|^{n+2\alpha}} \geqslant \frac{c_1 \varkappa^{n-1}}{\alpha} \left[r_D(x)^{-2\alpha} - c_2 r^{-2\alpha}\right] \tag{9.2}$$

where c_1 *and* c_2 *are positive constants depending only on* n *and* α^*.

Proof. — $\int_{F} \frac{dy}{|x-y|^{n+2\alpha}} \leqslant \int_{r_F(x)}^{\infty} \int_{\partial S(0,1)} \rho^{-1-2\alpha}\, d\theta\, d\rho \leqslant \frac{\omega_n}{2\alpha} r_F(x)^{-2\alpha}$.
To prove (9.2) we restrict the integration to the conoid and take polar coordinates with pole at x_0. Since the angular opening with respect to x_0 of the conoid at any distance less than r from x_0 is greater than $\frac{\omega_{n-1}}{n-1} \varkappa^{n-1}$ we have

$$\int_{D} \frac{dy}{|x-y|^{n+2\alpha}} \geqslant \frac{\omega_{n-1}}{n-1} \varkappa^{n-1} \int_0^r \frac{\rho^{n-1}\, d\rho}{(r_D(x)+\rho)^{n+2\alpha}}.$$

By repeated integration by parts it is not difficult to show

$$\int_0^r \frac{\rho^{n-1}\, d\rho}{(r_D(x)+\rho)^{n+2\alpha}} \geqslant - r^{-2\alpha} \left[\frac{\Gamma(n)\Gamma(2\alpha+1)}{\Gamma(n+2\alpha)2\alpha} + \sum_{j=1}^{n-1} \frac{\Gamma(j+2\alpha)}{\Gamma(j+1)}\right] + r_D(x)^{-2\alpha} \frac{\Gamma(n)\Gamma(2\alpha+1)}{\Gamma(n+2\alpha)2\alpha}$$

and (9.2) follows.

LEMMA 2. — *For $\alpha \geqslant 0$, a) if $x \in R^n - \overline{D}$ then*

(9.3) $$\int_{\partial S(0,1)} r_{D,\theta}(x)^{-2\alpha}\, d\theta \leqslant \omega_n r_D(x)^{-2\alpha}.$$

b) *If $x \in R^n - \overline{D}$ and at $x_0 \in \partial D$ where $r_D(x) = |x - x_0|$ there exists a conoid lying in D with vertex x_0, radius r and opening $\varkappa$, then*

(9.4) $$r_D(x)^{-2\alpha} \leqslant \frac{9^\alpha (n-1) 2^{n-1}}{\omega_{n-1}\varkappa^{n-1}} \int_{\partial S(0,1)} r_{D,\theta}(x)^{-2\alpha}\, d\theta + r^{-2\alpha}.$$

Proof. — Since $r_D(x) \leqslant r_{D,\theta}(x)$, (9.3) *is clear.* If $r_D(x) \geqslant r$ then (9.4) is clearly true. Suppose $r_D(x) < r$. Then from the hypotheses it is clear that there exists on the axial arc of the conoid a point $x_1 \in D$ such that $|x_0 - x_1| = r_D(x)$ and

$$S = S(x_1, \varkappa r_D(x)) \subset D.$$

Therefore $r_{S,\theta}(x) \geqslant r_{D,\theta}(x)$ and if $r_{S,\theta}(x) \neq \infty$, then

$$\frac{r_D(x)}{r_{S,\theta}(x)} \geqslant \frac{r_D(x)}{r_D(x) + (1+\varkappa) r_D(x)} \geqslant \frac{1}{3}.$$

Hence

$$\int_{\partial S(0,1)} r_{D,\theta}(x)^{-2\alpha}\, d\theta \geqslant r_D(x)^{-2\alpha} \int_{r_{S,\theta}(x)<\infty} \left(\frac{r_D(x)}{r_{S,\theta}(x)}\right)^{2\alpha} d\theta$$

$$\geqslant \frac{r_D(x)^{-2\alpha}}{3^{2\alpha}} \int_{r_{S,\theta}(x)<\infty} d\theta \geqslant r_D(x)^{-2\alpha} \frac{\chi^{n-1}\omega_{n-1}9^{-\alpha}}{2^{n-1}(n-1)}$$

and (9.4) follows.

LEMMA 3. — *For* $\frac{1}{2} < \alpha < 1$ *and* $u \in \check{P}^\alpha(a,b)$ (*b possibly infinite*),

$$(9.5) \quad \int_a^b |u(x) - u(a)|^2 (x-a)^{-2\alpha}\, dx \leqslant \frac{\pi}{\left(\alpha - \frac{1}{2}\right)^2} d_{\alpha,(a,b)}(u),$$

where $u(a) = \lim_{x \downarrow a} u(x)$.

Proof (30). — Suppose $b < \infty$. Clearly we may assume that $a = 0$ and by changing x into bx we may assume $b = 1$. Then for $0 < t < 1$

$$\left[\int_0^1 |u(x) - u(t^n x)|^2 x^{-2\alpha}\, dx\right]^{\frac{1}{2}}$$

$$\leqslant \sum_{k=0}^{n-1} \left[\int_0^1 |u(t^k x) - u(t^{k+1}x)|^2 x^{-2\alpha} dx\right]^{\frac{1}{2}}$$

$$\leqslant \left[\int_0^1 |u(x) - u(tx)|^2 x^{-2\alpha} dx\right]^{\frac{1}{2}} \sum_{k=0}^{n-1} t^{k\left(\alpha - \frac{1}{2}\right)}.$$

Since $\check{P}^\alpha(a,b) = P^\alpha(a,b)$ by Prop. 4'), § 3, there exists an extension of u, $\tilde{u} \in P^\alpha(R^1)$. For $\alpha > \frac{1}{2}$ the functions in $P^\alpha(R^1)$ are continuous (cf. Prop. 4), § 5, II). Hence

$$\lim_{x \downarrow 0} u(x) = u(0)$$

exists. Letting $n \to \infty$ in the above we have

$$\left(1 - t^{\alpha - \frac{1}{2}}\right)^2 \int_0^1 |u(x) - u(0)|^2 x^{-2\alpha}\, dx$$

$$\leqslant \int_0^1 |u(x) - u(tx)|^2 x^{-2\alpha}\, dx$$

and since

$$d_{\alpha,(0,1)}(u) = \frac{2}{C(1,\alpha)} \int_0^1 \int_0^1 \frac{|u(x) - u(tx)|^2}{x^{2\alpha}(1-t)^{1+2\alpha}}\, dx\, dt,$$

(30) The proof of the lemma is based on an idea from [1].

we have

$$\left[\frac{2}{C(1,\alpha)}\int_0^1 \frac{\left(1-t^{\alpha-\frac{1}{2}}\right)^2}{(1-t)^{1+2\alpha}}\,dt\right]\int_0^1 |u(x)-u(0)|^2 x^{-2\alpha}\,dx \leqslant d_{\alpha,(0,1)}(u).$$

It is easy to show that the square bracketed term is bounded from below by $\left(\alpha-\frac{1}{2}\right)^2\Big/\pi$ which yields the inequality.

If $b=\infty$ then $u \in \check{P}^\alpha(0,N)$ for every positive integer N; letting $N \to \infty$ yields the inequality in this case.

Lemma 3 will often be applied in the following situation. The function u belongs to $\check{P}^\alpha(I)$, $\frac{1}{2}<\alpha<1$, I being an interval of the line $l = l_{\theta,z'} = [z : z = z' + s\theta, -\infty < s < \infty] \subset R^n$. Furthermore, $u(z'+s\theta)=0$ for $z'+s\theta \in D$, D a relatively open subset of I. Then it is immediately checked that with our definition of $r_{D,\theta}(z)$, Lemma 3 gives

$$(9.6)\qquad \int_{I-\bar{D}} |u(z'+s\theta)|^2 r_{D,\theta}(z'+s\theta)^{-2\alpha}\,ds \leqslant \frac{\pi}{\left(\alpha-\frac{1}{2}\right)^2}\, d_{\alpha,I-\bar{D}}(u(z'+s\theta)).$$

We define for functions $u \in \check{P}^\alpha(D_1)$ the quadratic form

$$(9.7)\qquad J_{\alpha,D_1,D_2}(u) = \sum_{|i|\leqslant\alpha^*}\int_{D_1} |D_i u(x)|^2 r_{D_2}(x)^{-2\alpha+2|i|}\,dx$$

where $D_2 \subset R^n$ is an open set. If $D_1 \cap \overline{D}_2 \neq 0$ we adopt the convention that $|D_i u(x)|^2\, r_{D_2}(x)^{-2\alpha+2|i|} = 0$ wherever

$$D_i u(x) = 0$$

regardless of whether $r_{D_2}(x)^{-2\alpha+2|i|}=\infty$ or not. Also, if $\alpha = 0$ we adopt the customary convention that $J_{0,D_1,D_2}(u) = 0$.

THEOREM I. — *If* $D = [D_1 \cup D_2 \cup (\partial D_1 \cap \partial D_2)]^0$, $m=[\alpha]$, *and* $\beta = \alpha - m$, *then:*

a) If $u \in \check{P}^\alpha(D_1)$ *and* $J_{\alpha,D_1,D_2}(u) < \infty$, *then u has an extension* $\tilde{u}$ *in* $\check{P}^\alpha(D)$ *such that* $\tilde{u} = 0$ *on* D_2 *and*

$$(9.8)\qquad |\tilde{u}|^2_{\alpha,D} \leqslant |u|^2_{\alpha,D_1} + 2n(1-\beta)[J_{\alpha,D_1,D_2}(u) + |u|^2_{m,D_1}]\ ([31]).$$

[31] The second term on the right side of the inequality may be omitted if $\beta = 0$ and the inequality becomes an equality.

b) If D *is* L-*convex with constants* δ, p, *and* M, D_2 *is a* (C)-*domain with constants* r *and* $\varkappa$ *(and we shall assume* $r \leqslant \delta$*) and* u *is a function in* $\check{P}^\alpha(D)$ *such that* $u = 0$ on D_2, *then*

$$J_{\alpha, D_1, D_2}(u) \leqslant c|u|^2_{\alpha, D} \tag{9.9}$$

where c *depends only on* n, m, $\varkappa$, r, M *and* p.

Proof of a). — The proof is trivial if $\alpha = 0$ so we assume $\alpha > 0$. Define $v = u$ on D_1 and $= 0$ on $D - D_1$. We will show by an application of Prop. 2′), § 9, II that v is equivalent to a function in $P^\alpha_{loc}(D)$. Consider the lines l parallel to the x_k-axis such that

i) $$\sum_{j=0}^{m} \int_{l \cap D_1} \left| \frac{\partial^j u}{\partial x_k^j} \right|^2 dx_k < \infty,$$

ii) $$\sum_{j=0}^{\alpha^*} \int_{l \cap D_1} \left| \frac{\partial^j u}{\partial x_k^j} \right|^2 r_{D_2}(x)^{-2\alpha+2j}\, dx_k < \infty$$

and

iii) $\dfrac{\partial^j u}{\partial x_k^j}$, $j = 0, \ldots, m-1$, *is absolutely continuous on* $l \cap D_1$.

These conditions hold for almost all lines l.

Let $l \cap D_1 = \bigcup_\nu I_\nu$ a union of mutually disjoint open intervals. By virtue of (i) and (iii) $\dfrac{\partial^j u}{\partial x_k^j}$, $j = 0, \ldots, m-1$, is absolutely and uniformly continuous on each I_ν and in view of (ii) it must converge to 0 at the endpoints of the I_ν's which lie in $\overline{D}_2 \cap D$. Hence if we extend $\dfrac{\partial^j u}{\partial x^j}$, $j = 0, \ldots, m$ to the whole of $l \cap D$ by 0 and denote this extension by v_j, then v_j, $j = 0, \ldots, m-1$, is absolutely continuous on $l \cap D$ and is the indefinite integral of v_{j+1} on each interval of $l \cap D$. If $\beta = 0$ then $v_m \in P^0_{loc}(D)$ and if $\beta > 0$ then by Lemma 1, *a)* and (2.7), and since $D - D_1 \subset \overline{D}_2$,

$$\begin{aligned} &d_{\beta, D}(v_m) \tag{9.10} \\ &= d_{\beta, D_1}\left(\frac{\partial^m u}{\partial x_k^m}\right) + \frac{2}{C(n, \beta)} \int_{D_1} \left| \frac{\partial^m u}{\partial x_k^m}(x) \right|^2 \int_{D-D_1} \frac{dy}{|x-y|^{n+2\beta}}\, dx \\ &\leqslant d_{\beta, D_1}\left(\frac{\partial^m u}{\partial x_k^m}\right) + 2n \int_{D_1} \left| \frac{\partial^m u}{\partial x_k^m} \right|^2 r_{D_2}(x)^{-2\beta}\, dx < \infty, \end{aligned}$$

and φ_m is equivalent to a function in $P^\beta_{\text{loc}}(D)$. Hence φ is equivalent to a function in $P^\alpha_{\text{loc}}(D)$ by the cited proposition.

If $\tilde{u}$ is the correction of φ then $\tilde{u} = 0$ on D_2 since $\varphi = 0$ on $D - D_1$ by definition, and $= 0$ on $D_1 \cap \overline{D}_2$ by the finiteness of $J_{\alpha, D_1, D_2}(u)$. Furthermore, $\tilde{u} \in P^\alpha_{\text{loc}}(D)$, and to complete the proof we need only to prove (9.8).

By a calculation similar to (9.10) (for $\beta > 0$),

$$|\tilde{u}|^2_{\alpha, D} = |u|^2_{\alpha, D_1} + \frac{2}{C(n, \beta)} \sum_{k=0}^{m} \binom{m}{k} \sum_{|i|=k} \int_{D_1} |D_i u(x)|^2 \int_{D-D_1} \frac{dy}{|x-y|^{n+2\beta}}\, dx$$
$$\leqslant |u|^2_{\alpha, D_1}$$
$$+ 2n(1-\beta) \sum_{k=0}^{m} \binom{m}{k} \sum_{|i|=k} \int_{D_1} |D_i u(x)|^2 (r_{D_2}(x)^{-2\alpha+2|i|} + 1)\, dx$$
$$\leqslant |u|^2_{\alpha, D_1} + 2n(1-\beta)(J_{\alpha, D_1, D_2}(u) + |u|^2_{m, D_1}).$$

Proof of b). — i) Suppose we have two open coverings $\{V_k\}$ and $\{W_k\}$ of $(D_1 - \overline{D}_2) \cap S(D_2, \delta/8)$, with $V_k \subset W_k$ and such that

(9.11) *any point in* R^n *is in at most* $p' = 5^n + p$ *sets* W_k.

In addition suppose that for $|i| < m$,

(9.12*a*) $\int_{V_k \cap D_1} |D_i u|^2 r_{D_2}(x)^{-2\alpha+2|i|}\, dx$

$$\leqslant c' \left(\int_{W_k \cap D_1} |D_i u|^2\, dx + \int_{W_k \cap D_1} \sum_{l=1}^{n} \left| \frac{\partial}{\partial x_l} D_i u \right|^2 r_{D_2}(x)^{-2\alpha+2|i|+2}\, dx \right),$$

and for $|i| = m$

(9.12*b*) $\int_{V_k \cap D_1} |D_i u|^2 r_{D_2}(x)^{-2\beta}\, dx \leqslant c'' |D_i u|^2_{\beta, W_k \cap D},$

where c' and c'' depend only on n, m, $\varkappa$, r and M.

Then for $|i| < m$, since $|\partial D_2| = 0$ and $u = 0$ on D_2, we have by (9.12*a*)

$$\int_{D_1} |D_i u|^2 r_{D_2}(x)^{-2\alpha+2|i|} dx \leqslant (\delta/8)^{-2\alpha+2|i|} \int_{D_1} |D_i u|^2\, dx$$
$$+ \int_{(D_1 - \overline{D}_2) \cap S(D_2, \delta/8)} |D_i u|^2 r_{D_2}(x)^{-2\alpha+2|i|}\, dx$$
$$\leqslant (\delta/8)^{-2\alpha+2|i|} \int_{D_1} |D_i u|^2\, dx$$
$$+ \sum_{k=1}^{\infty} \int_{V_k \cap D_1} |D_i u|^2 r_{D_2}(x)^{-2\alpha+2|i|}\, dx$$
$$\leqslant (c'p' + (\delta/8)^{2\alpha+2|i|}) \int_{D_1} |D_i u|^2\, dx$$
$$+ c'p' \sum_{l=1}^{n} \int_{D_1} \left| \frac{\partial}{\partial x_l} D_i u \right|^2 r_{D_2}(x)^{-2\alpha+2|i|+2}\, dx.$$

Hence $I = \int_{D_1} |D_i u|^2 r_{D_2}(x)^{-2\alpha+2|i|}\,dx$ is majorated by $|D_i u|^2_{0,D_1}$ and $\sum_{|j|=|i|+1} \int_{D_1} |D_j u|^2 r_{D_2}(x)^{-2\alpha+2|j|}\,dx$ and by an easy inductive argument we have finally, that I is majorated by $|D_i u|^2_{m-|i|,D_1}$ and $\sum_{|j|=m} \int_{D_1} |D_j u|^2 r_{D_2}(x)^{-2\beta}\,dx$.

For $|i| = m$ we have, by (9.12 b),

$$\int_{D_1} |D_i u|^2 r_{D_2}(x)^{-2\beta}\,dx \leqslant ((\delta/8)^{-2\beta} + 2c''p')|D_i u|^2_{\beta,D},$$

and thus we obtain finally (9.9).

ii) Before we construct the coverings $\{V_k\}$ and $\{W_k\}$ and establish the properties (9.12a) and (9.12b) we prove the following.

LEMMA 4. — *Suppose* D^* *is a convex domain,* D_1^* *and* D_2^* *are disjoint open subsets of* D^*. *Furthermore, suppose*

(*) *There is a* $\varkappa^*$ *and* r^* *such that for any* $x \in D_1^*$ *there is an* $x_0 \in \partial D_2^*$ *with* $r_{D_2^*}(x) = |x - x_0|$ *and a conoid with vertex* x_0, *opening* $\varkappa^*$ *and radius* r^* *lying in* D_2^*.

If $v \in \check{P}^1(D^*)$ *and* $v = 0$ *on* D_2^* *then for any* $\gamma \geqslant 1$

$$(9.13a)\quad \int_{D_1^*} |v(x)|^2 r_{D_2^*}(x)^{-2\gamma}\,dx \leqslant c9^{\gamma}\varkappa^{*(1-n)}\left[r^{*-2\gamma}|v|^2_{0,D^*} + \sum_{l=1}^{n}\int_{D^*}\left|\frac{\partial v}{\partial x_l}\right|^2 r_{D_2^*}(x)^{-2\gamma+2}\,dx\right].$$

If $v \in \check{P}^{\beta}(D^*)$, $0 < \beta < 1$, *and* $v = 0$ *on* D_2^* *then*

$$(9.13b)\quad \int_{D_1^*} |v(x)|^2\, r_{D_2^*}(x)^{-2\beta}\,dx \leqslant c\varkappa^{*(1-n)}[r^{*-2\beta}|v|^2_{0,D^*} + d_{\beta,D^*}(v)]$$

where c *in both cases depends only on* n.

To prove (9.13) we need an inequality due to Hardy (Part. I, § 12) which will be used in other proofs of this section.

Hardy's Inequality. — *If* $f(s)$ *is absolutely continuous for* $a \leqslant s < b$, *then for any* $\gamma > \dfrac{1}{2}$

$$(9.14)\quad \int_a^b |f(s) - f(a)|^2 (s-a)^{-2\gamma}\,ds \leqslant \left(\gamma - \frac{1}{2}\right)^{-2} \int_a^b |f'(s)|^2 (s-a)^{-2\gamma+2}\,ds.$$

Let $l_{\theta,z'} = [z : z = z' + s\theta, -\infty < s < \infty]$. Then if φ satisfies the hypothesis of (9.13a), $\varphi(z' + s\theta) \in P^1(l_{\theta,z'} \cap D^*)$ by Prop. 2'), § 3 for almost all θ and z'. Since $\varphi = 0$ on D_2^*, and in view of the definition of $r_{D_2^*,\theta}(x)$ we have from (9.14) following the notation of (4.1)

$$\int_{\partial S}\int_{D^*} |\varphi(x)|^2 r_{D_2^*,\theta}(x)^{-2\gamma}\, dx\, d\theta$$
$$= \int_{\partial S}\int_{E_\theta(D^*)}\int_{I(\theta,z')} |\varphi(z' + s\theta)|^2 r_{D_2^*,\theta}(z' + s\theta)^{-2\gamma}\, ds\, dz'\, d\theta$$
$$\leqslant \left(\gamma - \frac{1}{2}\right)^{-2} \int_{\partial S}\int_{E_\theta(D^*)}\int_{I(\theta,z')} \left|\frac{\partial\varphi}{\partial_s}(z' + s\theta)\right|^2 r_{D_2^*,\theta}(z' + s\theta)^{-2\gamma+2}\, ds\, dz'\, d\theta$$
$$\leqslant \omega_n \left(\gamma - \frac{1}{2}\right)^{-2} \sum_{l=1}^{n} \int_{D^*} \left|\frac{\partial\varphi}{\partial x_l}\right|^2 r_{D_2^*}(x)^{-2\gamma+2}\, dx.$$

The last inequality was obtained by Lemma 2, a) (note that $\gamma \geqslant 1$). An application of Lemma 2, b) completes the proof of (9.13a).

To prove (9.13b) we consider two cases

$$0 < \beta \leqslant \frac{3}{4}, \quad \frac{3}{4} < \beta < 1.$$

In the first case we have by Lemma 1, b),

$$d_{\beta,D^*}(\varphi) \geqslant \frac{2}{C(n,\beta)} \int_{D_1^*}\int_{D_2^*} \frac{|\varphi(x)|^2}{|x-y|^{n+2\beta}}\, dx\, dy$$
$$\geqslant \frac{2\varkappa^{*(n-1)} c_1}{\beta C(n,\beta)} \left[\int_{D_1^*} |\varphi(x)|^2\, r_{D_2^*}(x)^{-2\beta}\, dx - c_2 r^{*-2\beta} \int_{D_1^*} |\varphi(x)|^2\, dx\right].$$

Since $\dfrac{1}{\beta C(n,\beta)}$ is uniformly bounded from below in $0 < \beta \leqslant \dfrac{3}{4}$, (9.13$b$) is proved in this case.

If $\dfrac{3}{4} < \beta < 1$ we write by (4.1)

$$d_{\beta,D^*}(\varphi) = \frac{C(1,\beta)}{2C(n,\beta)} \int_{\partial S}\int_{E_\theta(D^*)} d_{\beta,I(\theta,z')}(\varphi(z' + s\theta))\, dz'\, d\theta.$$

Applying Lemma 3 in the form (9.6) to $d_{\beta,I(\theta,z')}(\varphi(z' + s\theta))$ and then restricting the integration to D_1^* (9.13b) follows by an application of Lemma 2,b) and (4.2).

iii) We pass now to the construction of the coverings $\{V_k\}$ and $\{W_k\}$ used in part i).

Consider a maximal set $A \subset R^n$ with the property that for any $a' \in A$, $a'' \in A$, $a' \neq a''$, we have $|a' - a''| \geqslant \delta/4$. Such sets obviously exist and are enumerable. Furthermore we note the following four elementary facts (a will denote an arbitrary point in A) : 1° the spheres $S(a, \delta/8)$ are mutually disjoint; 2° the spheres $S(a, \delta/2)$ form a $\delta/4$-loose covering of R^n; 3° any point in R^n lies in at most 5^n distinct spheres $S(a, \delta/2)$; 4° for $a \in D^{\delta/2}$ the spheres $S(a, \delta/2)$ form a $\delta/4$-loose covering of $D^{3\delta/4}$ and each of these lies in D.

By hypothesis of part b) of our theorem there exists a covering $\{U_l\}$ of ∂D, δ-loose and of rank p, such that each $U_l \cap D$ is the image of a convex domain by a $C^{(0,1)}$-homeomorphism with constant $M(\geqslant 1)$. It follows that $\{U_l\}$ is a $\delta/4$-loose covering of $\overline{S(\partial D, 3\delta/4)}$; hence, combined with $S(a, \delta/2)$ for $a \in D^{\delta/2}$, it forms a $\delta/4$-loose covering of $\overline{D}$. The sets of this combined covering will be denoted by W_k. Obviously each point of R^n lies in at most $p' = 5^n + p$ of the sets W_k. Furthermore each $W_k \cap D$ is transformed by a $C^{(0,1)}$-homeomorphism T_k with constant M on a convex domain (if $W_k = S(a, \delta/2)$, T_k is the identity). We put

$$V_k = W_k^{\delta/4} \cap S(\overline{D}_2, \delta/8) \cap (D_1 - \overline{D}_2).$$

Obviously $\{V_k\}$ and $\{W_k\}$ satisfy the conditions of part i) of our proof and it remains to prove (9.12a) and (9.12b).

If $V_k = 0$ there is nothing to prove. Therefore we assume $V_k \neq 0$. Set

$$D^* = T_k(W_k \cap D),\ \ D_1^* = T_k(V_k),\ \ D_2^* = T_k(W_k \cap D_2).$$

We will consider T_k as extended by continuity to a homeomorphism of $\overline{W_k \cap D}$ onto $\overline{D}^*$.

If $x \in V_k$ then $r_{D_2}(x) < \delta/8$ and there exists $x_0 \in \overline{D}_2$ with $|x - x_0| = r_{D_2}(x) < \delta/8$. Hence $x_0 \in W_k^{\delta/8}$, $x_0 \in \overline{W_k \cap D_2}$, (this proves that $W_k \cap D_2$ and D_2^* are $\neq 0$), $T_k(x_0) \in \overline{D}_2^*$ and

$$r_{D_2^*}(T_k(x)) \leqslant |T_k(x) - T_k(x_0)| \leqslant M|x - x_0| = Mr_{D_2}(x).$$

On the other hand if $x \in W_k \cap D$, $r_{D_2^*}(T_k(x)) = |T_k(x) - x_0^*|$ with $x_0^* \in \overline{D}_2^*$, hence

$$r_{D_2}(x) \leqslant |x - T^{-1}(x_0^*)| \leqslant M|T_k(x) - x_0^*| = Mr_{D_2^*}(T_k(x)).$$

Since D^*, D_1^*, and D_2^* obviously satisfy the conditions of Lemma 4 with $\varkappa^* = \varkappa/M^2$ and $r^* = \min[\delta/16M, r/M]$, the formulas (9.13*a*) and (9.13*b*) with $\varphi(x^*) = D_i u(T^{-1}(x^*))$ are transformed immediately into (9.12*a*) and (9.12*b*) respectively by virtue of the above relations between $r_{D_2}(x)$ and $r_{D_2^*}(T_k x)$ and known properties of $C^{(0,1)}$-homeomorphisms (f.i. Prop. 8), § 2).

Remark 2. — The L-convexity of D in part *b*) of Theorem I was needed in order to allow us to apply Lemma 4. For $\alpha < 1$ only the (C)-condition for D_2 is needed to obtain (9.9) by direct application of Lemma 1, *b*). However, the constant *c* in (9.9), so obtained, blows up when $\alpha \uparrow 1$.

In the next two theorems we will consider open sets D_1 and D_2 and the open sets $U_\eta \equiv U_\eta(\overline{D}_1, \overline{D}_2)$ introduced in § 8. We remind the reader that $U_\eta = [x: \ r_{D_1}(x) < \eta r_{D_2}(x)]$ and that for $\eta > 0$, $D_1 - D_2 \subset U_\eta \subset R^n - \overline{D}_2$. We shall use the notation

$$(+) \qquad U_\eta^+ = D_1 \cup U_\eta \equiv D_1 \cup U_\eta(\overline{D}_1, \overline{D}_2).$$

THEOREM II. — *If* D_1 *is a* (C)*-domain with opening* $\varkappa$ *and radius r, and if* $u \in \check{P}^\alpha(U_\eta^+)$ *then for* $\varepsilon = \eta/(9\eta + 4)$ [32]

$$(9.15) \qquad J_{\alpha, U_\varepsilon^+, D_2}(u) \leqslant c[|u|^2_{\alpha, U_\eta^+} + J_{\alpha, D_1, D_2}(u)]$$

where c depends only on η, α^*, *n*, $\varkappa$ *and r.*

Proof. — We will write $r_1(x)$ and $r_2(x)$ instead of $r_{D_1}(x)$ and $r_{D_2}(x)$.

Let $m = [\alpha]$ and $\beta = \alpha - m$. The proof will be divided into three parts. Parts 1° and 2° will be concerned with showing that for $|i| = m$ and $\varepsilon' = \eta/(3\eta + 2)$

$$(9.16) \quad \int_{U_{\varepsilon'}^+} |D_i u(x)|^2 r_2(x)^{-2\beta}\, dx \leqslant c' \left[\int_{D_1} |D_i u(x)|^2 r_2(x)^{-2\beta}\, dx + |D_i u|^2_{\beta, U_\eta^+}\right].$$

Part 3° deals with lower derivatives: we shall show that with $\varepsilon = \varepsilon'/(3\varepsilon' + 2) = \eta/(9\eta + 4)$

$$(9.17) \quad \sum_{|i|<m} \int_{U_\varepsilon^+} |D_i u(x)|^2 r_2(x)^{-2\alpha + 2|i|}\, dx \leqslant c'' \left[J_{\alpha, D_1, D_2}(u) + \sum_{|i|=m} \int_{U_{\varepsilon'}^+} |D_i u|^2 r_2(x)^{-2\beta}\, dx\right].$$

[32] By a more elaborate proof we could prove (9.15) with $\varepsilon = \eta/(3\eta + 2)$.

The constants c' and c'' in (9.16) and (9.17) depend only on n, η, α^*, r, and k. Inequality (9.15) then follows from (9.16) and (9.17) since $U_\varepsilon \subset U_{\varepsilon'}$.

In parts 1° and 2° we shall let $\varphi = D_i u$, $|i| = m$. Then $\varphi \in \check{P}^\beta(U_\eta^+)$. If $\beta = 0$, (9.16) is obviously true with $c' = 1$ so we assume $\beta > 0$.

1° Suppose $0 < \beta \leqslant 3/4$. For $x \in D_1$ and $y \in U_{\varepsilon'}^+ - \overline{D}_1$ we write the inequality

$$(9.18)\quad I_1 \equiv \iint\limits_{\substack{D_1 U_{\varepsilon'}^+ - \overline{D}_1 \\ |x-y| > \frac{1}{2} r_2(x)}} \frac{|\varphi(y)|^2}{|x-y|^{n+2\beta}}\, dy\, dx$$

$$\leqslant 2 \iint\limits_{\substack{D_1 U_{\varepsilon'}^+ - \overline{D}_1 \\ |x-y| > \frac{1}{2} r_2(x)}} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{n+2\beta}}\, dy\, dx$$

$$+ 2 \iint\limits_{\substack{D_1 U_{\varepsilon'}^+ - \overline{D}_1 \\ |x-y| > \frac{1}{2} r_2(x)}} \frac{|\varphi(x)|^2}{|x-y|^{n+2\beta}}\, dy\, dx \equiv I_2 + I_3.$$

Clearly,

$$I_2 \leqslant C(n, \beta)\, d_{\beta, U_\eta^+}(\varphi),$$

$$I_3 \leqslant 2\omega_n \int_{D_1} |\varphi(x)|^2 \int_{\rho > \frac{1}{2} r_2(x)} \rho^{-1-2\beta}\, d\rho\, dx$$

$$\leqslant \frac{\omega_n}{\beta} 2^{2\beta} \int_{D_1} |\varphi(x)|^2 r_2(x)^{-2\beta}\, dx.$$

We notice now that if $y_0 \in \partial D_1$ satisfies $|y_0 - y| = r_1(y) < \varepsilon' r_2(y)$ and if $|x - y_0| > (1 + \varepsilon')r_2(y)$, then $|x - y| > \frac{1}{2} r_2(x)$ (since $|x - y| \geqslant |x - y_0| - |y_0 - y| \geqslant r_2(y) \geqslant r_2(x) - |x - y|$). Hence, by restricting the integration in I_1 to y with $r_2(y) < r/2$ and to x lying in the conoid with vertex y_0 at distance

$$|x - y_0| > (1 + \varepsilon')r_2(y)$$

and noticing that the conoid cuts out on $\partial S(y_0, \rho)$ for $\rho < r$

an angular area $> \dfrac{\omega_{n-1}}{(n-1)2^{n-1}} \varkappa^{n-1}$, we get

$$\begin{aligned}
I_1 &\geqslant \frac{\omega_{n-1}}{(n-1)2^{n-1}}\varkappa^{n-1}\int_{\substack{U^+_{\varepsilon'}-\overline{D}_1\\ r_2(y)<r/2}} |\varphi(y)|^2 \int_{(1+\varepsilon')r_2(y)}^{r} \frac{\rho^{n-1}\,d\rho}{(\varepsilon' r_2(y)+\rho)^{n+2\beta}}\,dy \ \text{[33]}\\
&\geqslant \frac{1}{2\beta}\frac{\omega_{n-1}}{(n-1)2^{n-1}}\varkappa^{n-1}(1+\varepsilon')^{-n-4\beta}\\
&\times\Big[\int_{\substack{U^+_{\varepsilon'}-\overline{D}_1\\ r_2(y)<r/2}} |\varphi(y)|^2 r_2(y)^{-2\beta}\,dy - r^{-2\beta}(1+\varepsilon')^{2\beta}\int_{\substack{U^+_{\varepsilon'}-\overline{D}_1\\ r_2(y)<r/2}} |\varphi(y)|^2\,dy\Big].
\end{aligned}$$

Since $\beta C(n, \beta)$ is uniformly bounded for $0 \leqslant \beta < \dfrac{3}{4}$ and since

$$\begin{aligned}
\int_{U^+_{\varepsilon'}} |\varphi|^2 r_2(y)^{-2\beta}\,dy \leqslant{}& \int_{\substack{U^+_{\varepsilon'}-\overline{D}_1\\ r_2(y)<r/2}} |\varphi(y)|^2 r_2(y)^{-2\beta}\,dy\\
&+\left(\frac{r}{2}\right)^{-2\beta}\int_{U^+_{\varepsilon'}} |\varphi|^2\,dy + \int_{D_1} |\varphi(y)|^2 r_2(y)^{-2\beta}\,dy,
\end{aligned}$$

the inequality (9.16) is proved in the present case.

2° Suppose $\dfrac{3}{4} \leqslant \beta < 1$. Let l be a line in the direction of the unit vector θ and $z' = \pi_\theta \cap l$ where π_θ is the subspace orthogonal to θ. As before we write $l \equiv l_{\theta,z'} = [z : z' + t\theta]$. Suppose $l \cap D_1 \neq 0$ and $d_{\beta, l\cap U^+_\eta}(\varphi(z'+t\theta)) < \infty$. We shall restrict our attention temporarily to this line and so shall write for $\varphi(z'+t\theta)$ and $r_2(z'+t\theta)$ simply $\varphi(t)$ and $r_2(t)$ respectively.

Define $s_1 = s + \dfrac{\eta}{1+\eta} r_2(s)$ for $s \in l \cap D_1$. Since

$$|r_2(s) - r_2(s')| \leqslant |s - s'|,$$

s_1 is an increasing function of s. For

$$s < t < s_1, \quad |s-t| < \frac{\eta}{1+\eta} r_2(s),$$

$$r_2(s) - \frac{\eta}{1+\eta} r_2(s) < r_2(t) < r_2(s) + \frac{\eta}{1+\eta} r_2(s),$$

$$r_1(t) \leqslant |s-t|$$

[33] Note that the inner integrand is

$$\geqslant \rho^{-1-2\beta}\left(\frac{\varepsilon' r_2(y)}{\rho}+1\right)^{-n-2\beta} > \rho^{-1-2\beta}(1+\varepsilon')^{-n-2\beta}.$$

(s being in D_1), hence

(9.19) *if* $s < t < s_1$, *then* $\frac{1}{1+\eta} r_2(s) < r_2(t) < \frac{1+2\eta}{1+\eta} r_2(s)$

and $z' + t\theta \in U_\eta$.

Applying the above and (9.5) we get

$$\left(\frac{\eta}{1+\eta} r_2(s)\right)^{-2\beta} \int_s^{s_1} |\varphi(t)|^2 \, dt \leqslant 2\left(\frac{\eta}{1+\eta} r_2(s)\right)^{-2\beta} \int_s^{s_1} |\varphi(s)|^2 \, dt$$
$$+ 2\left(\frac{\eta}{1+\eta} r_2(s)\right)^{-2\beta} \int_s^{s_1} |\varphi(t) - \varphi(s)|^2 \, dt$$
$$\leqslant 2\left(\frac{\eta}{1+\eta} r_2(s)\right)^{-2\beta} (s_1 - s)|\varphi(s)|^2 + 2\int_s^{s_1} |\varphi(t) - \varphi(s)|^2 |t-s|^{-2\beta} \, dt$$
$$\leqslant 2\left(\frac{\eta}{1+\eta} r_2(s)\right)^{-2\beta+1} |\varphi(s)|^2 + 32\pi \, d_{\beta,(s,s_1)}(\varphi).$$

Dividing by $\eta^{-2\beta}(1+\eta)^{-1} r_2(s)$ and noticing that by (9.19) $r_2(s) < (1+\eta) r_2(t)$, we get

$$(9.20) \quad \int_s^{s_1} |\varphi(t)|^2 r_2(t)^{-2\beta-1} \, dt \leqslant 2\eta(1+\eta)^{2\beta} |\varphi(s)|^2 \, r_2(s)^{-2\beta}$$
$$+ 32\pi\eta^{2\beta}(1+\eta) r_2(s)^{-1} \, d_{\beta,(s,s_1)}(\varphi).$$

In view of (9.19) the set of points $s \in l \cap D_1$ such that $s < t < s_1$ for a fixed t on l has a measure

$$\mu_\theta(t) \equiv \mu_\theta(z' + t\theta)$$

satisfying $\mu_\theta(t) \leqslant \eta r_2(t)$ and for all such points s,

$$\frac{1+\eta}{1+2\eta} r_2(t) < r_2(s) < (1+\eta) r_2(t).$$

Let $I'_\theta = \left[y = z' + t\theta : \text{there is an } x \in D_1,\ x = z' + s\theta, \text{ with } s < t < s + \frac{\eta}{1+\eta} r_2(s) = s_1\right]$.

By (9.19), $I'_\theta \subset U_\eta$. We perform now three integrations on both sides of (9.20) (compare notations with (4.1)),

$$\int_{\partial S} \int_{E_\theta(D_1)} \int_{l \cap D_1} \ldots \, ds \, dz' \, d\theta,$$

and obtain, following the above remarks,

$$(9.21)\quad \int_{\partial S}\int_{I'_\theta} |\varphi(y)|^2 r_2(y)^{-2\beta-1}\mu_\theta(y)\,dy\,d\theta \leqslant 2\omega_n\,\eta(1+\eta)^{2\beta}\int_{D_1}|\varphi(x)|^2 r_2(x)^{-2\beta}\,dx + 64\pi\,\eta^{2\beta+1}(1+2\eta)\frac{C(n,\beta)}{C(1,\beta)}\,d_{\beta,U_\eta}(\varphi).\ [34]$$

We aim now towards a lower bound for the left side of (9.21). Consider a point $y \in U_{\varepsilon'} - \overline{D}_1$ with $r_2(y) < r/2$. By a simple geometric argument we notice that if $y_0 \in \partial D_1$ with $|y - y_0| = r_1(y) < \varepsilon' r_2(y)$ and if on the axial arc of the conoid with vertex y_0, lying in D_1, we choose a point $y(\tau)$ with

$$|y(\tau) - y_0| = \frac{\varepsilon'}{4}\,r_2(y) < r$$

then for every z in the corresponding sphere $S\left(y(\tau), \frac{\varkappa\varepsilon'}{4}\,r_2(y)\right)$ we have $|z - y| < \frac{\eta}{\eta+1}\,r_2(z)$. It follows that for θ in the direction $\overline{zy}$, $y \in I'_\theta$. Furthermore, if z is restricted to

$$S\left(y(\tau), \frac{\varkappa\varepsilon'}{8}\,r_2(y)\right),$$

the measure $\mu_\theta(y) > \frac{\varkappa\varepsilon'}{4}\,r_2(y)$; the corresponding θ's form an angular area $> \frac{\omega_{n-1}}{n-1}\left(\frac{\varkappa}{10}\right)^{n-1}$. Hence the lower bound for the left side of (9.21) is given by

$$\frac{\omega_{n-1}}{n-1}\left(\frac{\varkappa}{10}\right)^{n-1}\frac{\varkappa\varepsilon'}{4}\int_{\substack{U_{\varepsilon'}-\overline{D}_1\\ r_2(y)<r/2}} |\varphi(y)|^2 r_2(y)^{-2\beta}\,dy$$

and thus we get formula (9.16) by the same concluding argument as in part 1°.

[34] The last term is obtained by using the following evaluation:

$$\int_{l\cap D_1} r_2(s)^{-1}\,d_{\beta,(s,s_1)}(\varphi)\,ds = \int_{l\cap(D_1-\overline{D}_2)} \cdots\,ds$$
$$= \frac{1}{C(1,\beta)}\int_{l\cap U_\eta}\int_{l\cap U_\eta}\frac{|\varphi(t)-\varphi(t')|^2}{|t-t'|^{1+2\beta}}\int_{l\cap(D_1-D_2)} r_2(s)^{-1}\chi_s(t)\,\chi_s(t')\,ds\,dt\,dt'$$
$$\leqslant \frac{1}{C(1,\beta)}\int_{l\cap U_\eta}\int_{l\cap U_\eta}\frac{|\varphi(t)-\varphi(t')|^2}{|t-t'|^{1+2\beta}}\,\frac{1+2\eta}{1+\eta}\,r_2(t)^{-1}\mu_\theta(t)\,dt\,dt'$$
$$= \frac{1+2\eta}{1+\eta}\,\eta\,d_{\beta,l\cap U_\eta}(\varphi),$$

where $\chi_s(t)$ is the characteristic function of the interval (s, s_1).

3° We use the same notation as in part 2° except we replace η by ε' and ε' by $\varepsilon = \varepsilon'/(3\varepsilon' + 2) = \eta/(9\eta + 4)$. We now deal with the function $\varphi = D_i u$, $|i| = q < m$. Hence $\varphi \in \check{P}^{m-q}(U^+_{\varepsilon'})$ and on almost all lines parallel to θ, $\frac{\partial^l \varphi}{\partial \theta^l}$, $l = 0,1, \ldots, m - q-1$, are absolutely continuous on $l \cap U^+_{\varepsilon'}$, with L^2-derivatives. We obtain succesively for $s \in l \cap D_1$,

$$s < t < s_1 = s + \frac{\varepsilon'}{1 + \varepsilon'} r_2(s),$$

$$\varphi(t) = \sum_{k=0}^{m-q-1} \frac{1}{k!} (t - s)^k \frac{\partial^k \varphi}{\partial \theta^k}(s) + \frac{1}{(m - q - 1)!} \int_s^t (t - \tau)^{m-q-1} \frac{\partial^{m-q} \varphi}{\partial \theta^{m-q}}(\tau)\, d\tau,$$

$$|\varphi(t)|^2 \leqslant (m - q + 1) \left(\sum_{k=0}^{m-q-1} \frac{(t - s)^{2k}}{(k!)^2} \left| \frac{\partial^k \varphi}{\partial \theta^k}(s) \right|^2 + \frac{(t - s)^{2m-2q-1}}{(2m - 2q - 1)((m - q - 1)!)^2} \int_s^t \left| \frac{\partial^{m-q} \varphi}{\partial \theta^{m-q}}(\tau) \right|^2 d\tau \right),$$

$$\int_s^{s_1} |\varphi(t)|^2\, dt \leqslant (m - q + 1) \left\{ \sum_{k=0}^{m-q-1} \frac{1}{(k!)^2(2k + 1)} \times \left(\frac{\varepsilon'}{1 + \varepsilon'} r_2(s) \right)^{2k+1} \left| \frac{\partial^k \varphi}{\partial \theta^k}(s) \right|^2 \right.$$
$$\left. + \frac{1}{(2m - 2q - 1)(2m - 2q)((m - q - 1)!)^2} \times \left(\frac{\varepsilon'}{1 + \varepsilon'} r_2(s) \right)^{2m-2q} \int_s^{s_1} \left| \frac{\partial^{m-q} \varphi}{\partial \theta^{m-q}}(\tau) \right|^2 d\tau \right\}.$$

Multiplying now by $\left(\frac{1}{1 + \varepsilon'} r_2(s) \right)^{-2\alpha+2q-1}$ and applying (9.19) we get

$$\int_s^{s_1} |\varphi(t)|^2 r_2(t)^{-2\alpha+2q-1}\, dt$$
$$\leqslant (1 + \varepsilon')^{2\alpha-2q+1}(m - q + 1) \left\{ \sum_{k=0}^{m-q-1} \frac{1}{(k!)^2(2k+1)} \times \left(\frac{\varepsilon'}{1 + \varepsilon'} \right)^{2k+1} r^2(s)^{-2\alpha+2(k+q)} \left| \frac{\partial^k \varphi}{\partial \theta^k}(s) \right|^2 \right.$$
$$\left. + \frac{\varepsilon'^{2m-2q}(1 + 2\varepsilon')^{2\beta+1}(1 + \varepsilon')^{-2\alpha+2q-1}}{(2m - 2q - 1)(2m - 2q)((m - q - 1)!)^2} \times \int_s^{s_1} \left| \frac{\partial^{m-q} \varphi}{\partial \theta^{m-q}}(\tau) \right|^2 r_2(\tau)^{-2\beta-1}\, d\tau \right\}.$$

We integrate over s, z' and θ and obtain in an analogous manner to the proof in part 2°:

$$\int_{\partial S}\int_{I'_\theta}|\varphi(y)|^2 r_2(y)^{-2\alpha+2q-1}\mu_\theta(y)\,dy\,d\theta$$
$$\leqslant \omega_n(1+\varepsilon')^{2\alpha-2q+1}(m-q+1)\sum_{|j|=k\leqslant m-q-1}\frac{1}{(k\,!)^2(2k+1)}$$
$$\times\left(\frac{\varepsilon'}{1+\varepsilon'}\right)^{2k+1}\int |D_j\varphi(x)|^2 r_2(x)^{-2\alpha+2(k+q)}dx$$
$$+\frac{\varepsilon'^{2m-2q}(1+2\varepsilon')^{2\beta+1}(m-q+1)}{(2m-2q-1)(2m-2q)((m-q-1)\,!)^2}\,\omega_n\varepsilon'$$
$$\times\sum_{|j|=m-q}\int_{U_{\varepsilon'}}|D_j\varphi(y)|^2 r_2(y)^{-2\beta}\,dy.$$

Here we used again the fact that $\mu_\theta(y)<\varepsilon' r_2(y)$. The lower bound for the left side is obtained as in part 2°. We sum up the inequalities for $\varphi = D_i u$ over all indicial systems, with $|i|<m$ and arrive at (9.17).

Theorem III. — *If* $(\overline{D}_2)^\circ = D_2$ [35], $U_\eta = U_\eta(\overline{D}_1, \overline{D}_2)$, $u \in \check{P}^\alpha(U_\eta^+)$, $J_{\alpha, U_\eta^+, D_2}(u) < \infty$ *and* φ *is the singular multiplier corresponding to the triple* $\{D_1, D_2, \eta/2\}$ *then* φu *extended by* 0 *outside* U_η^+ *is in* $P^\alpha(R^n)$ *and*

$$|\varphi u|^2_{\alpha, R^n} \leqslant c(J_{\alpha, U_\eta^+, D_2}(u) + |u|^2_{\alpha, U_\eta^+}) \tag{9.22}$$

where c *depends only on* α^*, η, *and* n.

Proof. — Without further mention we will use the properties of the singular multiplier φ as given in Theorem III, § 8 and we shall use c as a generic constant which depends only on α^*, η, and n.

By Theorem I, *a*) (with $D_1 = R^n - \overline{D}_2$) it is enough to prove the following three statements:

$$\varphi u \in P^\alpha_{\mathrm{loc}}(R^n - \overline{D}_2), \tag{9.23}$$
$$J_{\alpha, R^n - \overline{D}_2, D_2}(\varphi u) \leqslant c J_{\alpha, U_\eta^+, D_2}(u), \tag{9.24}$$
$$|\varphi u|^2_{\alpha, R^n - \overline{D}_2} \leqslant c(J_{\alpha, U_\eta^+, D_2}(u) + |u|^2_{\alpha, U_\eta^+}). \tag{9.25}$$

[35] This condition is needed to guarantee that if $D_1 = R_n - \overline{D}_2$ then

$$(D_1 \cup D_2 \cup (\partial D_1 \cap \partial D_2))^\circ = R^n,$$

cf. Theorem I.

As before, we shall let $r_1(y) = r_{D_1}(y)$, $r_2(y) = r_{D_2}(y)$, $m = [\alpha]$ and $\beta = \alpha - m$.

Proof of (9.23). — If $x \in R^n - \overline{D}_2$ then there is a neighborhood of x, N_x, such that the ratio $\dfrac{r_1(y)}{r_2(y)}$ is either $> \eta/2$ or $< \eta$ for $y \in N_x$. From the properties of φ it is clear, then, that $\varphi u \in P^{\alpha}_{loc}\ (R^n - \overline{D}_2)$.

Proof of (9.24). — Clearly it is sufficient to consider only points in U_η. Let $|i| \leqslant m$ then (cf. (2.5) for the summation notation)

$$r_2(x)^{-2\alpha+2|i|}|D_i(\varphi u)|^2 = \Big(\sum_{j\cup k=i}(D_j\varphi)r_2(x)^{|j|}(D_k u)r_2(x)^{-\alpha+|k|}\Big)^2$$
$$\leqslant \sum_{j\subset i}|D_j\varphi|^2\, r_2(x)^{2|j|}\sum_{k\subset i}|D_k u|^2\, r_2(x)^{-2\alpha+2|k|}.$$

Since $\sum_{j\subset i}|D_j\varphi|^2\, r_2(x)^{2|j|} \leqslant c$, (9.24) is immediate.

Proof of (9.25). — Let $|i| \leqslant m$. By Prop. 6), § 2 and (9.24) we have

$$(9.26)\quad \int_{R^n-\overline{D}_2}|D_i(\varphi u)|^2\,dx = \int_{\substack{U_\eta\\ r_2(x)<1}} + \int_{\substack{U_\eta\\ r_2(x)>1}}$$
$$\leqslant J_{\alpha,U_\eta,D_2}(\varphi u) + c|u|^2_{m,U_\eta}$$
$$\leqslant c(J_{\alpha,U^+_\eta,D_2}(u) + |u|^2_{m,U^+_\eta}).$$

If $\beta = 0$ then (9.25) is immediate so we suppose $\beta > 0$. Put $\gamma = \dfrac{\eta}{2(1+\eta)}$. Then (cf. (2.5) for the summation notation),

$$(9.27)\quad d_{\beta,R^n-\overline{D}_2}(D_i(\varphi u))$$
$$\leqslant \frac{1}{C(n,\beta)}\Big[2\int_{U_\eta}\int_{R^n-\overline{U}_\eta}\frac{|D_i(\varphi u)(x)|^2}{|x-y|^{n+2\beta}}\,dy\,dx$$
$$+ \int_{U_\eta}\int_{\substack{U_\eta\\ |x-y|>\gamma r_2(x)}}\frac{|D_i(\varphi u)(x) - D_i(\varphi u)(y)|^2}{|x-y|^{n+2\beta}}\,dy\,dx$$
$$+ 2^{|i|+1}\int_{U_\eta}\int_{\substack{U_\eta\\ |x-y|<\gamma r_2(x)}}\sum_{j\cup k=i}\frac{|D_k u(y)|^2|D_j\varphi(x) - D_j\varphi(y)|^2}{|x-y|^{n+2\beta}}\,dy\,dx$$
$$+ 2^{|i|+1}\int_{U_\eta}\int_{\substack{U_\eta\\ |x-y|<\gamma r_2(x)}}\sum_{j\cup k=i}\frac{|D_j\varphi(x)|^2|D_k u(x) - D_k u(y)|^2}{|x-y|^{n+2\beta}}\,dy\,dx\Big].$$

We denote the integrals by I_1, I_2, I_3, and I_4 respectively. If $x \in U_{\eta/2}$ and $|y - x| < \gamma r_2(x)$ then

$$r_1(y) \leqslant r_1(x) + \gamma r_2(x) < (\eta/2 + \gamma) r_2(x) = \frac{\eta(2+\eta)}{2(1+\eta)} r_2(x)$$

and $r_2(y) \geqslant r_2(x) - \gamma r_2(x) = \dfrac{2+\eta}{2(1+\eta)} r_2(x)$; hence $r_1(y) < \eta r_2(y)$ and:

(9.28) *If* $x \in U_{\eta/2}$ *and* $|x - y| < \gamma r_2(x)$ *then* $y \in U_\eta$.

Since φu vanishes outside $U_{\eta/2}$ it follows from (9.28), (9.26) and (9.24) that

$$\begin{aligned}
I_1 &\leqslant \int_{U_{\eta/2}} |D_i(\varphi u)(x)|^2 \int_{|x-y|>\gamma r_2(x)} |x-y|^{-n-2\beta}\, dy\, dx \\
&\leqslant \frac{\gamma^{-2\beta}\omega_n}{2\beta}\Bigg[\int_{\substack{U_{\eta/2}\\ r_2(x)<1}} |D_i(\varphi u)(x)|^2 r_2(x)^{-2\beta}\, dx \\
&\qquad\qquad + \int_{\substack{U_{\eta/2}\\ r_2(x)\geqslant 1}} |D_i(\varphi u)(x)|^2 dx\Bigg] \\
&\leqslant \frac{c}{\beta}\left[\int_{U_{\eta/2}} |D_i(\varphi u)(x)|^2 r_2(x)^{-2\alpha+2|i|} dx + \int_{U_{\eta/2}} |D_i(\varphi u)(x)|^2\, dx\right] \\
&\leqslant \frac{c_1}{\beta}\,(J_{\alpha, U_\eta, D_2}(u) + |u|^2_{m,U_\eta}).
\end{aligned}$$

For $|x - y| > \gamma r_2(x)$ we have $|x - y| > \gamma(r_2(y) - |x - y|)$ or $|x - y| > \dfrac{\gamma}{1+\gamma} r_2(y)$ and in an analogous manner to the above,

$$\begin{aligned}
I_2 &\leqslant 2\int_{U_{\eta/2}} |D_i(\varphi u)(x)|^2 \int_{|x-y|>\gamma r_2(x)} |x-y|^{-n-2\beta}\, dy\, dx \\
&\quad + 2\int_{U_{\eta/2}} |D_i(\varphi u)(y)|^2 \int_{|x-y|>\frac{\gamma}{1+\gamma} r_2(y)} |x-y|^{-n-2\beta}\, dx\, dy \\
&\leqslant \frac{c}{\beta}\,(J_{\alpha, U_\eta, D_2}(u) + |u|^2_{m,U_\eta}).
\end{aligned}$$

We note now two geometric properties immediately proved.

(9.29) *If* $|x - y| < \gamma r_2(x)$ *then* $|x - y| < \gamma' r_2(y)$ *with*

$$\gamma' = \frac{\eta}{2+\eta}.$$

(9.30) *If* $|x - y| < \gamma r_2(x)$ *and* z *lies in the segment* $[x; y]$ *then* $r_2(z) \geqslant \dfrac{2+\eta}{2(1+\eta)} r_2(x)$.

Evaluating $|D_j\varphi(x) - D_j\varphi(y)|$ for $|j| \leqslant m$ and $|x - y| < \gamma r_2(x)$ in terms of derivatives along the segment $[x; y]$ we obtain from (9.30), $|D_j\varphi(x) - D_j\varphi(y)|^2 \leqslant c|x - y|^2 r_2(y)^{-2|j|-2}$.

Therefore a typical term in I_3 is majorated by (cf. (9.29))

$$c\int_{U_\eta} |D_k u(y)|^2 r_2(y)^{-2|j|-2} \int_{|x-y|<\gamma' r_2(y)} |x - y|^{-n+2-2\beta}\, dx\, dy$$
$$\leqslant \frac{c_1}{1-\beta}\left[\int_{U_\eta} |D_k u|^2 r_2(y)^{-2\alpha+2|k|}\, dy + \int_{U_\eta} |D_k u|^2\, dy\right].$$

Consider now a typical term in I_4 with $|k| < m$. Since $\varphi(x) = 0$ outside of $U_{\eta/2}$ we can restrict x to $U_{\eta/2}$ and majorate this term as follows:

$$c\int_{U_{\eta/2}}\int_{\substack{U_\eta \\ |x-y|<\gamma r_2(x)}} r_2(x)^{-2|j|}\left(\frac{\gamma r_2(x)}{|x-y|}\right)^{1-\beta} \frac{|D_k u(y) - D_k u(x)|^2}{|x-y|^{n+2\beta}}\, dy\, dx$$
$$= c_1 \int_{\partial S}\int_{E_\theta(U_{\eta/2})}\int_{I(\theta,z)'} r_2(z' + s\theta)^{-2|j|+1-\beta}$$
$$\times \int_s^{s+\gamma r_2(z'+s\theta)} \frac{|D_k u(z' + t\theta) - D_k u(z' + s\theta)|^2}{(t-s)^{2+\beta}}\, dt\, ds\, dz'\, d\theta,$$

where we have used the change of variables given in (4.1). Now since $|k| < m$, $D_k u(z' + t\theta)$ is absolutely continuous for almost all θ and z'. By applying Hardy's inequality (9.14) to the integration with respect to t, then returning to the previous variables and using (9.29) we have the majoration

$$c\int_{U_{\eta/2}}\int_{\substack{U_\eta \\ |x-y|<\gamma r_2(x)}} r_2(x)^{-2|j|+1-\beta}\left(\sum_{l=1}^n \left|\frac{\partial}{\partial y_l} D_k u(y)\right|^2 |x - y|^{-n+1-\beta}\, dy\right) dx$$
$$\leqslant c_1 \int_{U_\eta} \sum_{l=1}^n \left|\frac{\partial}{\partial y_l} D_k u(y)\right|^2 r_2(y)^{-2|j|+1-\beta} \int_{|x-y|<\gamma' r_2(y)} |x - y|^{-n+1-\beta}\, dx\, dy$$
$$\leqslant \frac{c_2}{1-\beta}\sum_{l=1}^n\left[\int_{\substack{U_\eta \\ r_2(y)<1}} \left|\frac{\partial}{\partial y_l} D_k u(y)\right|^2 r_2(y)^{-2\alpha+2|k|+2}\, dy + \int_{\substack{U_\eta \\ r_2(y)>1}} \left|\frac{\partial}{\partial y_l} D_k u(y)\right|^2 dy\right].$$

If in I_4, $|k| = m$, then the corresponding term is obviously majorated by $C(n, \beta)\, d_{\beta, U_\eta}(D_k u)$ since $0 \leqslant \varphi(x) \leqslant 1$. Finally, we put all the majorations for the different terms on the right hand side of (9.27) and obtain from (2.7) and (9.26),

$$|D_i(\varphi u)|^2_{\beta, R^n - \bar{D}_2} \leqslant c(J_{\alpha, U_\eta^+, D_2}(u) + |u|^2_{\beta, U_\eta^+})$$

from which (9.25) follows.

Remarks on spaces $\check{P}^{\alpha,p}(D)$. For $1 < p < \infty$, all theorems of this section are valid with standard changes in proofs. The expression $J_{\alpha, D_1, D_2}(u)$ is now replaced by

$$(9.31) \quad J_{\alpha, p, D_1, D_2}(u) = \sum_{|i| \leqslant \alpha^*} \int_{D_1} |u(x)|^p r_{D_2}(x)^{-p(\alpha - |i|)}\, dx.$$

The inequality of Lemma 3 is now replaced by

$$(9.32) \quad C \int_a^b |u(x) - u(a)|^p (x - a)^{-p\alpha}\, dx < d_{\alpha, p, (a, b)}(u) \equiv \frac{1}{C(1, \alpha)} \int_a^b \int_a^b \frac{|u(x) - u(y)|^p}{|x - y|^{1+p\alpha}}\, dx\, dy$$

for $\dfrac{1}{p} < \alpha < 1$. The constant C is given by

$$(9.32') \quad C = \frac{2}{C(1, \alpha)} \int_0^1 \frac{(1 - t^{\alpha - 1/p})^p}{(1 - t)^{1+p\alpha}}\, dt > \frac{2(\alpha - 1/p)^p}{p(1 - \alpha)C(1, \alpha)}.$$

Hardy's inequality (9.14) becomes now, for $\gamma > 1/p$

$$(9.33) \quad \int_a^b |f(s) - f(a)|^p (s - a)^{-p\gamma}\, ds \leqslant (\gamma - 1/p)^{-p} \int_a^b |f'(s)|^p (s - a)^{-p\gamma + p}\, ds.$$

In case $p = \infty$ all the theorems hold; their proofs can be shortened considerably. As usual, the corresponding expressions and formulas are obtained by taking p-th roots in case $p < \infty$ and putting $p \nearrow \infty$. This gives

$$J_{\alpha, \infty, D_1, D_2}(u) = \max_{|i| \leqslant \alpha^*} \sup_{x \in D_1} [|D_i u(x)| r_{D_2}(x)^{-\alpha + |i|}],$$

$$\sup_{a < x < b} [|u(x) - u(a)|(x - a)^{-\alpha}] \leqslant d_{\alpha, \infty (a, b)}(u) \equiv \sup_{a < x, y < b} [|u(x) - u(y)||x - y|^{-\alpha}], \quad 0 < \alpha < 1,$$

$$\sup_{a < s < b} [|f(s) - f(a)|(s - a)^{-\gamma}] \leqslant \gamma \sup_{a < s < b} [|f'(s)|(s - a)^{-\gamma + 1}], \quad \gamma > 0.$$

For $p = 1$ the results are much less satisfactory. It can be stated as a general rule that each part of our theorems where the norm of u is evaluated remains valid; however, where the expression J is evaluated, the inequality is still valid for α non-integer but the constants obtained by our proofs converge to ∞ when α approaches an integer. This is due to the fact that Lemma 3 (inequality (9.32)) is not applicable because of the condition $1/p < \alpha < 1$ and that Hardy's inequality (9.33) is not valid for $\gamma = 1 = 1/p$. Thus, Theorem I, *a*) is valid whereas Theorem I, *b*) is not valid for α integer ([36]). Theorem II is valid for all α; however our proof gives constants converging to ∞ when α approaches an integer from below (this may be the fault of the proof). Theorem III is valid without exceptions.

10. Extension theorems.

In this section we will describe procedures for the construction of rather general domains with extension theorems. We build them by putting together a finite or infinite number of domains for which the extension theorems hold. First we shall give a few definitions.

A closed (bounded or unbounded) set Q is called a *q-cell* (*quasi-cell*) if $Q = \overline{Q^\circ}$ and its interior Q° satisfies the (C)-condition. The opening $\varkappa$ and radius r of conoids involved in this condition will be called the *(C)-constants* of Q. If G is any (C)-domain, then $\overline{G}$ is a q-cell with interior which in general is larger than G.

To simplify notation we will often write $\check{P}^\alpha(Q)$ for $\check{P}^\alpha(Q^\circ)$ and similarly for other classes defined for open sets.

We will be interested in systems $\{Q_k\}$ of q-cells, finite or enumerable. The system is *δ-loose*, $\delta > 0$, if for any k, l, either $Q_k \cap Q_l \neq 0$ or $\mathrm{dist}(Q_k, Q_l) \geqslant 4\delta$. The system is of rank $p < \infty$ if for every k, $Q_k \cap Q_l \neq 0$ for at most p indices l. Obviously every finite system of bounded q-cells is loose and of finite rank.

It is clear that for a loose system $\{Q_k\}$ of finite rank, $\bigcup_k Q_k$ is a closed set. We will want it to be a q-cell, and for this we need a uniformity condition. We say that the system $\{Q_k\}$

([36]) Simple examples show that Theorem 1, *b*) is not true when $p = 1$, $\alpha = 1$ $n = 1$, $D_1 = (0; 1)$, $D_2 = (-1; 0)$.

is *uniform* if it is loose, of finite rank and the (C)-constants of all Q_k's have uniform positive lower bounds. These lower bounds are the (C)-constants of $\bigcup_k Q_k$ which is then a q-cell. For a uniform system $\{Q_k\}$, the looseness-constant δ, the rank p and the lower bounds for the (C)-constants of the Q_k are called the *uniformity constants* of the system. Obviously again, every finite system of bounded q-cells is uniform. We also note that if $\{Q_k\}$ is a uniform system then

$$|\partial(\cup Q_k)| = |\cup \partial Q_k| = 0.$$

For a given uniform system $\{Q_k\}$ we define the *star* σ_k of Q_k as $\sigma_k = \Big(\bigcup_{Q_l \cap Q_k \neq 0} Q_l\Big)^{\circ}$ and the δ-star U_k (δ being the looseness-constant of the system) as $U_k = \sigma_k \cup S(Q_k, 2\delta)$ [37]. Clearly $Q_k^{\circ} \subset \sigma_k$ and

1) $\{U_k\}$ *is a δ-loose open covering of* $\bigcup_l Q_l$ *of rank* p^3 *(p being the rank of $\{Q_k\}$);* $U_k \cap (\cup Q_l)^{\circ} = \sigma_k$.

We can now state our first theorem.

THEOREM I. — *Let $\{Q_k\}$ be a uniform system of q-cells. Denote $D_k = Q_k^{\circ}$ and let $u_k \in \check{P}^{\alpha}(D_k)$. In order that there exist a simultaneous extension $\tilde{u} \in P^{\alpha}(R^n)$ for all the u_k's it is necessary and sufficient that there exist for each u_k an extension $\tilde{u}_k \in P^{\alpha}(R^n)$ such that*

$$(10.1a) \qquad \sum_k |\tilde{u}_k|^2_{\alpha, R^n} < \infty \quad \textit{and} \quad \sum_{\substack{k,l \\ Q_k \cap Q_l \neq 0}} J_{\alpha, D_k, D_l}(u_k - \tilde{u}_l) < \infty.$$

If $\tilde{u}$ is given then the $\tilde{u}_k$'s can be chosen explicitly as linear expressions in $\tilde{u}$ so that

$$(10.1b) \qquad \sum_k |\tilde{u}_k|^2_{\alpha, R^n} + \sum_{\substack{k,l \\ Q_k \cap Q_l \neq 0}} J_{\alpha, D_k, D_l}(u_k - \tilde{u}_l) \leqslant c' |\tilde{u}|^2_{\alpha, R^n}.$$

If the $\tilde{u}_k$'s are given, then a $\tilde{u}$ can be constructed explicitly as a linear expression in the $\tilde{u}_k$'s with

$$(10.1c) \qquad |\tilde{u}|^2_{\alpha, R^n} \leqslant c'' \Big[\sum_k |\tilde{u}_k|^2_{\alpha, R^n} + \sum_{\substack{k,l \\ Q_k \cap Q_l \neq 0}} J_{\alpha, D_k, D_l}(u_k - \tilde{u}_l)\Big].$$

[37] $S(E, \delta) = [x : \operatorname{dist}(x, E) < \delta]$.

The constants c' and c'' depend only on n, α^* and the uniformity constants of $\{Q_k\}$.

Proof. — 1° Suppose that $\tilde{u}$ exists. Denote by $\tau_k(x)$ the function constructed in Lemma 1, § 1 which is in $C^\infty(R^n)$, *is* $=1$ on $S\left(Q_k, \frac{\delta}{2}\right)$ and $=0$ outside of $S(Q_k, \delta)$. We put

$$\tilde{u}_k = \tau_k \tilde{u}. \tag{10.2}$$

By Prop. 7), § 2, we have $|\tilde{u}_k|^2_{\alpha, R^n} \leqslant c|\tilde{u}|^2_{\alpha, S(Q_k, 2\delta)}$, c depending only on α^*, n, and δ. Furthermore, on D_k,

$$u_k(x) - \tilde{u}_l(x) = (1 - \tau_l(x))\tilde{u}(x)$$

and since $1 - \tau_l$ is a multiplier of order α^* whose Lipschitz constant depends only on n, α^* and δ, and $1 - \tau_l$ vanishes for $r_{D_l}(x) < \delta/2$ we have by Prop. 6), § 2

$$J_{\alpha, D_k, D_l}(u_k - \tilde{u}_l) = J_{\alpha, D_k, D_l}((1 - \tau_l)\tilde{u}) \leqslant \left(\frac{\delta}{2}\right)^{-2\alpha} |(1 - \tau_l)\tilde{u}|^2_{m, D_k}$$

$$\leqslant c_1 \left(\frac{\delta}{2}\right)^{-2\alpha} |\tilde{u}|^2_{m, D_k},$$

c_1 depending only on n, α^* and δ. Thus

$$\sum_{\substack{k \\ Q_k \cap Q_l \neq 0}} J_{\alpha, D_k, D_l}(u_k - \tilde{u}_l) \leqslant pc_1 \left(\frac{\delta}{2}\right)^{-2\alpha} |\tilde{u}|^2_{m, U_l},$$

and by Prop. 1)

$$\sum_k |\tilde{u}_k|^2_{\alpha, R^n} + \sum_{\substack{k, l \\ Q_k \cap Q_l \neq 0}} J_{\alpha, D_k, D_l}(u_k - \tilde{u}_l)$$

$$\leqslant p^3 \left(pc_1 \left(\frac{\delta}{2}\right)^{-2\alpha} + c\right) |\tilde{u}|^2_{\alpha, R^n},$$

which proves (10.1*b*).

2° Suppose now that the $\tilde{u}_k$'s exist. Fix an index k and consider all indices l such that $Q_k \cap Q_l \neq 0$. Let these indices be $l_1 < l_2 < \ldots < l_q$, $q \leqslant p$ (k is among these indices). We construct successively functions $\varphi_1, \varphi_2, \ldots, \varphi_q$ all in $P^\alpha(R^n)$, the function φ_i being a simultaneous extension of all u_{l_j} with $j \leqslant i$. We put $\varphi_1 = \tilde{u}_{l_1}$. Suppose that φ_i is already constructed for $i < q$. If $D_{l_{i+1}} \subset \bigcup_{j=1}^{i} Q_{l_j}$, we can obviously put $\varphi_{i+1} = \varphi_i$ since ∂Q_l has measure 0 and $u_{l_{i+1}} = u_{l_j}$ on $D_{l_{i+1}} \cap D_{l_j}$ (in view of (10.1a)). If now $D_{l_{i+1}} \not\subset \bigcup_{j=1}^{i} Q_{l_j}$ we proceed as follows. We notice first that

$$J_{\alpha, D_{l_{i+1}}, \bigcup_{j \leqslant i} D_{l_j}}(\tilde{u}_{l_{i+1}} - \varphi_i) \leqslant \sum_{j=1}^{i} J_{\alpha, D_{l_{i+1}}, D_{l_j}}(\tilde{u}_{l_{i+1}} - \varphi_i)$$
$$\leqslant 2 \sum_{j=1}^{i} [J_{\alpha, D_{l_{i+1}}, D_{l_j}}(u_{l_{i+1}} - \tilde{u}_{l_j}) + J_{\alpha, D_{l_{i+1}}, D_{l_j}}(\tilde{u}_{l_j} - \varphi_i)].$$

By Theorem I, b), § 9,

$$J_{\alpha, D_{l_{i+1}}, D_{l_j}}(\tilde{u}_{l_j} - \varphi_i) \leqslant J_{\alpha, R^n, D_{l_j}}(\tilde{u}_{l_j} - \varphi_i) \leqslant c|\tilde{u}_{l_j} - \varphi_i|^2_{\alpha, R^n},$$

and hence $J_{\alpha, D_{l_{i+1}}, \bigcup_{j \leqslant i} D_{l_j}}(\tilde{u}_{l_{i+1}} - \varphi_i) < \infty$. Applying now Theorems II and III of § 9, and using the singular multiplier φ_i corresponding to the triple $\left\{Q_{l_{i+1}}, \bigcup_{j=1}^{i} Q_{l_j}, 1/26\right\}$ [37] we can put

$$\varphi_{i+1} = \varphi_i(\tilde{u}_{l_{i+1}} - \varphi_i) + \varphi_i. \tag{10.3}$$

The function φ_q will be denoted $\varphi^{(k)}$. It is easy to check by following the inductive definition of the φ_i's and using the evaluations in Theorems Ib), II, and III, § 9 that

$$|\varphi^{(k)}|^2_{\alpha, R^n} \leqslant c_1 \left[\sum_{i=1}^{q} |\tilde{u}_{l_i}|^2_{\alpha, R^n} + \sum_{1 \leqslant j < i \leqslant q} J_{\alpha, D_{l_i}, D_{l_j}}(u_{l_i} - \tilde{u}_{l_j})\right]$$

with c_1 depending only on n, α^*, and the uniformity constants of $\{Q_k\}$. In the second sum there might be terms corresponding to $Q_{l_i} \cap Q_{l_j} = 0$. Such a term can obviously be majorated by $(1 + (4\delta)^{-2\alpha})\ 2(|\tilde{u}_{l_i}|^2_{\alpha, R^n} + |\tilde{u}_{l_j}|^2_{\alpha, R^n})$. Replacing c_1 by

$$c_2 = c_1[1 + 2p(1 + (4\delta)^{-2\alpha})]$$

we can write

$$|\varphi^{(k)}|^2_{\alpha, R^n} \leqslant c_2 \left[\sum_{\substack{l \\ Q_l \cap Q_k \neq 0}} |\tilde{u}_l|^2_{\alpha, R^n} + \Sigma J_{\alpha, D_{l'}, D_{l''}}(u_{l'} - \tilde{u}_{l''})\right], \tag{10.4}$$

[37] We choose $\eta = 1$ in Theorem II, § 9, hence $\varepsilon = 1/13$ and in Theorem III the multiplier corresponds to $\left\{Q_{l_{i+1}}, \bigcup_{j=1}^{i} Q_{l_j}, 1/26\right\}$.

the last sum extended over all indices l', l'', with $Q_{l'} \cap Q_{l''} \neq 0$, $Q_{l'} \cap Q_k \neq 0$ and $Q_{l''} \cap Q_k \neq 0$.

We now take the partition of unity ψ_k corresponding to the δ-loose covering $\{U_k\}$ with rank p^3 of $\bigcup_k Q_k$ (see Lemma 2, § 5) and form

$$\tilde{u} = \sum_k \psi_k v^{(k)}. \tag{10.5}$$

Since for $x \in D_l$, $\psi_k(x) = 0$ except when $Q_l \cap Q_k \neq 0$ and then $v^{(k)}(x) = u_l(x)$ by construction, $\tilde{u}$ is an extension of u_l for every l. We obtain by using Prop. 1),

$$|\tilde{u}|^2_{\alpha,R^n} \leqslant 2p^3 \sum_k |\psi_k v^{(k)}|^2_{\alpha,R^n} \leqslant 2p^3 c \sum_k |v^{(k)}|^2_{\alpha,R^n}$$
$$\leqslant 2p^3 cc_2 \left[p \sum_l |\tilde{u}_l|^2_{\alpha,R^n} + p \sum_{Q_{l'} \cap Q_{l''} \neq 0} J_{\alpha,D_{l'},D_{l''}}(u_{l'} - \tilde{u}_{l''})\right]$$

where c depends only on n, $\alpha^\star$ and δ, hence (10.1c). The inductive definition of the $v^{(k)}$'s (see (10.3)) and formula (10.5) show, finally, that $\tilde{u}$ is linear in the $\tilde{u}_k$'s.

Remark 1. — In the subsequent theorems of this section we shall use Theorem I to construct extension theorems for domains which are unions of q-cells. To formalize this construction let $\{Q_k\}$ be a uniform system of q-cells such that $Q_k^\circ \in \mathcal{E}(I)$ and $D = \left(\bigcup_k Q_k\right)^\circ$. Let E_k be the linear extension mapping defined on $\mathfrak{D}_{E_k} \subset \mathfrak{M}(Q_k^\circ)$ into $\mathfrak{M}(R^n)$ and

$$\mathfrak{D}_E = \{u : u \in \mathfrak{M}(D),\ u|_{Q_k^\circ} \in \mathfrak{D}_{E_k} \text{ for all } k\}.$$

(Cf. the proof of Theorem II, §7). Clearly $\bigcup_{\alpha \in I} \check{P}^\alpha(D) \subset \mathfrak{D}_E$ and $\mathfrak{D}_E$ is a linear space. We set $u_k = u|_{Q_k^\circ}$, $\tilde{u}_k = E_k u_k$ and $Eu = \tilde{u}$ as given by (10.3) and (10.5). Eu so defined gives a linear extension mapping of $\mathfrak{D}_E$ into $\mathfrak{M}(R^n)$, i.e. this mapping satisfies condition (7.1a). However, additional conditions have to be imposed on the system $\{Q_k\}$ in order to guarantee that condition (7.1b) be satisfied.

THEOREM II. — *Let* $\{Q_k\}$ *be a uniform system of q-cells such that* Q_k° *and* $(Q_k \cup Q_l)^\circ$ *for* $Q_k \cap Q_l \neq 0$ *belong to* $\mathcal{E}(I, \Gamma)$. *Then* $D = \left(\bigcup_k Q_k\right)^\circ \in \mathcal{E}(I, c\Gamma)$ *where* $c \equiv c(I') \equiv c([\alpha_1, \alpha_2])$ *is a function only of* α_2^*, n, *and the uniformity constants of the system.*

Proof. — Let E be the mapping given in Remark 1. Then if $I' = [\alpha_1, \alpha_2] \subset I$, $\alpha \in I'$ and $u \in \check{P}^\alpha(D)$, it follows that (in the notation of Remark 1)

$$\Sigma|\tilde{u}_k|^2_{\alpha, R^n} \leqslant \Gamma(I')^2 \, \Sigma|u_k|^2_{\alpha, Q_k^o} \leqslant 2p\Gamma(I')^2|u|^2_{\alpha, D}.$$

If $u_{k,l}$ is the restriction of u to $(Q_k \cup Q_l)^o$ for $Q_k \cap Q_l \neq 0$, let $\tilde{u}_{k,l} \in P^\alpha(R^n)$ be the extension of $u_{k,l}$ with

$$|\tilde{u}_{k,l}|_{\alpha, R^n} \leqslant \Gamma(I')|u_{k,l}|_{\alpha, (Q_k \cup Q_l)^o}.$$

Then by Theorem 1, *b*), § 9,

$$\begin{aligned} J_{\alpha, Q_k^o, Q_l^o}(u_k - \tilde{u}_l) &= J_{\alpha, Q_k^o, Q_l^o}(\tilde{u}_{k,l} - \tilde{u}_l) \\ &\leqslant J_{\alpha, R^n, Q_l^o}(\tilde{u}_{k,l} - \tilde{u}_l) \leqslant c|\tilde{u}_{k,l} - \tilde{u}_l|^2_{\alpha, R^n} \\ &\leqslant 2c\Gamma(I')^2[|u_{k,l}|^2_{\alpha, (Q_k \cup Q_l)^o} + |u|^2_{\alpha, Q_l^o}] \end{aligned}$$

where c depends only on α_2^*, n and the uniformity constants. Hence by Prop. 1),

$$\begin{aligned} \sum_{\substack{k,l \\ Q_k \cap Q_l \neq 0}} J_{\alpha, Q_k^o, Q_l^o}(u_k - \tilde{u}_l) &\leqslant 2c\Gamma(I')^2 \sum_l \, [2p|u|^2_{\alpha, \sigma_l} + p|u|^2_{\alpha, Q_l^o}] \\ &\leqslant 4c\Gamma(I')^2(2p^4 + p^2)|u|^2_{\alpha, D}. \end{aligned}$$

and the proof is completed by (10.1*c*).

For our next theorem we must introduce some additional definitions.

For two closed non-empty sets $F_1 \neq F_2$ we define the *slope* $\omega \equiv \omega(F_1, F_2) = \omega(F_2, F_1)$ as follows

$$(10.6) \qquad \omega = \omega(F_1, F_2) = \inf_{x \notin F_1 \cap F_2} \frac{r_{F_1}(x) + r_{F_2}(x)}{\min(1, r_{F_1 \cap F_2}(x))}.$$

When $\omega = 0$, F_1 and F_2 are said to be *tangential*, otherwise *non-tangential.*

If $F_1 \cap F_2 = 0$ then $\omega(F_1, F_2) = \text{dist}(F_1, F_2)$. From this definition we deduce the following useful facts.

2) *If* F_1 *and* F_2 *are tangential then at least one of the two statements is true:* a) *There exists a sequence* $\{x^{(k)}\} \subset F_1$ *such that* $|x^{(k)}| \to \infty$, $r_{F_2}(x^{(k)}) \to 0$ *and* $r_{F_2}(x^{(k)})/r_{F_1 \cap F_2}(x^{(k)}) \to 0$; b) *there exists a sequence* $\{x^{(k)}\} \subset F_1$ *such that*

$$x^{(k)} \to x^{(0)} \in F_1 \cap F_2 \qquad \textit{and} \qquad r_{F_2}(x^{(k)})/r_{F_1 \cap F_2}(x^{(k)}) \to 0.$$

3) *If* F_1 *and* F_2 *are non-tangential then:* a) *for* $x \in F_1$ *and* $r_{F_2}(x) < \omega$, $r_{F_2}(x) \leqslant r_{F_1 \cap F_2}(x) \leqslant \frac{1}{\omega} r_{F_2}(x)$; b) *for* $x \in F_1$ *and* $r_{F_1 \cap F_2}(x) \geqslant 1$, $r_{F_2}(x) \geqslant \omega$.

A system $\{Q_k\}$ of q-cells is called *regular* if 1° *it is uniform*, 2° there is an ω_0, $0 < \omega_0 \leqslant 1$, such that for any distinct Q_k and Q_l with $Q_k \cap Q_l \neq 0$, $\omega(Q_k, Q_l) \geqslant \omega_0$ and 3° for every $y \in \partial(\cup Q_k)$ there exist arbitrarily small neighborhoods V_y such that $V_y \cap (\cup Q_k)^o$ is connected (i.e. the boundary $\partial(\cup Q_k)$ does not cut locally $(\cup Q_k)^o$). The uniformity-constants of $\{Q_k\}$, together with the bound ω_0 will be called the *regularity constants* of the system [38].

Two q-cells are called *adjacent* if their intersection is at least $(n - 1)$-dimensional.

THEOREM III. — *Let* $\{Q_k\}$ *be a regular system such that* $Q_k^o \in \mathcal{E}(I,\Gamma)$, $k = 1, 2, \ldots,$ *and* $(Q_k \cup Q_l)^o \in \mathcal{E}(I,\Gamma)$ *for all couples of adjacent q-cells. Then* $D = (\cup Q_k)^o \in \mathcal{E}(I, c\Gamma)$ *where* $c \equiv c(I') \equiv c([\alpha_1, \alpha_2])$ *depends only on* α_2^*, n, *and the regularity constants of* $\{Q_k\}$.

Remark 2. — We do not know of any union of two intersecting q-cells with an extension theorem for $\alpha \geqslant 1$, where the q-cells are non-adjacent or tangential; in fact, if n is odd, and D_1 and D_2 are arbitrary open sets such that $(\overline{D}_1 \cup \overline{D}_2)^o \in \mathcal{E}(I)$ with $(n + 1)/2 \in I$ then it can be shown that $\omega(\overline{D}_1, \overline{D}_2) \geqslant c/\Gamma$ where c, $0 < c \leqslant 2/3$, depends only on n and Γ is the extension constant of D for $\alpha = (n + 1)/2$. Furthermore, if ∂D cuts D locally, it can be shown that if $D \in \mathcal{E}(I)$ then $I \subset [0, n/2]$ (and we do not know of any domain which does have an extension theorem for $1 \leqslant \alpha \leqslant n/2$ if ∂D cuts D locally). This indicates that conditions 2° and 3° in the definition of a regular system are essentially necessary for the validity of Theorem III and that in practice Theorem III is stronger than Theorem II.

Proof of Theorem III. — We will need a topological lemma. We introduce first the notion of a *chain:* a sequence $Q_{l_0}, Q_{l_1}, \ldots, Q_{l_i}$ is a *chain connecting* Q_{l_0} *with* Q_{l_i} if any two consecutive q-cells are adjacent.

[38] Since $\omega(F_1, F_2) = \mathrm{dist}(F_1, F_2)$ for $F_1 \cap F_2 = 0$, we could combine the looseness of the uniform system and 2° in the single requirement that $\omega(Q_k, Q_l) \geqslant 4\delta$ for $k \neq l$.

LEMMA 1. — *In a regular system* $\{Q_k\}$ *for any* $y \in Q_{k'} \cap Q_{k''}$ *there exists a chain* $Q_{l_0}, Q_{l_1}, \ldots, Q_{l_i}$, $l_0 = k'$, $l_i = k''$ *such that* $y \in Q_{l_j}$, $j = 0, 1, \ldots, i$.

Proof. — Consider the class Λ_y of all Q_k's containing y. Obviously Λ_y contains at most p q-cells. For two Q_k's to be connected by a chain in Λ_y is an equivalence relation. We claim that there is only one equivalence class in Λ_y. In fact, suppose that there is more than one, and let Λ' be one such class. Put

$$\overline{G}_1 = \bigcup_{Q_k \in \Lambda'} Q_k, \qquad \overline{G}_2 = \bigcup_{Q_k \in \Lambda_y - \Lambda'} Q_k,$$
$$G_1 = (\overline{G}_1)^o \neq 0, \qquad G_2 = (\overline{G}_2)^o \neq 0.$$

Obviously $\overline{G}_1 \cap \overline{G}_2$ is of dimension $< n - 1$. Since $\{Q_k\}$ is a *regular* system we can find a small neighborhood V_y of y such that $V_y \cap (\cup Q_k)^o$ is connected and also that

$$\begin{aligned}(V_y \cap G_1) \cup (V_y \cap G_2) &\subset V_y \cap (\cup Q_k)^o \\ &= V_y \cap (\overline{G}_1 \cup \overline{G}_2)^o \subset (V_y \cap G_1) \cup (V_y \cap G_2) \cup (V_y \cap \overline{G}_1 \cap \overline{G}_2).\end{aligned}$$

$(V_y \cap G_1)$ and $(V_y \cap G_2)$ being non empty and disjoint this would mean that the open connected set $V_y \cap (\cup Q_k)^o$ is disconnected by an at most $(n-2)$-dimensional closed set $\overline{V}_y \cap \overline{G}_1 \cap \overline{G}_2$ which is impossible. Thus the lemma is proved.

Consider any $Q_{k'}$ and $Q_{k''}$ with $Q_{k'} \cap Q_{k''} \neq 0$. If we restrict ourselves only to *minimal* chains Γ connecting $Q_{k'}$ with $Q_{k''}$ (i.e. such that there is no chain $\Gamma' \subset \Gamma$ of smaller length connecting $Q_{k'}$ with $Q_{k''}$) it is clear that there are at most 2^p minimal chains Γ connecting $Q_{k'}$ with $Q_{k''}$ and satisfying $\bigcap_{Q_l \in \Gamma} Q_l \neq 0$. Number them $\Gamma^1, \ldots, \Gamma^N$, $N \leqslant 2^p$ and put

$$F^q = \bigcap_{Q_l \in \Gamma^q} Q_l, \quad q = 1,2, \ldots, N, \quad F^q \neq 0. \tag{10.8}$$

Obviously, by Lemma 1

$$Q_{k'} \cap Q_{k''} = \bigcup_{q=1}^{N} F^q. \tag{10.9}$$

Let $u \in \check{P}^\alpha(D)$, $\alpha \in I' = [\alpha_1, \alpha_2] \subset I$ and $\tilde{u} = Eu$ as given in Remark 1. We use the notation of Remark 1 and in addition denote by $\tilde{u}_{k,l}$ the extension of $u|_{(Q_k \cup Q_l)^o}$ for Q_k and Q_l adjacent as given in the hypotheses of the theorem.

We shall prove for $Q_{k'}$ and $Q_{k''}$ for which $Q_{k'} \cap Q_{k''} \neq 0$ that (σ_k being the star of Q_k)

$$(10.10) \quad J_{\alpha, Q^\circ_{k'}, Q^\circ_{k''}}(u - \tilde{u}_{k''}) \leqslant \omega_0^{-2\alpha}(2 + 16p^2 2^p c)\Gamma(I')^2 |u|^2_{\alpha, \sigma_{k'}} \text{ [39]}$$

where c depends only on α_2^*, n, and the uniformity constants of the system. Thus by (10.1c) and Prop. 1)

$$|\tilde{u}|^2_{\alpha, R^n} \leqslant c''\Gamma(I')^2 \left[\sum_k |u|^2_{\alpha, Q^\circ_k} + p\omega_0^{-2\alpha}(2 + 16p^2 2^p c) \sum_{k'} |u|^2_{\alpha, \sigma_{k'}}\right]$$
$$\leqslant 2c''\Gamma(I')^2 [p + p^4 \omega_0^{-2\alpha}(2 + 16p^2 2^p c)] |u|^2_{\alpha, D}.$$

which will complete the proof.

We pass now to the proof of (10.10). In the following we shall use the expression $J_{\alpha, D, F}(v)$ when F is not necessarily an open set and also the notation of (10.8) and (10.9).

By virtue of condition 2° of regular systems, Prop. 3a) and (10.9) we have

$$(10.11a) \quad J_{\alpha, Q^\circ_{k'}, Q^\circ_{k''}}(u - \tilde{u}_{k''}) \leqslant \omega_0^{-2\alpha}[|u - \tilde{u}_{k''}|^2_{\alpha, Q^\circ_{k'}} + J_{\alpha, Q^\circ_{k'}, Q_{k'} \cap Q_{k''}}(u - \tilde{u}_{k''})],$$

$$(10.11b) \quad J_{\alpha, Q^\circ_{k'}, Q_{k'} \cap Q_{k''}}(u - \tilde{u}_{k''}) \leqslant \sum_{q=1}^{N} J_{\alpha, Q^\circ_{k'}, F^q}(u - \tilde{u}_{k''}).$$

Let $Q_{l_0}, Q_{l_1}, \ldots, Q_{l_i}$, $l_0 = k'$, $l_i = k''$, $i \leqslant p - 1$ be the chain Γ^q. Then

$$J_{\alpha, Q^\circ_{k'}, F^q}(u - \tilde{u}_{k''}) = J_{\alpha, Q^\circ_{k'}, F^q}(\tilde{u}_{l_0} - \tilde{u}_{l_i})$$
$$\leqslant p \sum_{j=0}^{i-1} J_{\alpha, Q^\circ_{k'}, F^q}(\tilde{u}_{l_j} - \tilde{u}_{l_{j+1}})$$
$$\leqslant 2p \sum_{j=0}^{i-1} [J_{\alpha, Q^\circ_{k'}, F^q}(\tilde{u}_{l_j} - \tilde{u}_{l_j, l_{j+1}}) + J_{\alpha, Q^\circ_{k'}, F^q}(\tilde{u}_{l_{j+1}} - \tilde{u}_{l_j, l_{j+1}})]$$
$$\leqslant 2p \sum_{j=0}^{i-1} [J_{\alpha, R^n, Q^\circ_{l_j}}(\tilde{u}_{l_j} - \tilde{u}_{l_j, l_{j+1}}) + J_{\alpha, R^n, Q^\circ_{l_{j+1}}}(\tilde{u}_{l_{j+1}} - \tilde{u}_{l_j, l_{j+1}})].$$

Applying Theorem I b), § 9 to the square bracketed terms we have by the hypotheses of our theorem

$$J_{\alpha, Q^\circ_{k'}, F^q}(u - \tilde{u}_{k''}) \leqslant 2pc \sum_{j=0}^{i-1} [|\tilde{u}_{l_j} - \tilde{u}_{l_j, l_{j+1}}|^2_{\alpha, R^n} + |\tilde{u}_{l_{j+1}} - \tilde{u}_{l_j, l_{j+1}}|^2_{\alpha, R^n}]$$
$$\leqslant 16pc\Gamma(I')^2 \sum_{j=0}^{i-1} |u|^2_{\alpha, (Q_{l_j} \cup Q_{l_{j+1}})^p}$$
$$\leqslant 16p^2 c\Gamma(I')^2 |u|^2_{\alpha, \sigma_{k'}},$$

[39] We assume without loss of generality that $\omega_0 \leqslant 1$.

where c depends only on α_2^*, n and the uniformity constants of the system.

Combining the above with (10.11a) and (10.11b) we get (10.10) and the proof is complete.

The next theorem and its corollary aim at replacing in Theorem III the condition $(Q_k \cup Q_l)^\circ \in \mathcal{E}(I, \Gamma)$ for adjacent q-cells by a geometric property of the couple of cells.

THEOREM IV. — *Let* D_1 *and* D_2 *be such that* $\overline{D}_1 \cap \overline{D}_2 \neq 0$, $\overline{D}_1^\circ = D_1$, $\overline{D}_2^\circ = D_2$ *and* $\omega \equiv \omega(\overline{D}_1, \overline{D}_2) > 0$ *and let*

$$D = (\overline{D}_1 \cup \overline{D}_2)^\circ.$$

Suppose that one of the following two conditions holds:

a) $\overline{D}_1$ *and* $\overline{D}_1 \cap \overline{D}_2$ are q-cells;

b) *there is an* L-*convex domain* G *such that*

$$\overline{D}_1 \cap \overline{D}_2 \subset \overline{G} \subset \overline{D}_1 \cup \overline{D}_2$$

and $\overline{D}_1$, $\overline{D}_2$, $\overline{D}_1 \cap \overline{G}$ *and* $\overline{D}_2 \cap \overline{G}$ *are q-cells.*

If $u \in \check{P}^\alpha(D)$ *and* $u|_{D_k}$ *has an extension* $\tilde{u}_k \in P^\alpha(R^n)$, $k = 1,2$, *then* u *has an extension* $\tilde{u}$ *in* $P^\alpha(R^n)$ *which depends linearly on* $\tilde{u}_1$ *and* $\tilde{u}_2$ *and satisfies*

$$(10.12) \qquad |\tilde{u}|^2_{\alpha, R^n} \leqslant c_\alpha[|\tilde{u}_1|^2_{\alpha, R^n} + |\tilde{u}_2|^2_{\alpha, R^n} + |u|^2_{\alpha, D}].$$

In case a) c_α *depends only on* ω, n, α^*, *and the* (C)-*constants of* $\overline{D}_1$ *and* $\overline{D}_1 \cap \overline{D}_2$; *in case* b) c_α *depends only on* ω, n, α^*, *the* (C)-*constants of* $\overline{D}_1$, $\overline{D}_2$, $\overline{D}_1 \cap \overline{G}$ *and* $\overline{D}_2 \cap \overline{G}$ *and the* L-*convexity constants of* G.

This theorem has as an immediate consequence.

COROLLARY I. — *If* D_1 *and* D_2 *satisfy the hypotheses of Theorem* IV *either in case* a) *or case* b) *and in addition* D_1 *and* D_2 *are in* $\mathcal{E}(I, \Gamma)$ *then* $D = (\overline{D}_1 \cup \overline{D}_2)^0 \in \mathcal{E}(I, c'\Gamma)$ *where* $c'(I') \equiv c'([\alpha_1, \alpha_2]) = (3 \max_{\alpha \leqslant \alpha_2} c_\alpha)^{1/2}$ *and* c_α *is the constant of Theorem IV.*

Proof of Theorem IV. — Let $\omega_0 = \min(1, \omega)$. a) Since $\tilde{u}_1 - \tilde{u}_2$ vanishes in $D_1 \cap D_2$ we have by Prop. 3 (and noting that $r_{\overline{D}_1 \cap \overline{D}_2}(x) = r_{(\overline{D}_1 \cap \overline{D}_2)^0}(x) = r_{D_1 \cap D_2}(x)$),

$$J_{\alpha, D_1, D_2}(\tilde{u}_1 - \tilde{u}_2) \leqslant \omega_0^{-2\alpha} [|\tilde{u}_1 - \tilde{u}_2|^2_{\alpha, D_1} + J_{\alpha, D_1, D_1 \cap D_2}(\tilde{u}_1 - \tilde{u}_2)]$$

and by Theorem Ib), § 9,

$$J_{\alpha,D_1,D_1\cap D_2}(\tilde{u}_1 - \tilde{u}_2) \leqslant J_{\alpha,R^n,D_1\cap D_2}(\tilde{u}_1 - \tilde{u}_2) \leqslant c|\tilde{u}_1 - \tilde{u}_2|^2_{\alpha,R^n}.$$

Thus if φ is the singular multiplier corresponding to the triple $\{D_1, D_2, 1/26\}$ we have by Theorems II and III of § 9 [40] that $\tilde{u} = \varphi(\tilde{u}_1 - \tilde{u}_2) + \tilde{u}_2$ is an extension of u, is in $P^\alpha(R^n)$ and by an application of the bounds in the cited theorems satisfies (10.12) [41] which proves a).

b) Sinces $u - \tilde{u}_2$ vanishes on $G \cap D_2$ (and $\tilde{u}_1 - \tilde{u}_2 = u - \tilde{u}_2$ on $G \cap D_1$) we have by Theorem Ib), § 9,

$$J_{\alpha,G\cap D_1,G\cap D_2}(\tilde{u}_1 - \tilde{u}_2) \leqslant J_{\alpha,G,G\cap D_2}(u - \tilde{u}_2) \leqslant c|u - \tilde{u}_2|^2_{\alpha,G}.$$

Therefore if φ_1 is the singular multiplier corresponding to the triple $\{G \cap D_1, G \cap D_2, 1/26\}$ we have by Theorems II and III of § 9 that $v_1 = \varphi_1(\tilde{u}_1 - \tilde{u}_2) + \tilde{u}_2$ is in $P^\alpha(R^n)$, is an extension of $u|_G$ and satisfies $|v_1|^2_{\alpha,R^n} \leqslant c[|\tilde{u}_1|^2_{\alpha,R^n} + |\tilde{u}_2|^2_{\alpha,R^n} + |u|^2_{\alpha,D}]$.

Thus with the couple D_1 and G we are in case a) [42] and obtain an extension v_2 of $u|_{(\overline{G}\cup\overline{D}_1)^0}$ to $P^\alpha(R^n)$. Now with the couple D_2 and $(\overline{G} \cup \overline{D}_1)^0$ we again are in case a) and obtain finally the required extension. The bound in (10.12) is obtained by applying the bounds of the cited theorems and case a). This completes the proof of Theorem IV.

Another useful result for applications is:

Corollary II. — *If* Q_1 *and* Q_2 *are intersecting q-cells in* $\mathcal{E}(I, \Gamma)$ *and* $D = (Q_1 \cup Q_2)^0$ *is* L-*convex then* $D \in \mathcal{E}(I, c\Gamma)$ *where* $c(I') \equiv c([\alpha_1, \alpha_2])$ *depends only on* n, α_2^*, *the* (C)-*constants of* Q_1 *and* Q_2, *and the* L-*convexity constants of* D.

Proof. — If $u \in \check{P}^\alpha(D)$ and in the first paragraph of the proof of Theorem IV, b) we replace $G \cap D_1$, $G \cap D_2$ and G by Q_1^0, Q_2^0 and D respectively, then v_1 is the desired extension [43]. It is to be noticed that this part of the proof of Theorem IV b) does not need the hypothesis $\omega(\overline{D}_1, \overline{D}_2) > 0$.

[40] We choose $\eta = 1$ in Theorem II. Thus in Theorem III, $\eta/2 = 1/26$.

[41] The term $|u|^2_{\alpha,D}$ on the right side of (10.12) may be omitted in this case.

[42] We use here the easily proved fact that if F_1, F_2, and F_3 are closed sets satisfying $0 \neq F_1 \cap F_2 \subset F_3 \subset F_1 \cup F_2$ then $\omega(F_3, F_1) \geqslant \frac{1}{2}\,\omega(F_2, F_1)$.

[43] This corollary can be considered as a special case of Corollary I by virtue of the easily proved lemma: if F_1 and F_2 are closed sets such that $(F_1 \cup F_2)^0$ is L-convex with L-convexity constants δ, p, and M, then $\omega(F_1, F_2) \geqslant \frac{1}{2}\min(\delta, M^{-2})$.

We finish this section by giving a simple extension theorem for orders $\alpha < 1$.

THEOREM V. — *Let $0 \leqslant \alpha_1 \leqslant \alpha_2 < 1$. If $\{Q_k\}$ is a uniform system and each Q_k is in $\mathcal{E}[\alpha_1, \alpha_2]$ then also $(\bigcup Q_k)^0 \in \mathcal{E}[\alpha_1, \alpha_2]$.*

The proof follows immediately from Theorem I since for $\alpha < 1$, Lemma 1 b), § 9 allows the majorations of

$$J_{\alpha, Q_k^0, Q_l^0}(u - u_l) \qquad \text{by} \qquad c|u - u_l|^2_{\alpha, (Q_k \cup Q_l)^0}$$

c depending only on n and the (C)-constants of Q_l.

Remarks on spaces $\check{P}^{\alpha,p}(D)$. — Since the results of the present section are based on those of section 9, all of them are valid for $1 < p \leqslant \infty$ when $\check{P}^{\alpha}(D)$, $P^{\alpha}(R^n)$, and $\mathcal{E}(I, \Gamma)$ are replaced by $\check{P}^{\alpha,p}(D)$, $\check{P}^{\alpha,p}(R^n)$ and $\mathcal{E}^{(p)}(I, \Gamma)$ respectively (see the corresponding remarks at the end of sections 7 and 9).

For $p = 1$ the situation is more complicated in view of the exceptions mentioned at the end of § 9. The extension mapping defined in Remark 1 of the present section gives still a simultaneous extension for all non-integral α, but for α integer $\check{P}^{\alpha,1}(D)$ will not in general be transformed into $\check{P}^{\alpha,1}(R^n)$ under the hypotheses of Theorems II or III. Furthermore there will be uniform extension constants only for closed intervals I' which do not contain any integer.

11. The generalized Lichtenstein extension.

In § 3 we introduced the Lichtenstein reflection of order $q < \infty$ across a hyperplane. We mentioned also that quite recently R. T. Seeley [11] defined a Lichtenstein reflection of order ∞ across a hyperplane. Using Seeley's basic idea we will define a Lichtenstein reflection of infinite order across a Lipschitzian graph. This will allow us to obtain corresponding simultaneous extensions for the whole infinite interval $0 \leqslant \alpha < \infty$ for SLG-domains and then, by localization, to LG-domains (see § 8 for definition).

To define the reflection of infinite order we put in (3.1) $h_\mu = 2^{\mu+2} - 1$, $\mu = 0, \ldots, q$, and obtain by (3.2) the corresponding coefficients $a_\mu(q)$. One sees immediately that for $q \to \infty$, $(-1)^\mu a_\mu(q) \nearrow (-1)^\mu a_\mu$, where

$$(11.1a)\quad 0 < (-1)^\mu a_\mu = \prod_{\sigma=1}^{\infty} \frac{1}{1-2^{-\sigma}} \prod_{\nu=1}^{\mu} \frac{1}{1-2^{-\nu}} 2^{-\mu(\mu+1)/2} < 2^4 2^{-\mu(\mu+1)/2}.$$

We set

$$(11.1b)\qquad b_\mu = h_\mu + 1 = 2^{\mu+2}, \quad \mu = 0,1, \ldots$$

The equations (3.1) transform then into $\sum_{\mu=0}^{q} a_\mu(q) b_\mu^p = 1$ or 0 depending on whether the integer $p = 0$ or $0 < p \leqslant q$. Going to the limit $q \to \infty$ (which is obviously permissible) we get immediately

$$(11.1c)\quad \sum_{\mu=1}^{\infty} a_\mu = 1, \quad \sum_{\mu=1}^{\infty} a_\mu b_\mu^p = 0 \qquad \textit{for} \qquad p = 1, 2, 3, \ldots$$

$$(11.1d)\quad \sum_{\mu=0}^{\infty} |a_\mu| b_\mu^p \leqslant 2^8\, 2^{p(p+3)/2} \qquad \textit{for} \qquad p \geqslant 0.$$

Consider now a Lipschitzian function f with positive lower bound and with Lipschitz constant $M \geqslant 1$ defined on a rectangle $B \subset R^{n-1}$, $B = [a_k < x_k < b_k,\ k = 1, \ldots, n-1]$. It is well known that there exist Lipschitzian extensions $\tilde{f}$ of f to the whole of R^{n-1} with the same lower bound and the same Lipschitz constant. We fix a standard procedure assigning to every f a well-determined $\tilde{f}$ [44]. We set then

(11.2a) $\rho(x)$ *is the regularized distance of Theorem II, § 8 corresponding to the domain* $[x : x_n < \tilde{f}(x')]$ *with the choice of* $\varepsilon = 1/2$.

$$(11.2b)\qquad \begin{cases} D = [x : x' \in B,\ x_n < f(x')], \\ \tilde{D} = [x : x' \in B], \\ D_+ = [x : x' \in B,\ x_n > f(x')], \\ G = [x : x' \in B,\ x_n = f(x')], \\ \lambda = \min(1, (b_k - a_k) \textit{ for } k = 1, \ldots, n-1). \end{cases}$$

[44] Such a standard procedure would be the one applied in the proof of Prop. 2, § 3, by using successively simple reflections (of order $q = 0$ with $h_0 = 1$) across all the faces of B. We obtain thus an extension $\tilde{f}^{(1)}$ of f to a rectangle $\tilde{B}^{(1)} \supset B$. Repeating this process we get successively $\tilde{f}^{(1)}, \tilde{f}^{(2)}, \ldots$, defined on $\tilde{B}^{(1)} \subset \tilde{B}^{(2)} \subset \cdots$ with $R^{n-1} = \cup \tilde{B}^{(k)}$.

The quantity λ introduced above is called the *minimal width* of the rectangle B.

Consider now functions $u \in \mathfrak{M}(D)$ such that

(11.2*c*) *For some* $d < 0$, $u(x) = 0$ *if* $x_n < d$ (*d depending on u*).

For such functions we define

$$(11.2d) \quad \begin{cases} \tilde{u}(x) = u(x) \quad \textit{for} \quad x \in D, \\ \tilde{u}(x) = \sum_{\mu=0}^{\infty} a_\mu u(x', x_n - b_\mu \rho(x)) \textit{ for } x \in D_+ \textit{ wherever it is defined,} \\ \tilde{u}(x) = \textit{correction of } \tilde{u} \textit{ (as defined above in } D \cup D_+) \\ \qquad \textit{for } x \in G \textit{ wherever the correction exists.} \end{cases}$$

The function $\tilde{u}$ restricted to D_+ is the Lichtenstein reflection of order ∞ of u across the Lispchitzian graph G.

By Theorem II, § 8, $\frac{\partial}{\partial x_n}(x_n - b_\mu \rho(x)) \leqslant 1 - \frac{1}{2} b_\mu < -1$; hence, if $x = (x', x_n) \in D_+$ then the points $(x', x_n - b_\mu \rho(x))$ are in D for all μ. Furthermore, in view of (11.2*c*),

$$u(x', x_n - b_\mu \rho(x)) \neq 0$$

for at most a finite number of μ's. Therefore (11.2 *d*) is defined a.e. and $\tilde{u} \in \mathfrak{M}(\tilde{D})$.

Let $\psi(x) \in C^\infty(R^n)$, $0 \leqslant \psi(x) \leqslant 1$, $\psi(x) = 0$ for $x_n \leqslant -2$ and $\psi(x) = 1$ for $x_n \geqslant -1$. Then we define the *generalized Lichtenstein extension* by

$$(11.3) \quad E : \mathfrak{M}(D) \to \mathfrak{M}(\tilde{D}), \qquad Eu = (1 - \psi)u + \widetilde{\psi u}$$

where $(1 - \psi)u$ is extended by 0 in $\tilde{D} - D$.

Our basic result in this section is

Theorem I. — *If* $u \in \check{P}^\alpha(D)$ *then* $Eu \in \check{P}^\alpha(\tilde{D})$ *and*

$$|Eu|_{\alpha,\tilde{D}} \leqslant c\lambda^{-\beta} M^{\alpha+n+3/2} |u|_{\alpha,D}$$

where $\beta = \alpha - [\alpha]$ *and c depend only on* α^* *and n.*

Since D is obviously a G-domain, Prop. 3), § 6 implies that $C_0^\infty(R^n)$ restricted to D is dense in $\check{P}^\alpha(D)$; hence by the functional space property of $\check{P}^\alpha(D)$ and $\check{P}^\alpha(\tilde{D})$ it is sufficient to consider only such restrictions for the proof of our theorem. It follows that it is enough to prove

THEOREM I′. — *If* u *is the restriction of a function in* $C_0^\infty(R^n)$ *to* D *then a)* $\tilde{u} \in C^\infty(\tilde{D})$; *b)* $|\tilde{u}|_{\alpha,\tilde{D}} \leqslant c\lambda^{-\beta}M^{\alpha+n+3/2}|u|_{\alpha,D}$ *where* c *depends only on* α^* *and* n.

In fact, if Theorem I′ is true and u is a restriction of a function in $C_0^\infty(R^n)$ then ψu is also such a function and by Prop. 6, § 2 and the present theorem

$$|\widetilde{\psi u}|_{\alpha,\tilde{D}} \leqslant C\lambda^{-\beta}M^{\alpha+n+3/2}|u|_{\alpha,D}$$

with C depending only on α^* and n. On the other hand, since $(1-\psi)u$ vanishes in $\tilde{D}$ for $x_n \geqslant -1$ we get, by applying again Prop. 6, § 2, and putting $m = [\alpha]$, $\beta = \alpha - m$,

$$|(1-\psi)u|^2_{\alpha,\tilde{D}} \leqslant |(1-\psi)u|^2_{\alpha,D} + \frac{\omega_n}{\beta C(n, \beta)}|(1-\psi)u|^2_{m,D} \leqslant C_1|u|^2_{\alpha,D}$$

and finally $|Eu|_{\alpha,D} \leqslant c\lambda^{-\beta}M^{\alpha+n+3/2}|u|_{\alpha,D}$ giving the statement of Theorem I for such functions u.

From now on till the end of the proof of Theorem I′, u will be a restriction to D of a function in $C_0^\infty(R^n)$. An inspection of the series (11.2d) giving $\tilde{u}(x)$ for $x \in D_+$ shows that for x in a compact $K \subset D_+$ the series contains only a finite number of non-vanishing terms, this number depending on the distance from K to G (and not on x in K). It follows that $\tilde{u}$ is in $C^\infty(D \cup D_+)$.

We establish next a formula for derivatives $D_i\tilde{u}(x)$, $x \in D_+$.

Let $y_l^{(\mu)}(x) = x_l$ for $1 \leqslant l < n$ and $y_n^{(\mu)}(x) = x_n - b_\mu\rho(x)$ in the formula (2.6). For $x \in D_+$ and any indicial set i, we have, following (11.2d) and (2.6),

$$\text{(11.4)} \quad D_i\tilde{u}(x) = \sum_{\mu=0}^{\infty} a_\mu \Big[\Sigma \frac{1}{|t|\,!}\Big\{\prod_{m=1}^{|t|} D_{s^{(m)}}y_{t_m}^{(\mu)}\Big\} (D_t u)(y_1^{(\mu)}(x), \ldots, y_n^{(\mu)}(x))\Big].$$

where the summation in square brackets is taken over all indicial sets t with $1 \leqslant |t| \leqslant |i|$, $1 \leqslant t_m \leqslant n$ and all indicial sets $s^{(m)}$, $m = 1, \ldots, |t|$, satisfying $\bigcup_{m=1}^{|t|} s^{(m)} = i$ and $|s^{(m)}| \geqslant 1$.

For any choice of $s^{(m)}$ and t satisfying the above conditions we will denote by $\tilde{r}$, $\hat{r}$, and $\bar{r}$, the three disjoint subsequences of the sequence $(1, 2, \ldots, |t|)$ defined as follows

$\tilde{r} = (\tilde{r}_1, \ldots, \tilde{r}_{|\tilde{r}|})$ *is formed by all integers* m, $1 \leqslant m \leqslant |t|$, *satisfying* $t_m = n$ *and* $s^{(m)} \neq (n)$,

$\hat{r} = (\hat{r}_1, \ldots, \hat{r}_{|\hat{r}|})$ *is formed by all integers* m, $1 \leqslant m \leqslant |t|$, *satisfying* $t_m = n$ *and* $s^{(m)} = (n)$,

$\bar{r} = (\bar{r}_1, \ldots, \bar{r}_{|\bar{r}|})$ *is formed by all integers* m, $1 \leqslant m \leqslant |t|$, *satisfying* $t_m \neq n$ *and* $s^{(m)} = (t_m)$.

We note that if there exists an integer

$$l \in (1, 2, \ldots, |t|) - (\tilde{r} \cup \hat{r} \cup \bar{r})$$

then $D_{s^{(l)}} y_{t_l}^{(\mu)} = 0$. Hence the only non-vanishing terms in the round brackets in (11.4) can be written

$$\prod_{m=1}^{|t|} D_{s^{(m)}} y_{t_m}^{(\mu)} = \prod_{m \in \tilde{r}} \prod_{m \in \hat{r}} = (-b_\mu)^{|\tilde{r}|} \prod_{m \in \tilde{r}} D_{s^{(m)}} \rho(x) \prod_{m \in \hat{r}} (1 - b_\mu D_{s^{(m)}} \rho(x))$$
$$= \sum_r (-b_\mu)^{|r|} \prod_{m \in r} D_{s^{(m)}} \rho(x),$$

where the summation is over all increasing sequences of integers satisfying $\tilde{r} \subset r \subset \tilde{r} \cup \hat{r}$. (Note that r may be empty if $\tilde{r}$ is empty.)

It follows that the whole sum in the square brackets of (11.4) can be written now in the form

$$(11.5) \qquad \sum_r \frac{1}{k!} (-b_\mu)^{|r|} \prod_{m \in r} D_{s^{(m)}} \rho(x) \, (D_t u) \, (y_1^{(\mu)}(x), \ldots, y_n^{(\mu)}(x))$$

where the summation is extended over all decompositions

$$(11.5a) \qquad i = \bigcup_{m=1}^{k} s^{(m)}, \qquad 1 \leqslant k \leqslant |i|, \qquad |s^{(m)}| \geqslant 1,$$

and all sequences r satisfying

$$(11.5b) \quad r \subset (1, 2, \ldots, k), \quad |s^{(m)}| = 1 \quad \textit{for} \quad m \notin r,$$

the indicial set t being determined by

$$(11.5c) \quad |t| = k, \quad (t_m) = s^{(m)} \;\; \textit{for} \;\; m \notin r, \quad t_m = n \;\; \textit{for} \;\; m \in r.$$

We shall separate in (11.5) the terms with $|r| = 0$. For such terms $|s^{(m)}| = 1$ for $m = 1, 2, \ldots, k$; hence, by (11.5a) and (11.5c), $k = |i|$ and the indicial set t is a permutation of the indicial set i. Since distinct permutations of the $|i|$ elements of i give rise by our conventions (see § 2) to distinct decompositions $i = \bigcup_{m=1}^{|i|} s^{(m)}$, we have exactly $|i|\,!$ terms with $|r| = 0$ each equal to $(|i|\,!)^{-1}(D_i u)(y_1^{(\mu)}, \ldots, y_n^{(\mu)})$. Therefore the formula (11.4) can now be written

$$(11.6)\quad D_i\tilde{u}(x) = \sum_{\mu=0}^{\infty} a_\mu (D_i u)(y_1^{(\mu)}(x), \ldots, y_n^{(\mu)}(x)) + \sum_{\mu=0}^{\infty} a_\mu \sum_{\{s^{(m)}\}, |r|\geqslant 1} \frac{1}{k\,!}(-b_\mu)^{|r|} \times \Big\{\prod_{m\in r} D_{s^{(m)}}\rho(x)\Big\}(D_t u)(y_1^{(\mu)}(x), \ldots, y_n^{(\mu)}(x)),$$

where the last summation is over all decompositions (11.5a) and all sequences r satisfying (11.5b) with $|r| \geqslant 1$, t being determined by (11.5c). From these conditions follows the relation

$$(11.6a)\qquad |i| = \sum_{m\in r} |s^{(m)}| + |t| - |r|.$$

By Taylor's theorem we may write for $\mu = 0,1, \ldots$ and any non-negative integer q

$$(11.7)\quad (D_t u)(x', x_n - b_\mu\rho(x)) = \sum_{l=0}^{q-1} \frac{(b_0 - b_\mu)^l}{l\,!}\rho(x)^l \left(\frac{\partial^l}{\partial x_n^l} D_t u\right)(x', x_n - b_0\rho(x)) + (b_0 - b_\mu)^q \rho(x)^q R(x, t, q, \mu)$$

where

$$(11.7a)\quad R(x, t, q, \mu) = \begin{cases} \dfrac{1}{(q-1)\,!}\displaystyle\int_0^1 \left(\frac{\partial^q}{\partial x_n^q} D_t u\right)(x', x_n - (\tau b_\mu + (1-\tau)b_0)\,\rho(x)) \times (1-\tau)^{q-1}\, d\tau, & \textit{for} \quad q \geqslant 1 \\ (D_t u)(x', x_n - b_\mu\rho(x)) & \textit{for} \quad q = 0. \end{cases}$$

In the double sum of (11.6) we change the order of summations and for each choice of $\{s^{(m)}\}$ and r replace

$$(D_t u)(y_1^{(\mu)}, \ldots, y_n^{(\mu)})$$

by the right-hand side of (11.7) with

$$q = \sum_{m\in r} |s^{(m)}| - |r| = |i| - |t|.$$

Summing then with respect to μ and using the second formula in (11.1c) we get our final formula for $D_i\tilde{u}(x)$, $x \in D_+$,

(11.8)

$$D_i\tilde{u}(x) = \sum_{\mu=0}^{\infty} a_\mu (D_i u)(x', x_n - b_\mu \rho(x)) + \sum_{\{s^{(m)}\}, |r| \geq 1} \frac{1}{|t|!} \rho(x)^{|i|-|t|} \times \prod_{m \in r} D_{s^{(m)}} \rho(x) \sum_{\mu=0} a_\mu (-b_\mu)^{|r|} (b_0 - b_\mu)^{|i|-|t|} R(x, t, |i| - |t|, \mu).$$

From the properties of $\rho(x)$ (see Theorem II, § 8) we deduce, in view of (11.6a)

$$\rho(x)^{|i|-|t|} \prod_{m \in r} |D_{s^{(m)}} \rho(x)| \leqslant CM^{|i|}, \tag{11.9}$$

where C depends only on $|i|$ and n.

Before we pass to the proof of Theorem I′ we prove three lemmas.

Lemma 1. — *The mapping* $T_{\tau,\mu} : D_+ \to D$ *defined by*

$$T_{\tau,\mu}(x', x_n) = (x', x_n - (\tau b_\mu + (1 - \tau) b_0)\rho(x)),$$
$$0 \leqslant \tau \leqslant 1, \quad \mu = 0,1, \ldots,$$

is a homeomorphism of D_+ *onto* D *and has the following properties:*

i) *the Jacobian of* $T_{\tau,\mu}^{-1}$ *is bounded by* 1,

ii) *for* $x, y \in D_+$, $|T_{\tau,\mu}(x) - T_{\tau,\mu}(y)| \leqslant cb_\mu M|x - y|$, *and for* $x \in \overline{D}$, $y \in D_+$, $|x - T_{\tau,\mu}(y)| \leqslant cb_\mu M|x - y|$, *where c depends only on n.*

Proof. — It is clear that $T_{\tau,\mu}$ is a continuous mapping of D_+ into D. For fixed x' : 1° $T_{\tau,\mu}$ transforms the half-line $[x', x_n > f(x')]$ into the half-line $[x', x_n < f(x')]$, 2° if $x_n \searrow f(x')$ then $x_n - (\tau b_\mu + (1 - \tau) b_0)\rho(x) \to f(x')$, and

3° $$\frac{\partial}{\partial x_n} [x_n - (\tau b_\mu + (1 - \tau) b_0)\rho(x)] < -1.$$

It follows that the first half-line is transformed homeomorphically onto the second and that $T_{\tau,\mu}$ is a homeomorphism of D_+ onto D.

Since $\frac{\partial}{\partial x_n}\rho(x) \geqslant \frac{1}{2}$ the Jacobian of $T_{\tau,\mu}$, $J(T_{\tau,\mu})$, satisfies

$$J(T_{\tau,\mu}) = \left|1 - (\tau b_\mu + (1-\tau)b_0)\frac{\partial}{\partial x_n}\rho(x)\right| \geqslant \frac{1}{2} b_0 - 1 \geqslant 1$$

which proves i).

By (11.2*a*) and Theorem II, §8, $\frac{\partial}{\partial x_l}\rho(x)$, $l = 1, \ldots, n$, are majorated by $B_1\left(\frac{1}{2}\right)^{-1} M$ (M is the Lipschitz constant of the graph function) where B_1 depends only on n. From this it follows that $\rho(x)$ is Lipschitzian on $\overline{D}_+$ with Lipschitz constant $2B_1M$. Hence for $x, y \in D_+$

$$\begin{aligned}&|T_{\tau,\mu}(x) - T_{\tau,\mu}(y)|^2\\ &= |x'-y'|^2 + |(x_n - y_n) - (\tau b_\mu + (1-\tau)b_0)(\rho(x)-\rho(y))|^2\\ &\leqslant |x'-y'|^2 + 2|x_n-y_n|^2 + 2(2B_1M)^2 b_\mu^2 |x-y|^2\end{aligned}$$

from which the first part of ii) follows.

From Theorem II and Remark I of § 8 we have for $y \in D_+$ that $\rho(y) \leqslant (1+M^2)^{1/2} r_D(y)$; hence for $x \in \overline{D}$, $y \in D_+$,

$$\begin{aligned}|x - T_{\tau,\mu}(y)|^2 &= |x'-y'|^2 + |(x_n - y_n) + (\tau b_\mu + (1-\tau)b_0)\rho(x)|^2\\ &\leqslant |x'-y'|^2 + 2|x_n-y_n|^2 + 2b_\mu^2(1+M^2)r_D(y)^2\end{aligned}$$

which completes the proof of ii).

LEMMA 2. — *For any indicial set* $i = (i_1, \ldots, i_{|i|})$, $1 \leqslant i_k \leqslant n$, *any* τ *with* $0 \leqslant \tau \leqslant 1$ *and any* $\mu = 0, 1, \ldots$ *define in* $\tilde{D}$ *the function* $v_{i,\tau,\mu}(x)$ *as follows:* 1° $v_{i,\tau,\mu}(x) = D_i u(x)$ *for* $x \in D$; 2° $v_{i,\tau,\mu}(x) = D_i u(x', x_n - (\tau b_\mu + (1-\tau)b_0)\rho(x)) \equiv D_i u(T_{\tau,\mu}(x))$ *for* $x \in D_+$; 3° $v_{i,\tau,\mu}(x)$ *so defined in* $D \cup D_+$ *is extended by continuity to the graph* G. *Then* $v_{i,\tau,\mu}$ *belongs to* $C^{(0,1)}(\tilde{D})$ *and to every* $\check{P}^\beta(\tilde{D})$ *for* $0 \leqslant \beta < 1$ *and we have*

$$(11.10a)\quad |v_{i,\tau,\mu}(x) - v_{i,\tau,\mu}(y)| \leqslant c b_\mu M^2 \sup_{\substack{|j|=|i|+1\\ z\in D}} |D_j u(z)||x-y|$$

for $x, y \in \tilde{D}$,

$$(11.10b)\quad |v_{i,\tau,\mu}\rceil_{\beta,\tilde{D}} \leqslant c(b_\mu M)^{n/2+\beta}|D_i u\rceil_{\beta,D} \quad \textit{for} \quad 0 \leqslant \beta < 1,$$

with constant c depending only on n.

Proof. — The following facts are immediate consequences of the properties of $\rho(x)$ and of our assumption that u is a restriction to D of a function in $C_0^\infty(R^n)$: 1° $\varphi_{i,\tau,\mu}$ is in $C^\infty(D \cup D_+)$ and vanishes outside of a bounded set, and 2°) $\varphi_{i,\tau,\mu}$ restricted to D or D_+ has a continuous extension to $\overline{D}$ or $\overline{D}_+$ respectively and these two extensions agree on the graph G. The last fact justifies our definition of $\varphi_{i,\tau,\mu}$ on G. To prove (11.10a), which also gives that $\varphi_{i,\tau,\mu} \in C^{(0,1)}(\tilde{D})$, we notice that any two points $x, y \in \tilde{D}$ can be connected by a polygonal line $P_{x,y} \subset \tilde{D}$ composed of two segments (one of them parallel to x_n-axis), such that $P_{x,y}$ intersects G in at most one point and is of total length $\leqslant 3M|x - y|$. Representing then $\varphi_{i,\tau,\mu}(y) - \varphi_{i,\tau,\mu}(x)$ as integral of derivative along $P_{x,y}$ and using the properties of $T_{\tau,\mu}$ (see Lemma 1, ii)) we arrive at (11.10a).

Since $\varphi_{i,\tau,\mu} \in C^{(0,1)}(\tilde{D})$, $\varphi_{i,\tau,\mu}$ belongs to $P^\beta_{\text{loc}}(\tilde{D})$ for $0 \leqslant \beta \leqslant 1$. To finish the proof it is enough now to show that (11.10b) holds. To this effect, we write for $0 \leqslant \beta < 1$

$$\begin{aligned}|\varphi_{i,\tau,\mu}]^2_{\beta,\tilde{D}} = \Big[\int_D |D_i u(x)|^2\,dx + \int_{D_+} |(D_i u)(T_{\tau,\mu}(x))|^2\,dx\Big] \\ + \frac{1}{C(n,\beta)}\Big[\int_D\int_D \frac{|D_i u(x) - D_i u(y)|^2}{|x-y|^{n+2\beta}}\,dx\,dy \\ + \int_{D_+}\int_{D_+} \frac{|(D_i u)(T_{\tau,\mu}(x)) - (D_i u)(T_{\tau,\mu}(y))|^2}{|x-y|^{n+2\beta}}\,dx\,dy \\ + 2\int_{D_+}\int_D \frac{|D_i u(x) - (D_i u)(T_{\tau,\mu}(y))|^2}{|x-y|^{n+2\beta}}\,dx\,dy\Big].\end{aligned}$$

We then transform the integrals over D_+ into integrals over D by using the transformation $T_{\tau,\mu}$. Lemma 1 gives then immediately

$$\begin{aligned}|\varphi_{i,\tau,\mu}]^2_{\beta,\tilde{D}} \leqslant 2\int_D |D_i u(x)|^2\,dx \\ + [1 + 3(cb_\mu M)^{n+2\beta}]\,\frac{1}{C(n,\beta)}\int_D\int_D \frac{|D_i u(x) - D_i u(y)|^2}{|x-y|^{n+2\beta}}\,dx\,dy,\end{aligned}$$

and this gives formula (11.10b).

LEMMA 3. — *Let* $\{s^{(m)}\}$ *and* r *be indicial sets, and* q *an integer,* $0 \leqslant q \leqslant |i| - 1$, *satisfying* $r \subset (1, 2, \ldots, |i|)$ *and*

$$\sum_{m \in r} |s^{(m)}| - |r| = q \qquad \textit{with} \qquad |s^{(m)}| \geqslant 1.$$

Then for any $v_{i,\tau,\mu}(x)$, *as given in Lemma* 2, *we define*

$$w_{i,\tau,\mu,\{s^{(m)}\},r,q}(x) \equiv w(x)$$

for $x \in \tilde{D}$ *by*

$$w(x) = \begin{cases} \rho(x)^q \prod_{m \in r} D_{s^{(m)}}\rho(x)(v_{i,\tau,\mu}(x) - v_{i,\tau,0}(x)) & \textit{for } x \in D_+ \\ 0 & \textit{for } x \in \tilde{D} - D_+. \end{cases}$$

Then w *belongs to* $C^{(0,1)}(\tilde{D})$ *and to every* $\check{P}^{\beta}(\tilde{D})$, $0 \leqslant \beta < 1$, *and*

$$(11.11a) \quad |w(x) - w(y)| \leqslant c' b_\mu M^{|i|+3} \sup_{\substack{|j|=|i|+1 \\ z \in D}} |D_j u(z)| \, |x - y|,$$

$$(11.11b) \quad |w\urcorner_{\beta,\tilde{D}} \leqslant c' b_\mu^{n/2+\beta} M^{\beta+|i|+n+3/2} \lambda^{-\beta} |D_i u\urcorner_{\beta, D}$$

for $0 \leqslant \beta < 1$, *where* c' *depends only on* $|i|$ *and* n.

Proof. — For notational convenience in this proof we let $\tilde{v}(x) = v_{i,\tau,\mu}(x) - v_{i,\tau,0}(x)$, $x \in \tilde{D}$, and

$$\Phi(x) = \rho(x)^q \prod_{m \in r} D_{s^{(m)}}\rho(x), \quad x \in D_+, \quad \text{and} \quad = 0 \quad \text{for} \quad x \in \tilde{D} - D_+.$$

Thus $w(x) = \Phi(x)\tilde{v}(x)$ for $x \in \tilde{D}$.

By Theorem II and Remark 1 of § 8,

1° $|\Phi(x)| \leqslant c_0 M^{|i|}$ *and* $|\nabla \Phi(x)| \leqslant c_0 M^{|i|+1} r_D(x)^{-1}$ *for* $x \in D_+$ *where* c_0 *depends only on* $|i|$ *and* n.

From (11.10a) it follows that

$$|\tilde{v}(x)| \leqslant 2cb_\mu^2 M^2 \sup_{\substack{|j|=|i|+1 \\ z \in D}} |D_j u(z)| r_D(x)$$

and $|\nabla \tilde{v}(x)| \leqslant 2cb_\mu M^2 \sup_{\substack{|j|=|i|+1 \\ z \in D}} |D_j u(z)|$ almost everywhere, c being the constant of Lemma 2.

Clearly $w(x)$ is absolutely continuous on $D \cup D_+$ and from the above, continuous on $\tilde{D}$ since

$$|\nabla w(x)| \leqslant 4cc_0 b_\mu M^{|i|+3} \sup_{\substack{|j|=|i|+1 \\ z \in D}} |D_j u(z)|$$

a.e. on $\tilde{D}$. Thus (11.11a) is satisfied and $w \in C^{(0,1)}(\tilde{D})$.
$w \in P^{\beta}_{\text{loc}}(\tilde{D})$, $0 \leqslant \beta < 1$, by virtue of the fact that $w \in C^{(0,1)}(\tilde{D})$. Thus to complete the proof of the lemma we need only prove

(11.11*b*). If $\beta = 0$ (11.11*b*) is clear from 1° and (11.10*b*). So we suppose $0 < \beta < 1$.

Let C be the cone defined by

$$C = [z = (z', z_n) : \quad z_n \geq 0, \ |z'| \leq |z|(1 + M^2)^{-1/2}].$$

Then it is easy to see that for $x = (x', x_n)$, $x \in D_+$ and $z \in C + x$ we have $r_D(z) \geq r_D(x)$. From this it follows that if x and y are points in D_+ there is a polygonal line $P_{x,y} \subset (C+x) \cup (C+y)$ composed of two segments of total lenth $\leq \sqrt{M^2 + 1}\, |x - y|$ and such that

$$\operatorname{dist}(P_{x,y}, D) \geq \min(r_D(x), r_D(y)).$$

Hence by 1° we have:

2° *If* $x,y \in D_+$, $r_D(x) \leq r_D(y)$ *then*

$$|\Phi(x) - \Phi(y)| \leq 2c_0 M^{|i|+2} r_D(x)^{-1} |x - y|.$$

If $x_0 \in \partial D$ it is easy to see by considering $D \cap (x_0 - C)$ that there is a $z \in D$, $|z - x_0| = 2\sqrt{n-1}\,\lambda M$ (λ is given by (11.2*b*)) such that $S(z, \lambda/2) \in D$. Hence:

3° *For any* $x \in D_+$ *and* $x_0 \in \partial D$ *such that* $r_D(x) = |x - x_0|$ *there is a conoid in* D *with vertex* x_0, *opening* $(4\sqrt{n-1}\,M)^{-1}$ *and radius* λM.

By 1° and 2° we have for $x \in D_+$,

$$\text{(11.12)} \quad \int_{\substack{D_+ \\ r_D(x) < r_D(y)}} \frac{|\Phi(x) - \Phi(y)|^2}{|x - y|^{n+2\beta}}\, dy$$
$$\leq \left[\int_{\substack{D_+, |x-y| < r_D(x) \\ r_D(x) < r_D(y)}} + \int_{\substack{D_+, |x-y| > r_D(x) \\ r_D(x) < r_D(y)}} \frac{|\Phi(x) - \Phi(y)|^2}{|x - y|^{n+2\beta}}\, dy \right]$$
$$\leq 4c_0^2 M^{2|i|} \Big[M^4 r_D(x)^{-2} \int_{|x-y| < r_D(x)} |x - y|^{2-n-2\beta}\, dy$$
$$+ \int_{|x-y| > r_D(x)} |x - y|^{-n-2\beta}\, dy \Big]$$
$$\leq \frac{2\omega_n c_0^2 M^{2|i|+4}}{\beta(1 - \beta)}\, r_D(x)^{-2\beta}.$$

By 3°, we have that $\tilde{D}$, D_+, and D satisfy the hypotheses of Lemma 4, § 9 (replacing D^*, D_1^*, and D_2^* respectively) with $\varkappa^* = (4\sqrt{n-1}\,M)^{-1}$ and $r^* = \lambda M$. Hence by 1°, (11.2), (9.13*b*), and (2.7),

$$(11.12b)\quad d_{\beta,\tilde{D}}(w) = \frac{1}{C(n,\beta)}\Bigg[\int_{D_+}\int_{D}\frac{|\Phi(x)\tilde{\varphi}(x)|^2}{|x-y|^{n+2\beta}}\,dy\,dx + 2\int_{D_+}\int_{\substack{D_+\\ r_D(x)<r_D(y)}}\frac{|\Phi(x)\tilde{\varphi}(x)-\Phi(y)\tilde{\varphi}(y)|^2}{|x-y|^{n+2\beta}}\,dx\,dy\Bigg]$$

$$\leqslant \frac{1}{C(n,\beta)}\Bigg[c_0^2 M^{2|i|}\int_{D_+}\int_{D}\frac{|\tilde{\varphi}(x)|^2}{|x-y|^{n+2\beta}}\,dy\,dx + 4\int_{D_+}|\Phi(y)|^2\int_{D_+}\frac{|\tilde{\varphi}(x)-\tilde{\varphi}(y)|^2}{|x-y|^{n+2\beta}}\,dx\,dy + 4\int_{D_+}|\tilde{\varphi}(x)|^2\int_{\substack{D_+\\ r_D(y)>r_D(x)}}\frac{|\Phi(x)-\Phi(y)|^2}{|x-y|^{n+2\beta}}\,dy\,dx\Bigg]$$

$$\leqslant c_0^2 M^{2|i|}d_{\beta,\tilde{D}}(\tilde{\varphi}) + 4c_0^2 M^{2|i|}d_{\beta,\tilde{D}}(\tilde{\varphi}) + 16nc_0^2 M^{2|i|+4}\int_{D_+}|\tilde{\varphi}(x)|^2 r_D(x)^{-2\beta}\,dx$$

$$\leqslant c_1 M^{2|i|+n+3}\lambda^{-2\beta}|\tilde{\varphi}|^2_{\beta,\tilde{D}},$$

where c_1 depends only on $|i|$ and n. The proof of (11.11*b*) is completed by (11.10*b*). This completes the proof of Lemma 3.

We shall use the following special convention for indicial sets:

(11.13) *If* $t = (t_1, \ldots, t_{|t|})$ *is an indicial set then*

$$t + \langle k \rangle = (t_1, \ldots, t_{|t|}, \underbrace{n, \ldots, n}_{k\text{-times}}).$$

Proof of Theorem I'. — Using the functions

$$\varphi_{i,\tau,\mu} \quad \text{and} \quad w_{i,\tau,\mu,\{s^{(m)}\},r,q}$$

introduced in Lemmas 2 and 3 we obtain from (11.7*a*), (11.8), (11.6*a*), and (11.1*c*) for $x \in \tilde{D}$

$$(11.14)\quad D_i\tilde{u}(x) = \sum_{\mu=0}^{\infty} a_\mu \varphi_{i,1,\mu}(x) + \sum_{\{s^{(m)}\},|r|\geqslant 1}\sum_{\mu=0}^{\infty} a_\mu(-b_\mu)^{|r|}(b_0-b_\mu)^{|i|-|t|}\frac{1}{|t|!(|i|-|t|-1)!} \times \int_0^1 w_{t+\langle|i|-|t|\rangle,\tau,\mu,\{s^{(m)}\},r,|i|-|t|}(x)(1-\tau)^{|i|-|t|-1}\,d\tau$$

Here, when $|i|-|t| = 0$, the expression

$$\frac{1}{(|i|-|t|-1)!}\int_0^1 \ldots d\tau$$

should be replaced by $w_{t,1,\mu,\{s^{(m)}\},r,0}(x)$. By (11.1$c$), it is clear for $x \in D_+$ that

$$\sum_{\mu=0}^{\infty} a_\mu(-b_\mu)^{|r|}(b_0 - b_\mu)^{|i|-|t|} w_{t+<|i|-|t|>,\dots,|i|-|t|}(x)$$
$$= \sum_{\mu=0}^{\infty} a_\mu(-b_\mu)^{|r|}(b_0 - b_\mu)^{|i|-|t|} \rho(x)^{|i|-|t|}$$
$$\times \prod_{m\in r} D_{s^{(m)}}\rho(x) \left(\frac{\partial^{|i|-|t|}}{\partial x_n^{|i|-|t|}} D_t u\right)(x', x_n - (\tau b_\mu + (1-\tau)b_0)\rho(x))$$

and so (11.14) is just a repetition of (11.8). Since for $x \in D$ all the $w_{t+<|i|-|t|>,\dots}(x)$'s vanish and $\varphi_{i,1,\mu}(x) = D_i u(x)$, hence by (11.1$c$) we have $D_i u(x)$ on the right side of (11.4) as we should.

Using Minkowski's inequality in (11.14) (for sums as well as for integrals) we obtain from (11.1d), (11.10a) and (11.11a)

$$(11.15a)\quad |D_i\tilde{u}(x) - D_i\tilde{u}(y)| \leqslant |x-y| \sup_{\substack{|j|=|i|+1\\ z\in D}} |D_j u(z)|$$
$$\times \left\{ cM^2 \sum_{\mu=0}^{\infty} |a_\mu| b_\mu + \sum_{\{s^{(m)}\},\,|r|\geqslant 1} \frac{c'M^{|i|+3}}{|t|\,!(|i|-|t|)\,!} \sum_{\mu=0}^{\infty} |a_\mu| b_\mu^{|r|+1}(b_\mu - b_0)^{|i|-|t|} \right\}$$

for $x, y \in \tilde{D}$.

From (11.1d), (11.10b), and (11.12b) we have

$$(11.15b)\quad \ulcorner D_i\tilde{u}\urcorner_{\beta,\tilde{D}} \leqslant cM^{n/2+\beta} \ulcorner D_i u\urcorner_{\beta,D} \sum_{\mu=0}^{\infty} |a_\mu| b_\mu^{n/2+\beta}$$
$$+ \sum_{\{s^{(m)}\},\,|r|\geqslant 1} \sum_{\mu=0}^{\infty} |a_\mu| b_\mu^{|r|+n/2+\beta}(b_\mu - b_0)^{|i|-|t|} \frac{1}{|t|\,!(|i|-|t|)\,!}$$
$$\times c'\lambda^{-\beta} M^{\beta+|i|+n+3/2} \ulcorner D_{t+<|i|-|t|>} u\urcorner_{\beta,D}$$
$$\leqslant c''\lambda^{-\beta} M^{\beta+|i|+n+3/2} \sum_{|j|=|i|} \ulcorner D_j u\urcorner_{\beta,D},$$

where c'' depends only on $|i|$ and n.

Formula (11.15a) shows that $D_i\tilde{u} \in C^{(0,1)}(\tilde{D})$. Since this is true for every indicial set i, $\tilde{u} \in C^\infty(\tilde{D})$. (11.15$b$) gives inequality b) of Theorem I′.

Remark 1. — For α an integer (i.e. $\beta = 0$) it is immediately seen by following the proofs of Lemmas 2 and 3 and of Theorem I′ that in inequality b) of Theorem I′ and the corresponding inequality of Theorem I, $\lambda^{-\beta}M^{\alpha+n+3/2}$ can by replaced by M^α.

Remark 2. — It is immediately checked that if the graph G is a hyperplane, i.e., $f(x') = c$ a constant, then we can replace $\rho(x)$ by $x_n - b_n$ and (11.14) becomes

(11.14')
$$D_i\tilde{u}(x) = \begin{cases} D_i u(x) & \text{for } x \in \tilde{D} - D_+ \\ \Sigma a_\mu (D_i u)(x', x_n - b_\mu(x_n - c)) & \text{for } x \in D_+, \end{cases}$$

and it is easy to see that $|D_i\tilde{u}\rceil_{\beta,\tilde{D}} \leqslant c|D_i u\rceil_{\beta,D}$ where c depends only on n.

Let D be an SLG-domain with basis B:

$$D = [x : a_k < x_k < b_k, k = 1,2, \ldots, n-1, a_n < x_n < a_n + f(x')]$$

with Lipschitz constant $M (M \geqslant 1)$, then we define:

$$(11.16) \quad \lambda_D = \min\left\{1/M, \; b_1 - a_1, \; \ldots, \; b_{n-1} - a_{n-1}, \; \inf_{x' \in B'} f(x')\right\}$$

the minimal width of the SLG-*domain* D.

THEOREM II. — *Let* D *be an* SLG-*domain with minimal width* λ_D. *Then* $D \in \mathcal{E}([0, \infty), \Gamma)$ *where*

$$\Gamma(I') \equiv \Gamma([\alpha_1, \alpha_2]) = c\lambda_D^{-(n+1)(\alpha_2^*+2)-3/2}$$

and c depends only on α_2^* *and n.*

Proof. — We suppose D is defined as above.

1° Consider the multiplier $\varphi_n(x_n)$ in one variable x_n (cf. Lemma 1, § 1) such that $\varphi_n(x_n) = 1$ for $x_n \leqslant a_n + \lambda_D/4$ and $= 0$ for

$$x_n \geqslant a_n + 3\lambda_D/4.$$

For a function $u \in \mathfrak{M}(D)$ consider the functions u' and u'' defined as follows: *i)* $u'(x) = \varphi_n(x_n)u(x)$ for $x \in D$, $u'(x) = 0$ for $x' \in B$, $x_n \geqslant a_n + 3\lambda_D/4$; *ii)* $u''(x) = (1 - \varphi_n(x_n))u(x)$ for $x \in D$, $u''(x) = 0$ for $x' \in B$, $x_n \leqslant a_n + \lambda_D/4$. Clearly, the function $u''(x)$ is defined in the domain $B \times [x_n < a_n + f(x')]$ which, if we shift the origin to the point $(x' = 0, x_n = a_n)$ becomes a function of the type treated in Theorem I. We can therefore apply the extension mapping of this theorem and obtain a function $\tilde{u}''$ defined in $B \times [-\infty < x_n < \infty]$. The function u' is defined in the domain $B \times [x_n > a_n]$. If we shift the origin to $[x' = 0, x_n = a_n + 1]$ and invert the direction of the x_n-axis this domain becomes a domain of the type treated in Theorem I (with graph function $f(x') = 1$) and by applying the extension mapping of this theorem we obtain a function $\tilde{u}'$ defined in $B \times [-\infty < x_n < \infty]$. Hence $\tilde{u} = \tilde{u}' + \tilde{u}''$ gives a linear extension mapping of $u \in \mathfrak{M}(D)$ into $\mathfrak{M}(B \times [-\infty < x_n < \infty])$. By applying Lemma 1, § 1, Prop. 6, § 2, and Theorem I, we obtain immediately for all $\alpha \geqslant 0$,

(11.17) *If* $u \in \check{P}^\alpha(D)$ *then* $\tilde{u} \in \check{P}^\alpha(B \times [-\infty < x_n < \infty])$

and

$$|\tilde{u}|_{\alpha, B\times[-\infty<x_n<\infty]} \leqslant c\lambda_D^{-(2\alpha+n+5/2)}|u|_{\alpha,D},$$

with c depending only on n and α^*.

2° If a function φ is defined in an n-dimensional rectangle $[x:\ a_k < x_k < b_k,\ k \leqslant n]$ and this rectangle is not the whole space R^n, then by interchanging the variables and, if necessary, shifting the origin and inverting the direction of the x_n-axis we can represent the rectangle in a form

$$[x:\ a'_k < x_k < b'_k,\ k \leqslant n]$$

where $a'_n \leqslant 0 < b'_n = \min(1,\ b'_n - a'_n)$. If a'_n is $-\infty$ we apply Remark 2 directly (the graph-function is now $f(x') = b_n$); if $a'_n > -\infty$ we apply the procedure given in the beginning of 1°. In both cases we obtain an extension

$$\tilde{\varphi} \in \mathfrak{M}([x:\ a'_k < x'_k < b',\ k \leqslant n-1]).$$

From Prop. 6, § 2, Lemma 1, § 1, and Remark 2 we get for any $\alpha \geqslant 0$:

(11.18) *If* $\varphi \in \check{P}^\alpha([x:\ a'_k < x_k < b'_k,\ k \leqslant n])$ *then*
$\tilde{\varphi} \in \check{P}^\alpha([x:\ a'_k < x_k < b'_k,\ k \leqslant n-1])$

and

$$|\tilde{\varphi}|_{\alpha,[x:a'_k<x_k<b'_k,k\leqslant n-1]} \leqslant c\lambda^{-(\alpha^*+1)}|\varphi|_{\alpha,[x:a'_k<x_k<b'_k,k\leqslant n]},$$

where $\lambda = \min(1,\ b'_n - a'_n)$ *and* c *depends only on* α^* *and* n.

3° Coming back to the hypotheses of Theorem II, we apply there the procedure of 1°, then, at most $n-1$ times, the procedure of 2° to obtain successively extension-mappings to larger and larger rectangles and arrive thus at an extension mapping of $\mathfrak{M}(D)$ into $\mathfrak{M}(R^n)$. Formula (11.17) in part 1° and formula (11.18) in part 2° guarantee that the extension constant $\Gamma(I')$ has the form stated in the theorem.

We now consider the property of a domain D to be an SLG-domain with minimal width $\geqslant \lambda_D > 0$. The strong localization of this property gives the property of being an LG-*domain.* The LG-constants of an LG-domain D are the looseness δ and the rank p of the corresponding covering $\{U_k\}$ and also the uniform bound of minimal widths of the SLG-domains $U_k \cap D$.

By the general localization theorem of § 7 (Theorem II) we obtain the

Corollary 1. — *Every* LG-*domain belongs to the class* $\mathcal{E}([0, \infty))$.

Remarks about spaces $\check{P}^{\alpha,p}(D)$. — All the results of this section are valid for $1 < p \leqslant \infty$, and all α, $\alpha \geqslant 0$. The proofs for $1 < p < \infty$ are essentially the same as those in case $p = 2$ presented in the text (with obvious changes dependent on the exponent p). The difference in proofs for $p = \infty$ is due to the fact that the restrictions to D of functions in $C_0^\infty(R^n)$ are not dense in $\check{P}^{\alpha,\infty}(D)$. However, the functions in $\check{P}^{\alpha,\infty}(D)$ are sufficient lyregular to allow all the developments preceding the proof of Theorem I′ without having to replace them by more regular functions.

For $p = 1$ one can follow the same line of argument as presented in the text for $p = 2$. A difficulty arises only in Lemma 3. This lemma is based on Lemma 4, § 9 which in turn relies on Lemma 3, § 9. The last lemma is not valid for $p = 1$. Without reference to Lemma 4, § 9, Lemma 3 of the present section is still valid for $\beta < 1$ but with constant which blows up when $\beta \nearrow 1$. Therefore our extension mapping will not have uniform extension constants $\Gamma([\alpha_1, \alpha_2])$ for intervals containing an integer. Curiously enough for α an integer, our extension mapping still transforms $\check{P}^{\alpha,1}(D)$ into $\check{P}^{\alpha,1}(R^n)$ since for an integer α we do not need Lemma 3 to prove Theorem I. We do not know if this seeming inconsistency is due to our line of proof or is in the essence of things.

12. Applications.

Associated with every convex domain D there is a unique *maximal cone* C with vertex at the origin such that if $x \in D$ then $x + C \subset D$ (or if $x \in \overline{D}$ then $x + C \in \overline{D}$) and $x + C$ contains all half lines issued from x which lie in D (or in $\overline{D}$). Such cones are closed and convex. If D is bounded then $C = (0)$ and conversely. The dimension of this maximal cone will enter in two of the following propositions.

A *spherical cone* with vertex at the origin is the closed cone generated by the sphere $S(\theta,\varkappa)$, $|\theta| = 1$ and $\varkappa \leqslant 1$. We call $\varkappa$ the *opening*, γ the *angular opening* where $\sin \gamma = \varkappa$ and the half line $l = [t\theta : t \geqslant 0]$ the *central axis* of the cone. If

$$C = C' \cap \overline{S(0, r)}$$

where C' is a spherical cone with opening $\varkappa$ and central axis l then we call C a *bounded spherical cone* with *opening* $\varkappa$, *central axis* l and *radius* r.

THEOREM I. — *If* D *is a convex domain then* $D \in \mathcal{E}([0, \infty))$ *if and only if* D *is a* (C)*-domain.*

The proof of Theorem I follows from Props. 2) and 4) below. In view of this theorem it will be of interest to give some sufficient criteria for D to be a (C)-domain ([45]).

1) *Let* D *be a convex domain. Then* D *is a* (C)*-domain if* $n \leqslant 2$ *or the dimension of the maximal cone* C *in* D *is* 0 *or* n.

If $\dim C = 0$ (*i.e.* D *is bounded*) *and* $S(x, r) \subset D \subset S(x, r/\varkappa)$ *then* r *and* $\varkappa$ *are* (C)*-constants of* D. *If* $\dim C = n$ *and* C *contains a spherical cone with opening* $\varkappa$ *then* $\varkappa$ *and* r *are the* (C)*-constants of* D *where* r *is an arbitrary positive number.*

Proof. — The proof is obvious if $n = 1$ or $\dim C = 0$, or $\dim C = n$, so we need only to consider the case when $n = 2$ and $\dim C = 1$.

If C is a straight line then ∂D is formed by two straight lines l_1 and l_2 parallel to C. Then clearly $r = 1/2 \operatorname{dist}(l_1, l_2)$ and $\varkappa = 1$ are (C)-constants of D.

([45]) In example 9, § 13 we construct a convex domain in R^3 which is not a (C)-domain.

The remaining case is when C is a half line, i.e.

$$C = [t\theta : t \geqslant 0, |\theta| = 1].$$

Take any point $x_0 \in D$ and let $D_+ = [x : (x - x_0, \theta) > 0, x \in D]$ and $D_- = [x : (x - x_0, \theta) < 0, x \in D]$. Then D_- is a bounded convex domain and by the above a (C)-domain. Clearly $\operatorname{dist}(C + x_0, \partial D \cap \overline{D}_+) = r_1 > 0$ and the half-strip with axis $C + x_0$ and width $2r_1$ is contained in D_+. Hence we see that for $x \in \partial D \cap \overline{D}_+$ there is a bounded spherical cone with angular opening $\pi/4$, radius $2r_1$, and vertex x contained in $\overline{D}_+$. From this it follows that D is a (C)-domain.

The following elementary lemma, which we give without proof, will be used in the proof of Prop. 2).

LEMMA 1. — *If the radius of the maximal sphere contained in the convex set* D *is* r *and* $D \subset S(0, R)$ *then* $|D| \leqslant C r R^{n-1}$ *with* C *dependent only on* n [46].

2) *If* D *is a convex domain but not a* (C)*-domain then* $n \geqslant 3$ *and the dimension of the maximal cone contained in* D *is* l *with* $1 \leqslant l \leqslant n - 1$. *Furthermore* $D \notin \mathcal{E}(\alpha)$ *for* $\alpha > (n - l)/2$ *if* $l > 1$ *and* $> (n - 2)/2$ *if* $l = 1$.

Proof. — That $n \geqslant 3$ and $1 \leqslant l \leqslant n - 1$ is implied by Prop. 1). Suppose that $D \in \mathcal{E}(\alpha)$ for α satisfying our conditions.

Let $x_k \in \partial D$ be such that the maximal sphere contained in $S(x_k, 1) \cap D$ has radius less than $1/k$. Obviously the x_k must exist if the (C)-condition is not satisfied with constants $r = 1/k$ and $\varkappa = 1/k$ for $k = 2, 3, \ldots$ Then, since $S(x_k, 1) \cap D$ is convex, we have by Lemma 1 that

$$|S(x_k, 1) \cap D| \equiv |S(0, 1) \cap (D - x_k)| \searrow 0$$

as $k \nearrow \infty$.

Let $u \in C^\infty(R^n)$, be $= 1$ on $S(0, 1/3)$ and $= 0$ outside $S(0, 2/3)$.

(46) For $n > 1$ we obtain $C = \dfrac{n+1}{n-1}\omega_{n-1}$. Probably the best constant C is

$$\frac{2}{n-1}\omega_{n-1}.$$

Then with $u_k(x) = u(x - x_k)$ for $x \in \mathrm{D}$, $u_k \in \check{\mathrm{P}}^\alpha(\mathrm{D})$ and

$$|u_k|_{\alpha, \mathrm{D}} \leqslant c|u_k|_{\alpha, \mathrm{S}(x_k, 1) \cap \mathrm{D}} = c|u|_{\alpha, \mathrm{S}(0,1) \cap (\mathrm{D} - x_k)}$$

where c depends only on n. From the previous paragraph we see that $|u_k|_{\alpha, \mathrm{D}} \searrow 0$. It follows that if $\tilde{u}_k \in \mathrm{P}^\alpha(\mathrm{R}^n)$ is the extension of u_k, $\|\tilde{u}_k\|_\alpha \to 0$.

Let C be the maximal cone associated with D and suppose $\dim \mathrm{C} = l$ with $2 \leqslant l \leqslant n - 1$. Clearly

$$[(\mathrm{C} \cap \mathrm{S}(0,1)) + x_k] \subset \overline{\mathrm{D}}$$

for every k. Put G = interior of $\mathrm{C} \cap \mathrm{S}(0,1)$ in the l-dimensional hyperplane containing it. Then $\mathrm{G} + x_k \subset \mathrm{D}$. Let u'_k be the restriction of u_k to $\mathrm{G} + x_k$ and $u'(x)$ the restriction of u to G. Since u is 1 on $\mathrm{S}(0,1/3)$, we have by Prop. 6), § 1

$$0 < \|u'\|_{\alpha - \frac{n-l}{2}, \mathrm{G}} = \|u'_k\|_{\alpha - \frac{n-l}{2}, \mathrm{G}+x_k} \leqslant \frac{\Gamma\left(\alpha - \frac{n-l}{2}\right)}{2^{n-l}\pi^{\frac{n-l}{2}}\Gamma(\alpha)} \|\tilde{u}_k\|_\alpha \to 0,$$

a contradiction.

If $\dim \mathrm{C} = 1$ then we fix some $x_0 \in \mathrm{D}$ and note that for k sufficiently large, say $k \geqslant k_0$, the orthogonal projection of x_k on the line containing $x_0 + \mathrm{C}$ lies in $x_0 + \mathrm{C}$. Furthermore, if $r_0 = \inf\{1/3, r_{x_0+\mathrm{C}}(x_k)(k \geqslant k_0)\}$ then $r_0 > 0$ and the intersection of D, $\mathrm{S}(x_k, r_0)$ and the plane containing x_k and $x_0 + \mathrm{C}$ contains a two-dimensional bounded spherical cone Γ_k with angular opening $\pi/4$, radius r_0 and vertex x_k. Let F_k be the interior of Γ_k in the plane which contains it.

We construct u, u_k and $\tilde{u}_k$ as in the beginning of the proof, denote by u'_k the restriction of u_k to F_k, and assume $\alpha > (n-2)/2$. Then since u is 1 on $\mathrm{S}(0,1/3)$ there is a $\tau > 0$ such that for $k \geqslant k_0$ (we use again Prop. 6), § 1)

$$0 < \tau \leqslant \|u'_k\|_{\alpha - \frac{n-2}{2}, \mathrm{F}_k} \leqslant \frac{\Gamma\left(\alpha - \frac{n-2}{2}\right)}{2^{n-2}\pi^{\frac{n-2}{2}}\Gamma(\alpha)} \|\tilde{u}_k\|_\alpha \to 0$$ a contradiction.

3) *If* D *is a convex domain and* $\mathrm{S}(0,r) \subset \mathrm{D} \subset \mathrm{S}(0, \mathrm{R})$ *then* $\mathrm{D} \in \mathcal{E}([0, \infty), \Gamma)$ *where* $\Gamma([\alpha_1, \alpha_2])$ *depends only on* n, α_2^*, r, *and* R.

Proof. — Consider the n-dimensional rectangle

$$B^{(n)} = [x : |x_k| < r/4\sqrt{n} \quad \text{for} \quad k \leqslant n-1, \ \ r/2 < x_n < \infty].$$

By rotating $B^{(n)}$ around the origin we cover the whole closed shell $[r \leqslant |x| \leqslant R]$. We can therefore find a finite number of such rotated rectangles — $B_1^{(n)}, \ldots, B_N^{(n)}$ — which cover this shell. The same rotations performed on the rectangle

$$\tilde{B}^{(n)} = [x : |x_k| < r/2\sqrt{n} \quad \text{for} \quad k \leqslant n-1, \ \ r/4 < x_n < \infty]$$

give rectangles $\tilde{B}_1^{(n)}, \ldots, \tilde{B}_N^{(n)}$ covering the shell with looseness-constant $\geqslant r/4\sqrt{n}$. Since N can be chosen depending only on n and R/r, the rank p of the covering $\{\tilde{B}_l^{(n)}\}$ depends also only on n and R/r [47]. $\{\tilde{B}_l^{(n)}\}$ is clearly a covering of ∂D, $r/4\sqrt{n}$ — loose and of rank p. The intersection $\tilde{B}_l^{(n)} \cap D$ is obviously a convex SLG-domain which by rotation becomes a domain

$$[x : |x_k| < r/2\sqrt{n} \quad \text{for} \quad k \leqslant n-1, \ \ r/4 < x_n < r/4 + f_l(x')]$$

with $r/2 < f_l(x') < R$. It follows that the Lipschitz constant of the convex function $f_l(x')$ is $< 2R/r$. The minimal width of the SLG-domain $\tilde{B}_l^{(n)} \cap D$ is $=$ min $(r/2\sqrt{n}, \ r/2R)$. Our proposition follows now from Theorem II, § 11 and the localization theorem of § 7 (Theorem II).

4) *If* D *is a convex* (C)*-domain with* (C)*-constants* $\varkappa$ *and* r *then* $D \in \mathcal{E}([0, \infty), \ \Gamma]$ *where* $\Gamma([\alpha_1, \alpha_2])$ *depends only on* α_2^*, n, $\varkappa$, *and* r.

Proof. — Let $\{x_l'\}$ be a maximal set of points in R^n such that $|x_l' - x_k'| \geqslant r$ for $k \neq l$. Then $\{S(x_l', r/2)\}$ are mutually disjoint and $\{S(x_l', r)\}$ is a covering of R^n. Consider the subset $\{x_k\}$ of $\{x_l'\}$ such that $r_{\partial D}(x_k) < r$. Then $\{S(x_k, r)\}$ is a covering of ∂D and $\{S(x_k, 3r)\}$ form a $2r$-loose covering of ∂D.

Next we prove that $\{S(x_k, 3r)\}$ is of rank less than 13^n. If $S(x_k, 3r) \cap S(x_l, 3r) \neq 0$ then $|x_k - x_l| < 6r$ and $S(x_l, r/2) \subset S(x_k, 13r/2)$. Since the $\{S(x_l, r/2)\}$ are mutually disjoint,

(47) By a more careful construction we could make the rank depend only on n.

$S(x_l, r/2) \subset S(x_k, 13r/2)$ for at most $\frac{\omega_n}{n}(13r/2)^n \Big/ \frac{\omega_n}{n}(r/2)^n = 13^n$ x_l's which completes this part of the proof.

Let $z_k \in S(x_k, r) \cap \partial D$. Since D is a (C)-domain there is a y_k such that $|z_k - y_k| = r$ and $S(y_k, \varkappa r) \subset D$. Thus

$$S(y_k, \varkappa r) \subset D \cap S(x_k, 3r) \subset S(y_k, 5r).$$

The proof of the proposition is completed by applying Prop. 3) and Theorem II, § 7 to the covering $\{S(x_k, 3r)\}$ of ∂D.

The next proposition gives some useful sufficient conditions for determining if the union of two adjacent non-tangential convex domains belongs to $\mathcal{E}([0, \infty))$.

5) *Let* D_1 *and* D_2 *be adjacent, bounded, convex domains with slope* $\omega(\overline{D}_1, \overline{D}_2) \geqslant \omega_0 > 0$. *Suppose that for some* $r > 0$ *and* $\varkappa$, $0 < \varkappa \leqslant 1$, *there exist points* $x_0 \in \overline{D}_1 \cap \overline{D}_2$ *and* $x_i \in D_i$ *such that* $S(x_i, r) \subset D_i \subset S(x_i, r/\varkappa)$ *and* $S(x_0, r) \subset \overline{D}_1 \cup \overline{D}_2$. *Then* $D = (\overline{D}_1 \cup \overline{D}_2)^\circ \in \mathcal{E}([0, \infty), \Gamma)$, *where* $\Gamma([\alpha_1, \alpha_2])$ *depends only on* n, α_2^*, ω_0, r, *and* $\varkappa$.

Proof. — Let $\overline{G}$ be the closed convex hull of

$$\overline{S(x_0, r)} \cup (D_1 \cap D_2) \quad \text{and} \quad G = (\overline{G})^\circ.$$

We shall show that $\overline{G} \cap \overline{D}_1$ and $\overline{G} \cap \overline{D}_2$ are q-cells with (C)-constants bounded from below by $\varkappa r/2$ and $\varkappa^2/4$. Since G is convex and $\overline{D}_1$ and $\overline{D}_2$ are q-cells with (C)-constants r and $\varkappa$ (cf. Prop. 1)) the proof of the proposition will be completed by Prop. 3) and Theorem IV, *b*), § 10. Since $G \cap D_1$ and $G \cap D_2$ are bounded convex domains $\overline{G} \cap \overline{D}_1$ and $\overline{G} \cap \overline{D}_2$ are q-cells and we need only determine their (C)-constants.

Consider $\overline{G} \cap \overline{D}_i$. Set $y_i = x_0 + \frac{r}{1+\varkappa}\frac{x_i - x_0}{|x_i - x_0|}$. The sphere $S\left(y_i, \frac{r\varkappa}{1+\varkappa}\right)$ is contained obviously in $S(x_0, r)$ and in D_i, hence $S\left(y_i, \frac{r\varkappa}{1+\varkappa}\right) \subset \overline{G} \cap \overline{D}_i$. On the other hand

$$S\left(y_i, \frac{2r}{\varkappa}\right) \supset S(x_0, r) \cup D_i$$

and therefore $S\left(y_i, \frac{2r}{k}\right) \supset \overline{G} \cap \overline{D}_i$. By Prop. 1) it follows that the (C)-constants of $\overline{G} \cap \overline{D}_i$ are $> \frac{r\varkappa}{1+\varkappa} \geqslant \frac{r\varkappa}{2}$ and $> \frac{r\varkappa\varkappa}{(1+\varkappa)2r} > \frac{\varkappa^2}{4}$ respectively which finishes the proof.

The next proposition gives conditions for D to be in $\mathcal{E}([0, \infty))$ when $D = \left(\bigcup_k Q_k\right)^\circ$ and the Q_k's are convex q-cells.

6) *Let* $\{Q_k\}$ *be a system of convex q-cells and* $D = \left(\bigcup_k Q_k\right)^\circ$. *Then* $D \in \mathcal{E}([0, \infty))$ *if* 1° $\{Q_k\}$ *is* δ*-loose*; 2° *there are* $r > 0$, R, *and* $x_k \in Q_k$ *such that* $S(x_k, r) \subset Q_k \subset S(x_k, R)$; 3° *for every pair of adjacent q-cells,* Q_k *and* Q_l, *there is an* $x_{k,l} \in Q_k \cap Q_l$ *such that* $S(x_{k,l}, r) \subset Q_k \cup Q_l$ (*r is the same as in* 2°); 4° $\{Q_k\}$ *is of finite rank* (*this can be deduced from* 2° *if the* $S(x_k, r)$ *are disjoint*), 5° *there is an* $\omega_0 > 0$ *such that* $\omega_0 \leqslant \omega(Q_k, Q_l)$ *for* $Q_k \cap Q_l \neq 0$ *and* 6° ∂D *is an* $(n-1)$*-manifold or, more generally,* ∂D *does not cut locally* D.

Proof. — 1°, 2°, 4°, 5° and 6° guarantee that $\{Q_k\}$ is a regular system. 2°, 3°, and 5° imply, by Props. 3) and 5) that Q_k° and $(Q_k \cup Q_l)^\circ$ for Q_k and Q_l adjacent, are in $\mathcal{E}([0, \infty), \Gamma)$, Γ independent of k and l. The proof is completed by Theorem III, § 10.

7) *If* $\overline{D}$ *is a finite geometric polyhedron and* ∂D *is an* $(n-1)$*-manifold* (*or does not cut* D *locally*), *then* $D \in \mathcal{E}([0, \infty))$.

Proof. — Since $\overline{D}$ is a finite geometric polyhedron it can be decomposed into $\overline{D} = \bigcup_{k=1}^{N} Q_k$ where the Q_k's are convex polyhedra (not necessarily bounded) having at most $(n-1)$-dimensional intersections. Elementary geometric considerations show that 1° the slope between any two polyhedra is positive, 2° for any two convex polyhedra Q_k and Q_l with $Q_k \cap Q_l$ $(n-1)$-dimensional, there exists an n-dimensional convex polyhedron $\overline{G}$ such that $Q_k \cap Q_l \subset G \subset Q_k \cup Q_l$. Our proposition then follows from Theorem III, Corollary 1*b*) of § 10 and Prop. 4) of the present section.

13. Examples.

In this chapter we are interested primarily in the classes $\check{P}^\alpha(D)$ and $P^\alpha(D)$. The following examples are designed to illustrate some of the basic differences between them.

We will only indicate the types of construction involved in each example and will leave the calculations to the reader.

Example 1.

We show that the inequalities (2.4) of Proposition 3), § 2 are the best possible in the following sense: If there are four constants M_1, M_2, M_3 and M_4 such that the inequalities

$$M_1|u\urcorner_{\alpha,D} \leq |u|_{\alpha,D} \leq M_2|u\urcorner_{\alpha,D}$$
$$M_3|u|^2_{m,D} \leq |u\urcorner^2_{\alpha,D} - |u|^2_{\alpha,D} \leq M_4|u|^2_{m,D}$$

hold for all $\alpha \geq 0$, for all open sets $D \subset R^n$ and for all functions $u \in \check{P}^\alpha(D)$ (as usual $m = [\alpha]$), then

$$M_1 \leq 2^{-1/2}, \qquad M_2 \geq 1, \qquad M_3 \leq 0, \qquad M_4 \geq 1.$$

This statement remains true if we restrict ourselves to $0 < \alpha < 1$ and $D = R^n$. In fact, using Fourier transforms we have, by Proposition 2, § 2

$$|u\urcorner^2_{\alpha,R^n} = \int_{R^n} (1 + |\xi|^{2\alpha})|\hat{u}(\xi)|^2\, d\xi$$
$$|u|^2_{\alpha,R^n} = \int_{R^n} (1 + |\xi|^2)^\alpha|\hat{u}(\xi)|^2\, d\xi.$$

We notice that.

1° For $\rho \geq 0$, $1 \geq (1+\rho^2)^\alpha/(1+\rho^{2\alpha}) \geq 2^{\alpha-1}$ the value 1 being attained at $\rho = 0$ and $\rho = \infty$, and the value $2^{\alpha-1}$ being attained at $\rho = 1$.

2° For $\rho \geq 0$, $1 + \rho^{2\alpha} - (1+\rho^2)^\alpha$ is an increasing function of ρ taking the value 0 at $\rho = 0$ and 1 at $\rho = \infty$.

It follows therefore that by choosing $\hat{u}(\xi)$ with support arbitrarily near the origin we will show the inequalities for M_2 and M_3. By choosing the support of $\hat{u}$ outside of an arbitrarily large sphere one proves the inequality for M_4. Finally, by choosing the support of $\hat{u}$ in an arbitrarily small neighborhood of $\partial S(0, 1)$ and taking α arbitrarily small one proves the inequality for M_1.

Example 2.

We give an example of a domain D *such that* $\overline{\check{P}^{\alpha}(D)} \not\supset \check{P}^{\gamma}(D)$ *with* $\alpha \leqslant \gamma$ *(cf. Prop.* 5, § 2) *and such that* $\overline{P^1(D)} \neq \check{P}^1(D)$.

In polar coordinates ($x = x_1 + ix_2 = \rho e^{i\theta}$) let

$$D = [\rho e^{i\theta}:\ 1 < \rho < 2,\ 0 < \theta < 2\pi], \quad D_+ = D \cap [x_2 > 0]$$

and $D_- = D \cap [x_2 < 0]$. Let $\omega(\theta) \in C^\infty(0,2\pi)$ and $= 1$ for $0 < \theta < \pi/4$, $= 0$ for $\pi/2 < \theta < 2\pi$. Define

$$u(x) \equiv u(\rho e^{i\theta}) = \omega(\theta).$$

Obviously $u \in \check{P}^m(D)$, m an integer. Since D_+ is a C^∞ homeomorphic image of a square, $u|_{D_+}$ is in $\check{P}^\alpha(D_+)$ for all α. Thus $u \in \check{P}^\alpha(D)$ if and only if

$$\frac{2}{C(2,\alpha)} \sum_{k=0}^{m} \binom{m}{k} \sum_{|i|=k} \int_{D_+}\int_{D_-} \frac{|D_i u(x)|^2}{|x-y|^{n+2\beta}}\,dx\,dy < \infty,$$

where $m = [\alpha]$, $\beta = \alpha - m$, and this is the case if and only if $0 \leqslant \beta < 1/2$, i.e. $u \in \check{P}^\alpha(D)$ *if and only if* $\alpha - [\alpha] < 1/2$. Suppose that $u \in \overline{P^1(D)}$, then there are $u_n \in P^1(D)$ such that $\|u_n - u\|_{1,D} \searrow 0$. Let $\tilde{u}_n \in P^1(R^2)$ be extensions of u_n, $\check{u}_n$ the restriction of $\tilde{u}_n$ to $I = [\rho e^{i\theta}:\ 1 < \rho < 2,\ \theta = 0]$. Then

$$|\tilde{u}_n - u|_{1,D_+} \searrow 0 \qquad \text{and} \qquad |\tilde{u}_n|_{1,D_-} \searrow 0.$$

Since D_+ and D_- are LG-domains they are in $\mathcal{E}([0,\infty))$ (cf. Corollary II, § 11). Hence if we extend $\tilde{u}_n - u|_{D_+}$ and $\tilde{u}_n$ to functions in $P^1(R^2)$ and then restrict the extensions to I, we have by the continuity properties of functions in $P^1(R^2)$, $|\check{u}_n - 1|_{1/2,I} \searrow 0$ and $|\check{u}_n|_{1/2,I} \searrow 0$, a contradiction.

Another example of a domain D such that $\overline{\check{P}^{\alpha}(D)} \not\supset \check{P}^{\gamma}(D)$ with $\alpha \leqslant \gamma$, but with a domain limited by a simple Jordan curve can be constructed as follows. Consider on the x_1-axis a sequence of points b_k with $b_0 = 0$ and such that

$$b_{2k+1} - b_{2k} = b_{2k+2} - b_{2k+1} = \frac{1}{k+1} 2^{-2k};$$

we notice that $b_i \nearrow b < 3$.

On the segment $[b_i; b_{i+1}]$ construct an isosceles triangle T_i with altitude h_i such that $h_{2k} = h_{2k+1} = \dfrac{1}{k+1}$. Now we join to $\bigcup_i T_i$ the rectangle $R = [0 \leqslant x_1 \leqslant 3, -1 \leqslant x_2 \leqslant 0]$ and obtain a domain D with $\overline{D} = \bigcup_i T_i \cup R$ such that ∂D is a Jordan curve. Define $u(x) = 2^k x_2$ for $x \in T_{2k+1}$ and $= 0$ otherwise. It is immediately checked that $u \in \check{P}^1(D)$ and $u \notin \check{P}^\alpha(D)$ for $1/2 < \alpha < 1$. To check the last statement it is enough to estimate the part of $d_{\alpha,D}(u)$ corresponding to couples $x \in T_{2k}$ and $y \in T_{2k+1} \cap \left[x_2 > \frac{1}{2} h_{2k+1}\right]$.

Example 3.

We give an example of a domain D *such that* $\tilde{P}^\alpha(D) \neq P^\alpha(D)$ (*cf. Remark* 4, § 4).

In R^n, $n \geqslant 2$, let $l = [t\theta : t \geqslant 0]$ for some $\theta, |\theta| = 1$, D_+ be the cone with central axis l and opening 1/2 and D_- the cone with central axis $-l$ and opening 1/2. Then we define $D = [|x| > 2] \cup D_+ \cup D_-$. Let $\varphi \in C^\infty(R^n)$, $= 1$ for $|x| \leqslant 1$ and $= 0$ for $|x| \geqslant 2$; define $u = \varphi$ if $x \in D_+$ and $= 0$ for $x \in D - D_+$. Then $u \in C^\infty(D)$ and $|u|_{\alpha,D}$ is uniformly bounded on any compact subinterval of $[0, \infty)$; thus $u \in \tilde{P}^\alpha(D)$ for arbitrary α. Since $J_{\alpha,D_+,D_-}(u)$ is infinite for $\alpha \geqslant n/2$, we see that $u \notin P^\alpha(D)$ for $\alpha \geqslant n/2$.

Example 4.

Our aim here is to obtain an example of

$$(*) \qquad \liminf_{\alpha \nearrow 1} d_{\alpha,D}(u) = a, \qquad \limsup_{\alpha \nearrow 1} d_{\alpha,D}(u) = b$$

for $0 \leqslant a < b < \infty$ (*cf. Prop.* 5, § 4).

On the real line R^1 consider for $k = 1, 2, \ldots$ the open intervals

$$D'_k = \left(\frac{1}{2k}, \frac{1}{2k} + \frac{1}{20k^3}\right) \qquad D''_k = \left(\frac{1}{2k} + \frac{2}{20k^3}, \frac{1}{2k} + \frac{3}{20k^3}\right).$$

Put $D_k = D'_k \cup D''_k \subset \left(\frac{1}{2k}, \frac{1}{2k-1}\right)$ and $D = \bigcup_{k=1}^{\infty} D_k$. On D we will consider functions u defined as follows. Let $\{q_k\}$ be a sequence of positive numbers bounded by some $q < \infty$. We define

$$u(x) = q_k k^{-2} \quad \textit{for} \quad x \in D'_k, \quad u(x) = 0 \quad \textit{for} \quad x \in D''_k.$$

Immediate calculations show that for $1/2 < \alpha < 1$:

$$d_{\alpha, D_k}(u) = \frac{(20)^{2\alpha-1}\,(1 - 3^{1-2\alpha})}{\alpha(2\alpha - 1)\,C(1, \alpha)} \frac{q_k^2}{k^{7-6\alpha}},$$

$$0 < d_{\alpha, D}(u) - \sum_{k=1}^{\infty} d_{\alpha, D_k}(u) \leqslant c(1 - \alpha)q^2, \text{ } c \textit{ an absolute constant.}$$

For $\alpha = 1 - \frac{1}{N}$, N *integer* $\geqslant 3$,

$$\sum_{k=1}^{N-1} d_{\alpha, D_k}(u) + \sum_{k=N^N-1}^{\infty} d_{\alpha, D_k}(u) \leqslant c\frac{1}{N}\log N.$$

For $q_k = q$, $k = 1, 2, \ldots$, $\lim_{\alpha \nearrow 1} d_{\alpha, D}(u) = \lim_{\alpha \nearrow 1} \sum_{k=1}^{\infty} d_{\alpha, D_k(u)} = \frac{20}{9}q^2$. Therefore, if we define $N_1 = 3$, $N_{i+1} = N^{N_i}$, $q_k = \sqrt{\frac{9a}{20}}$ for k lying in the intervals $N_{2j-1} \leqslant k < N_{2j}$, $j = 1, 2, \ldots$, and $q_k = \sqrt{\frac{9b}{20}}$ for all other k's, we get (*) with lim inf and lim sup attained by the sequences

$$\{\alpha_j\} = \{1 - N_{2j-1}^{-1}\} \qquad \text{and} \qquad \{\alpha_j\} = \{1 - N_{2j}^{-1}\}$$

respectively.

Obviously $d_{1, D}(u) = 0$ and $u \in \tilde{P}^1(D) - P^1(D)$, another example where $\tilde{P}^1(D) \neq P^1(D)$.

Example 5.

We construct a simply connected unbounded domain D *in the plane* $(z = x_1 + ix_2 = \rho e^{i\theta})$ *such that*

$$\liminf_{\alpha \searrow 0} \delta(\alpha) = a, \qquad \limsup_{\alpha \searrow 0} \delta(\alpha) = b, \qquad 0 \leqslant a \leqslant b \leqslant 1$$

(*see* (4.7) *and Prop.* 7), § 4).

The domain is symmetric relative to x_1-axis. Its part in the upper half-plane is formed by points z satisfying one of three conditions (not mutually exclusive): 1° $x_1 > 0$, $0 < x_2 < 1$; 2° $0 < \arg(z - i) < \pi a$; 3° $0 < \arg(z - i) < \pi b$, $N_{2j} < |z - i| < N_{2j+1}$ for $j = 1, 2, \ldots$. Here N_i are the integers introduced in the preceding example. The lim inf and lim sup are attained for $\{\alpha_j\} = \{N_{2j}^{-1}\}$ and $\{N_{2j+1}^{-1}\}$ respectively.

Example 6.

We construct a polyhedron $P \subset R^3$ *such that* ∂P *is an* $(n - 1)$-*manifold but* P *is not an* LG-*domain.*

Let θ_k be the unit vector in the direction of the positive x_k-axis. We construct two tetrahedrons T_1 and T_2 with vertices

$$T_1 : 0, \theta_1, \theta_1 + \theta_2, \theta_1 - \theta_2 + \theta_3,$$
$$T_2 : 0, -\theta_1, -\theta_1 - \theta_2, -\theta_1 + \theta_2 + \theta_3.$$

If Q is the cube $|x_1| < 2$, $|x_2| < 2$, $0 < x_3 < 4$, then

$$P = Q - (T_1 \cup T_2)$$

is the required polyhedron. At the origin its boundary has no local representation as a graph.

The next example shows that the condition — ∂D does not cut locally D — is weaker than the condition that ∂D is an $(n - 1)$-manifold.

Example 7.

We now construct a polyhedron $P \subset R^3$ *such that* ∂P *does not cut locally* P *but* ∂P *is not a* 2-*manifold.*

In the notation of Example 6 let T_1 and T_2 be two pyramids with vertices

$$T_1 : \theta_3, \theta_1, \theta_2, -\theta_1, -\theta_2$$
$$T_2 : \theta_3, 1/2\ \theta_1, 1/2\ \theta_2, -1/2\ \theta_1, -1/2\ \theta_2.$$

Then $P = T_1 - \overset{\circ}{T}_2$ is the desired polyhedron; in particular, ∂P is not a manifold at θ_3.

Example 8.

We give an example of a bounded convex set D *and a* $C^{(0,1)}$-*homeomorphism* T, *such that* T(D) *is not an* LG-*domain.*

Consider in polar coordinates in $R^2(\rho e^{i\theta} = x_1 + ix_2)$ the $C^{(0,1)}$-homeomorphism T given by

$$T(\rho e^{i\theta}) = \rho e^{i(\theta - \log \rho)}.$$

Then $D = [0 < \rho < 1, 0 < \theta < \pi/2]$ is convex but T(D) is not an LG-domain; in particular, T(D) is not locally a graph at the origin.

Example 9.

We construct an unbounded convex domain $D \subset R^3$ *which contains a* 2*-dimensional cone but is not a* (C)*-domain.*

Let $\pi_{1,k}$ and $\pi_{2,k}$ be the couple of hyperplanes which contain the line $l_k = [(x_1, x_2, x_3) : x_3 = kx_1 - 3k^2, x_2 = 0]$, $k = 1,2,\ldots$, and are tangent to the sphere $[|x| \leqslant 1]$. Let further $\pi^+_{1,k}$ and $\pi^+_{2,k}$ be the half spaces determined by $\pi_{1,k}$ and $\pi_{2,k}$ such that $[|x| < 1] \subset \pi^+_{1,k} \cap \pi^+_{2,k}$. Then $D = \bigcap_{k=1}^{\infty} (\pi^+_{1,k} \cap \pi^+_{2,k})$. It is not difficult to see that $x^k = (6k, 0, 3k^2) \in \partial D \cap \pi_{1,k} \cap \pi_{2,k}$ and that the largest sphere contained in $S(x^k, 1) \cap D$ has a radius less than $\frac{\sqrt{1+k^2}}{3k^2} \searrow 0$. Thus D is not a (C)-domain. It is also clear that the 2-dimensional cone $[x_1 < 0, x_2 = 0, x_3 > 0]$ is contained in D.

Appendix. Complete continuity of standard norms.

We restrict ourselves to bounded open sets in R^n.

For potential norms $\|u\|_{\alpha, D}$ the question of complete continuity is settled by Prop. 3), § 1: for $\beta < \alpha$, the α-norm is c.c. (completely continuous) rel. (relative to) the α-norm.

For standard norms the situation is much more complicated. The question of complete continuity for integral orders was investigated quite thoroughly by F. Rellich (see [6]): 1° There are bounded domains for which $|u|_{0,D}$ is not c.c. rel. $|u|_{1,D}$. 2° If $|u|_{0,D}$ is c.c. rel. $|u|_{1,D}$, then for any two integers $0 \leqslant m_1 < m$, the norm of order m_1 is c.c. rel. the norm of order m. 3° The complete continuity of the 0-norm rel. the 1-norm holds for domains whose boundaries are piecewise graph-manifolds ([48]).

When we turn to norms of non-integral order, the situation changes considerably. For domains of 3° above, in general the α-norm is not c.c. rel. to the 1-norm for $0 < \alpha < 1$ (see

([48]) i.e. those which in some coordinate system can be represented as a graph of a continuous function $x_n = f(x_1, \ldots, x_{n-1})$ (but the function is not necessarily Lipschitzian). The boundary is supposed to be covered by a *finite* number of closed graph-manifolds. Domains with this property form a larger class than the G-domains.

Example 2, § 13). However, several general results can be obtained which we describe here.

1) *For* $\beta \geqslant \alpha$, $|u|_{\beta, D}$ *is never c.c. rel.* $|u|_{\alpha, D}$.

This is obvious if we consider the norms on the subspace of functions C^∞ vanishing outside of a relatively compact subdomain of D.

2) *If* $|u|_{0, D}$ *is c.c. rel.* $|u|_{\alpha, D}$ *with* $0 < \alpha < 1$, *then for any* β *with* $0 < \beta < \alpha$, $|u|_{\beta, D}$ *is c.c. rel.* $|u|_{\alpha, D}$.

For the proof we decompose, for arbitrarily small ε,

$$|u|^2_{\beta, D} = |u|^2_{0, D} + \frac{1}{C(n, \beta)}\Bigg[\iint\limits_{|x-y|>\varepsilon} + \iint\limits_{|x-y|<\varepsilon}\Bigg]$$
$$\leqslant \left(1 + \frac{\varepsilon^{-2\beta}\omega_n}{\beta C(n, \beta)}\right)|u|^2_{0, D} + \frac{\varepsilon^{2\alpha}C(n, \alpha)}{\varepsilon^{2\beta}C(n, \beta)}|u|^2_{\alpha, D}.$$

3) *If* $D \in \mathcal{E}(\alpha)$ *and* $\beta < \alpha$, *then* $|u|_{\beta, D}$ *is c.c. rel.* $|u|_{\alpha, D}$.

Proof. — $\|u\|_{\beta, D}$ is c.c. rel. $\|u\|_{\alpha, D}$ by Prop. 3, § 1 and since $\|u\|_{\alpha, D}$ is equivalent to $|u|_{\alpha, D}$ and $|u|_{\beta, D} \leqslant \|u\|_{\beta, D}$ (Prop. 4, § 2) the proposition is clear.

4) *If* $|u|_{\beta, D}$ *is c.c. rel.* $|u|_{\alpha, D}$ *for all* α *and* β *with* $0 \leqslant \beta < \alpha \leqslant 1$, *then* $|u|_{\beta, D}$ *is c.c. rel.* $|u|_{\alpha, D}$ *for any* $\beta < \alpha$.

This becomes obvious if one compares each term in the expression of $|u|^2_{\beta, D}$ with suitable terms in the expression of $|u|^2_{\alpha, D}$.

5) *If* $|u|_{\beta, D}$ *is c.c. rel.* $|u|_{\alpha, D}$ *and if* T *is a* $C^{(\alpha^*, 1)}$*-homeomorphism of* D *onto* D^*, *then* $|u|_{\beta, D^*}$ *is c.c. rel.* $|u|_{\alpha, D^*}$.

This is immediate since the correspondence

$$u(x) \to u^*(x^*) = u(T^{-1}x^*)$$

is a linear and topological isomorphism of $\check{P}^\beta(D)$ onto $\check{P}^\beta(D^*)$ for all β with $\beta \leqslant \alpha^* + 1$ (see Prop. 8), § 2).

THEOREM I. — (*Localization theorem.*). *If* $\{U_k\}$, $k = 1, 2, \ldots, N$, *is an open covering of* $\overline{D}$ *such that* $|u|_{\beta, D \cap U_k}$ *is c.c. rel.* $|u|_{\alpha, D \cap U_k}$ *then* $|u|_{\beta, D}$ *is c.c. rel.* $|u|_{\alpha, D}$.

Proof. — If β is an integer our statement follows immediately from the following two inequalities:

$$\text{(i)} \quad |u|^2_{\beta, D} \leqslant \sum_{k=1}^{N} |u|^2_{\beta, D \cap U_k}, \qquad \text{(ii)} \quad \sum_{k=1}^{N} |u|^2_{\alpha, D \cap U_k} \leqslant N|u|^2_{\alpha, D}.$$

If β is not an integer then we have to consider the looseness δ of the covering U_k and replace the inequality (i) by:

$$|u\rceil^2_{\beta,D} \leqslant \frac{\omega_n}{(\beta-\beta^*)C(n,\beta-\beta^*)}\delta^{-2(\beta-\beta^*)}\sum_{k=1}^{N}|u\rceil^2_{\beta^*,D\cap U_k}+\sum_{k=1}^{N}|u\rceil^2_{\beta,D\cap U_k}.$$

Remark 1. — For β an integer the same proof would give a stronger theorem where $\{U_k\}$ would not be required to cover the whole of $\overline{D}$ but only to cover D up to a set of Lebesgue measure 0.

THEOREM II. — *If* D *is* L-*convex then* $|u|_{\beta,D}$ *is c.c. rel.* $|u|_{\alpha,D}$ *for all* $\beta < \alpha$.

The proof follows immediately from Theorem I and Prop. 5), 4), and 3), since the covering of a bounded L-convex set can be chosen finite, and bounded convex domains belong to $\mathcal{E}([0, \infty))$.

Remark 2. — The bounded L-convex domains together with those of class $\mathcal{E}(\alpha)$ form the most general class of domains we know in which the complete continuity theorem holds for all $\beta > \alpha$.

Remark 3. — The problem of complete continuity for unbounded domains seems unsolved. We were not able to find any examples of an unbounded domain where the complete continuity holds for any $\beta < \alpha$. For very large classes of unbounded domains we can prove that there cannot be complete continuity and it appears that the only doubtful cases are domains of finite measure with very rapidly and regularly decreasing measure when one approaches infinity.

BIBLIOGRAPHY

[1] N. ARONSZAJN and H. G. HARDY, Properties of a class of double integrals, *Ann. of Math.* 46 (1945), 220-241.

[2] N. ARONSZAJN, F. MULLA et P. SZEPTYCKI, On spaces of potentials connected with L^p classes, *Ann. de l'Inst. Fourier*, Vol. XIII (1963), 211-306.

[3] O. V. BESOV, On a family of functional spaces, Theorems about restrictions and extensions, *Dokl. Akad. Nauk SSSR*, 126, 6 (1959) 1163-1165.

[4] A. P. Calderon, Lebesgue spaces of differentiable functions and distributions, *Proc. of Symposium in Pure Math.* Vol. IV, Partial Differential Equations (1961), 33-49.

[5] A. P. Calderon, Proc. Symposium Diff. Equations, Berkeley, Calif. 1960.

[6] R. Courant and D. Hilbert, Methoden der Mathematischen Physik, 2 Band Springer Verlag, 1938.

[7] E. Gagliardo, Caratterizzazioni della tracce sulla frontiera relative ad alcune classi di funzioni in n variabili, *Rendiconti Sem. Mat. Padova*, Vol. 27 (1957), 248-305.

[8] M. R. Hestenes, Extension of the range of a differentiable function, *Duke Math. Journ.* Vol. 8 (1941), 183-192.

[9] L. Lichtenstein, Eine elementare Bemerkung zur reellen Analysis, *Math. Zeitschrift*, Vol. 30 (1929), 794-795.

[10] S. M. Nikolskii, Theorems about restrictions, extensions and approximation of differentiable functions of several variables (Survey Article), *Usp. Mat. Nauk.* Vol. 16, 5 (1961), 63-114.

[11] R. T. Seeley, Extension of C^∞ functions defined in a half-space, *Proc. Amer. Math. Soc.* 15 (1964), 625-626.

[12] L. I. Slobodeckii, Spaces of S. L. Sobolev of fractional order, *Dokl. Akad. Nauk. SSSR.*, Vol. 118 (1958), 243-246.

[13] M. H. Taibleson, Smoothness and differentiability conditions for functions and distributions in E_n, Dissertation. University of Chicago 1962.

SPECIAL NOTATION

SPECIAL SYMBOLS

α^* is the greatest integer $< \alpha$.
$[\alpha]$ is the greatest integer $\leqslant \alpha$.

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