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# ON ENTIRE FUNCTIONS REPRESENTED BY DIRICHLET SERIES (IV)

by Pawan Kumar KAMTHAN

1. Let

$$f(s) = \sum_{n=1}^{\infty} \mathcal{A}_n e^{s\lambda_n}, \qquad s = \sigma + it$$

represent an entire function, where

$$(1.1) \qquad \varlimsup_{n \to \infty} n/\lambda_n = D < \infty ;$$

$$(1.2) \qquad \varliminf_{n \to \infty} (\lambda_{n+1} - \lambda_n) = h > 0,$$

such that ([10], p. 201) $h\mathrm{D} \leqslant 1$, and

$$(1.3) \qquad 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty$$

as $n \to \infty$. Now $f(s)$ represents an entire function and so its abscissa of absolute convergence must be infinite, that is

$$(1.3') \qquad \varlimsup_{n \to \infty} \log |\mathcal{A}_n|/\lambda_n = -\infty .$$

Let us define $\chi_n$ as follows:

$$\chi_n = \frac{\log |\mathcal{A}_{n-1}/\mathcal{A}_n|}{\lambda_n - \lambda_{n-1}}.$$

Then $\chi_n$ is a non-decreasing function of $n$ (see [1]) and $\to \infty$ as $n \to \infty$. The fact is similar to what G. Valiron describes about rectified ratio in his book ([12], p. 32). So we have:

$$0 \leqslant \chi_1 \leqslant \chi_2 \leqslant \cdots \leqslant \chi_n \leqslant \cdots; \quad \chi_n \to \infty, n \to \infty.$$

Let $\mu(\sigma)$ be the maximum term in the representation of $\Sigma|\mathcal{A}_n|e^{\sigma\lambda_n}$ and call it as the maximum term of $f(\mathrm{S})$. Let $\lambda_{\nu(\sigma)}$

be that value of $\lambda_n$ which makes $|\mathfrak{a}_n|e^{\sigma\lambda_n}$ the maximum term and call $\lambda_{\nu(\sigma)}$ as the rank of $\mu(\sigma)$. Let us similarly correspond $\mu_{(m)}(\sigma)$ and $\lambda_{\nu(m)(\sigma)}$ to $f^{(m)}(S)$, the $m$-th derivative of $f(S)$ as we have done about $\mu(\sigma)$ and $\lambda_{\nu(\sigma)}$ connecting them with $f(S)$, where $\mu_{(0)}(\sigma) \equiv \mu(\sigma)$, $\lambda_{\nu(0)(\sigma)} \equiv \lambda_{\nu(\sigma)}$. It is well-known that ([13]; [4], pp. 1-2)

$$\log \mu(\sigma) = \int_1^\sigma \lambda_{\nu(x)}\, dx. \tag{1.4}$$

We define the order (R)$\rho$ and lower order (R)$\lambda$ of $f(s)$ as follows:

$$\varlimsup_{\sigma\to\infty} \frac{\log\log M(\sigma)}{\sigma} = \frac{\rho}{\lambda};$$

where $M(\sigma) = \underset{-\infty<t<\infty}{\text{l.u.b.}} |f(s)|$.

According to Mandelbrojt ([10], p. 216) we call $\rho$ as the Ritt order (to be written as order (R)$\rho$) of $f(s)$. We, therefore, naturally call the lower limit in $\log\log M(\sigma)/\sigma$ as $\sigma \to \infty$ to be the lower order (R) $\lambda$. However, I shall drop the word (R) in the sequel. The results starting after Theorem C and onwards are expected to be new; Theorems A and B have already appeared but the secretary wishes them to incorporate here. This paper is to be considered as a sequel to my previous papers [6; 7; 8 et 9]. For the sake of completeness I start with the following result ([4], Th. 1).

2. Theorem A. — *For an entire function* $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ *where* $\{\lambda_n\}$ *satisfies* (1.2), *then*

$$\mu(\sigma) \leqslant M(\sigma) < \mu(\sigma)\left[\left(1+\frac{1}{L\sigma}\right)\lambda_{\nu(\sigma+\sigma/\lambda_{\nu(\sigma)})} + 1\right], \tag{2.1}$$

*where* $L = h - \varepsilon$, $\varepsilon$ *being an arbitrarily taken small positive number.*

We now proceed to prove it. The left-hand inequality in (2.1) is obvious in view of Ritt's inequality:

$$|a_n|e^{\sigma\lambda_n} \leqslant M(\sigma).$$

Let

$$W(\sigma) = \sum_{n=1}^{\infty} e^{-G_n+\sigma\lambda_n}, \qquad G_n = -\log|a_n|.$$

Suppose $p$ is a positive integer $> \lambda_{\nu(\sigma)}$, such that $\chi_p > \sigma$. Let $q \geqslant p$. Now

$$e^{-G_q e^{\sigma\lambda_q}} < e^{-G_{p-1} e^{\sigma\lambda_{p-1}}} \exp\{(\sigma - \chi_p)(\lambda_q - \lambda_{p-1})\}$$
$$\leqslant \mu(\sigma) \exp\{(\sigma - \chi_p)(\lambda_q - \lambda_{p-1})\}.$$

Hence

$$W(\sigma) < \mu(\sigma) \left[p + \sum_{q=p}^{\infty} \left(\frac{e^{\sigma}}{e^{\chi_p}}\right)^{\lambda_q - \lambda_{p-1}}\right].$$

Hence in view of (1.2), if we write $x = \exp(\chi_p - \sigma)$, then $\chi > 1$ and so

$$\sum_{q=p}^{\infty} \left(\frac{e^{\sigma}}{e^{\chi_p}}\right)^{\lambda_q - \lambda_{p-1}} < \chi^{-L} + \chi^{-2L} + \cdots = \frac{1}{x^L - 1}$$

Therefore

$$W(\sigma) < \mu(\sigma) \left[p + \frac{e^{L\sigma}}{e^{Lx_p} - e^{L\sigma}}\right].$$

Let

$$p = \lambda_{\nu(\sigma + \sigma/\lambda_{\nu(\sigma)})} + 1,$$

we find that

$$e^{L\chi_p} - e^{L\sigma} > e^{L\sigma}\{e^{L\sigma/\lambda_{\nu(\sigma)}} - 1\}$$

and therefore the right-hand part in (2.1) follows.

Making use of Theorem A, we prove ([4], Th. 2, p. 5):

Theorem B. — *Let $f(s)$ be an entire function of order $\rho$ and lower order $\lambda$; $\lambda_n$ satisfies (1.2) in the expansion of $f(s)$. Then*

$$(2.2) \quad \overline{\lim_{\sigma \to \infty}} \frac{\log \lambda_{\nu(\sigma)}}{\sigma} = \frac{\rho}{\lambda}; \qquad (0 \leqslant \rho \leqslant \infty;\ 0 \leqslant \lambda \leqslant \infty).$$

As regards the proof, the upper limit is similar to a result proved by Valiron ([12], p. 33), care is only to be taken that during the course of proof, we use the fact that $\log \mu(\sigma)$ is a covex function of $\sigma$ [2]. From the previous theorem and the fact that if $\rho$ is finite, we notice that

$$\log M(\sigma) \sim \log \mu(\sigma), \qquad \sigma \to \infty.$$

Let

$$\overline{\lim_{\sigma \to \infty}} \frac{\log \lambda_{\nu(\sigma)}}{\sigma} = \rho < \infty,$$

so that from (1.4), for $\sigma \geqslant \sigma_0$

$$\log \mu(\sigma) < K + \frac{e^{(\rho+\varepsilon)\sigma}}{\rho+\varepsilon}.$$

Therefore

$$\overline{\lim_{\sigma\to\infty}} \frac{\log\log M(\sigma)}{\sigma} \leqslant \rho.$$

Let us suppose now

$$\overline{\lim_{\sigma\to\infty}} \frac{\log\log M(\sigma)}{\sigma} = \rho_1 \,(\leqslant \rho).$$

Therefore from (1.4) and the relation $\mu(\sigma) \leqslant M(\sigma)$, we find that

$$2\lambda_{\nu(\sigma)} \leqslant \int_\sigma^{\sigma+2} \lambda_{\nu(x)}\, dx < (1+\varepsilon)e^{(\sigma+2)(\rho_1+\varepsilon)},$$

and so we find that

$$\overline{\lim_{\sigma\to\infty}} \frac{\log \lambda_{\nu(\sigma)}}{\sigma} \leqslant \rho_1.$$

Therefore $\rho = \rho_1$. Therefore the ratios $\log\log M(\sigma)/\sigma$ and $\log \lambda_{\nu(\sigma)}/\sigma$ have the same upper limit. To prove that

$$\underline{\lim_{\sigma\to\infty}} \frac{\log \lambda_{\nu(\sigma)}}{\sigma} = \lambda,$$

we proceed in some other way. Let

$$\underline{\lim_{\sigma\to\infty}} \frac{\log \lambda_{\nu(\sigma)}}{\sigma} = \alpha.$$

With the help of (1.4) and $\mu(\sigma) \leqslant M(\sigma)$, one easily finds that for any constant $C > 0$.

$$C\lambda_{\nu(\sigma)} \leqslant \log \mu(\sigma + c) \leqslant \log M(\sigma + c) < e^{(\lambda+\varepsilon)(\sigma+c)},$$

for an arbitrarily large value of $\sigma$. This implies $\alpha \leqslant \lambda$. If $\lambda = 0$, then $\alpha = 0$ and there is nothing to prove. Let $0 \leqslant \alpha < \infty$. Choose $\beta$ and $\gamma$, such that $\alpha < \beta$ and $\alpha/\beta < \gamma < 1$. Hence

$$\lambda_{\nu(\sigma)} < e^{\beta\sigma}, \qquad (\gamma\sigma_n \leqslant \sigma \leqslant \sigma_n) \tag{2.3}$$

where $\{\sigma_n\}$ is a sequence of $\sigma$, such that $\sigma_n \to \infty$ as $n \to \infty$. We shall show that

$$\frac{\log M(\sigma)}{\log \mu(\sigma)} \to 1,$$

as $\sigma \to \infty$ through the sequence for which (2.3) holds (it is not assumed that $\rho$ is finite: if $\rho$ is finite we cannot claim necessarily that $\log M(\sigma) \sim \log \mu(\sigma)$).

Let $\delta$ and $\varepsilon'$ be two positive numbers such that

$$\gamma < \delta < 1; \qquad \gamma/\delta < \varepsilon' < 1.$$

Put $\delta\sigma_n = \xi_n$. Then for $n \geqslant n_0$, $\gamma\sigma_n < \varepsilon'\xi_n < \xi_n < \sigma_n - \frac{1}{2}$.

Further, let $\mu(0) = 1$, which we may without loss of generality. Then from (1.4)

$$\log \mu(\xi_n) = \log \mu(\xi_n\varepsilon') + \int_{\varepsilon'\xi_n}^{\xi_n} \lambda_{\nu(x)}\, dx.$$

But $\log \mu(\varepsilon'\xi_n) < \varepsilon'\xi_n\lambda_{\nu(\varepsilon'\xi_n)}$, so

$$\begin{aligned} \log \mu(\xi_n) &> \log \mu(\varepsilon'\xi_n) + (1 - \varepsilon')\xi_n\lambda_{\nu(\varepsilon'\xi_n)} \\ &> \frac{1}{\varepsilon'} \log \mu(\varepsilon'\xi_n). \end{aligned}$$

Hence

$$\begin{aligned} (1 - \varepsilon') \log \mu(\xi_n) &< \int_{\varepsilon'\xi_n}^{\xi_n} \lambda_{\nu(x)}\, dx \\ &< \frac{1}{\beta}\, [e^{\beta\xi_n} - e^{\beta\varepsilon'\xi_n}], \end{aligned} \tag{2.4}$$

for all $n \geqslant n_0$. But from Theorem A

$$\begin{aligned} \log M(\xi_n) &< \log \mu(\xi_n) + \log \lambda_{\nu(\xi_n + \xi_n/\lambda_{\nu(\xi_n)})} + 0(1) \\ &< \log \mu(\xi_n) + \log \lambda_{\nu(2\xi_n)} + 0(1) \\ &< \log \mu(\xi_n) + 2\beta\xi_n + 0(1). \end{aligned}$$

Hence we get for all $n \geqslant n_0$.

$$\begin{aligned} \log\log M(\xi_n) &< (1 + 0(1)) \log\log \mu(\xi_n) \\ &< (1 + 0(1))\beta\xi_n, \end{aligned}$$

from (2.4). Consequently $\lambda \leqslant \beta$ and as $(\beta - \alpha)$ can be made arbitrarily small we see that $\lambda \leqslant \alpha$; and this, when combined with the already established inequality: $\lambda \geqslant \alpha$, gives the required result.

Next, I give the following result ([5], p. 45).

THEOREM C. — *Let*

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$$

*be an entire function, where* $\{\lambda_n\}$ *satisfies* (1.2), *of order* $\rho$ *and lower order* $\lambda$ $(0 < \rho \leqslant \infty;\ 0 \leqslant \lambda < \infty)$. *Then*

$$\overline{\lim_{\sigma\to\infty}} \frac{\log/\mu(\sigma)}{\sigma\lambda_{\nu(\sigma)}} \leqslant 1 - \frac{\lambda}{\rho}.$$

*Proof.* — We have

$$\log \mu(\sigma) = \sum_{\chi_n \leqslant \sigma} (\lambda_n - \lambda_{n-1})(\sigma - \chi_n)$$
$$= \sigma\lambda_{\nu(\sigma)} - \sum_{\chi_n \leqslant \sigma} (\lambda_n - \lambda_{n-1})\chi_n.$$

But for all $n \geqslant n_0$ (from Th. B)

$$\log \lambda_n < (\rho + \varepsilon)\chi_n.$$

So we find

$$\sum_{\chi_n \leqslant \sigma} (\lambda_n - \lambda_{n-1})\chi_n > \sum_{\chi_n \leqslant \sigma,\, n \geqslant n_0} (\lambda_n - \lambda_{n-1}) \frac{\log \lambda_n}{\rho + \varepsilon}.$$

Let N be the largest integer such that $\chi \leqslant \sigma$, then

$$\sum_{\chi_n \leqslant \sigma} (\lambda_n - \lambda_{n-1})\chi_n > \frac{1}{\rho + \varepsilon} \{\lambda_N \log \lambda_N + 0(\lambda_n)\}$$
$$= \frac{1}{\rho + \varepsilon} \{\lambda_{\nu(\sigma)} \log {}_{\nu(\sigma)}\} + 0({}_{\nu(\sigma)}).$$

Si that for $\sigma \geqslant \sigma_0$

$$\log \mu(\sigma) < \sigma\lambda_{\nu(\sigma)}\left\{1 - \frac{\lambda - \varepsilon}{\rho + \varepsilon} + 0(1)\right\}$$

and the result follows.

**3.** Below I construct an example to exhibit that the result of Th. C is best possible in view of the fact that if $\lambda < \infty$, $\rho = \infty$, then

$$\overline{\lim_{\sigma\to\infty}} \frac{\log \mu(\sigma)}{\sigma\lambda_{\nu(\sigma)}} = 1. \tag{3.1}$$

*Example* 1. — Let

$$f(s) = \sum_{n=N}^{\infty} \left\{\frac{e^s}{I(\lambda_n)}\right\}^{\lambda_n},$$

where $\lambda_{n+1} = \lambda_n$; N is a positive integer, such that $I(\lambda_N) \geqslant e$ and that

$$\log I(x) = \int_{x_0}^{x} \frac{dt}{t\theta(t) \log t} \to \infty,$$

as $x \to \infty$, where further.

(*i*) $\theta(x)$ is a positive, continuous and non-decreasing function for $x \geqslant x_0$ and $\to \infty$ with $x$, and has a derivative;

$$(ii) \quad \frac{x\theta'(x)}{\theta(x)} \leqslant \frac{1}{\log x \log\log x \log\log\log x}, \qquad x \geqslant x_0.$$

*Demonstration of the aim.* — According to a result ([10], p. 217, eq. (94)) we see that the order $\rho$ of $f(s)$ is

$$= \overline{\lim_{n\to\infty}} \frac{\lambda_n \log \lambda_n}{\lambda_n \log I(\lambda_n)}$$
$$\geqslant \overline{\lim_{n\to\infty}} \frac{\log \lambda_n}{A \log\log \lambda_n},$$

from (ii) and the integral representation of $I(x)$, A being a finite number. Therefore the order $\rho$ of $f(s)$ is infinite. Let

$$\chi_n = \log\{\{I(\lambda_n)\}^{\lambda_n}/\{I(\lambda_{n-1})\}^{\lambda_{n-1}}\}/(\lambda_n - \lambda_{n-1}),$$

then it is easily found that $\chi_{n+1} > \chi_n (n > n_0)$ and that $\chi_n \to \infty$, as $n \to \infty$. Hence for $\chi_n \leqslant \sigma < \chi_{n+1}$,

$$\log \mu(\sigma) = \{\sigma - \log I(\lambda_n)\}\lambda_n, \qquad \lambda_n = \lambda_{\nu(\sigma)}.$$

Therefore

$$\frac{\log \mu(\chi_{n+1})}{\chi_{n+1}\, \lambda_{\nu(\chi_{n+1})}} = 1 - \frac{(1 + 0(1)) \log I(\lambda_n)}{\log I(\lambda_{n+1}) + 0(\log I(\lambda_n))}.$$

Further

$$\log I(\lambda_{n+1}) - \log I(\lambda_n) > (1 + 0(1)) \frac{l_2\lambda_{n+1}}{l_3\lambda_{n+1}},$$

and as $\log I(\lambda_n) < A l_2 \lambda_n$, $A = a$ constant, we find that

$$\frac{\log I(\lambda_{n+1})}{\log I(\lambda_n)} \to \infty, \qquad (n \to \infty)$$

and so

$$\frac{\log \mu(\chi_{n+1})}{\chi_{n+1}\lambda_{\nu(\chi_{n+1})}} \to 1, \qquad (n \to \infty)$$

and hence

$$\varlimsup_{n\to\infty} \frac{\log \mu(\sigma)}{\sigma\lambda_{\nu(\sigma)}} \geqslant 1. \tag{3.2}$$

Further

$$\begin{aligned}\log \mu(\chi_{n+1}) &= \frac{\lambda_n\lambda_{n+1} \log \{I(\lambda_{n+1})/I(\lambda_n)\}}{\lambda_{n+1} - \lambda_n} \\ &= (1 + 0(1))\lambda_n \log I(\lambda_{n+1}),\end{aligned}$$

and therefore

$$\log\log \mu(\chi_{n+1}) \sim \log\log I(\lambda_{n+1}) + \log \lambda_n,$$

and as $\chi_{n+1} \sim \log I(\lambda_{n+1})$, it follows that

$$\lambda = \varliminf_{\sigma\to\infty} \frac{\log\log \mu(\sigma)}{\sigma} = 0.$$

Hence from Theorem C

$$\varlimsup_{\sigma\to\infty} \frac{\log \mu(\sigma)}{\sigma\lambda_{\nu(\sigma)}} \leqslant 1. \tag{3.3}$$

Inequalities (3.2) and (3.3) provide the demonstration of our aim.

*Example* 2. — Let us consider the function defined by (see Theorem 6 [3], p. 22 where I put $\beta = 1$)

$$f(s) = \sum_{n=1}^{\infty} \left(\frac{e^s}{\lambda_n}\right)^{\lambda_n}, \quad \lambda_{n+1} = \alpha^{\lambda_n}; \qquad \alpha \geqslant e; \qquad \lambda_1 = \alpha.$$

The function $f(s)$ is certainly an entire function on account of (1.3)′. The order $\rho$ of $f(s)$ is in this case

$$= \varlimsup_{n\to\infty} \frac{\lambda_n \log \lambda_n}{\lambda_n \log \lambda_n} = 1.$$

Also

$$\mu(\sigma) = \{e^\sigma/\lambda_n\}^{\lambda_n}; \qquad \lambda_n = \lambda_{\nu(\sigma)},$$

for $\chi_n \leqslant \sigma < \chi_{n+1}$, where

$$\chi_n = \frac{\lambda_n \log \lambda_n - \lambda_{n-1} \log \lambda_{n-1}}{\lambda_n - \lambda_{n-1}}.$$

Then

$$\log \mu(\chi_n) = \lambda_n(\lambda_n - \log \lambda_n)$$
$$= \frac{\lambda_n \lambda_{n-1}}{\lambda_n - \lambda_{n-1}} \log (\lambda_n/\lambda_{n-1})$$
$$(3.4) \qquad = (1 + 0(1))\, \lambda_{n-1} \log \lambda_n;$$
$$\log\log \mu(\chi_n) = (1 + 0(1)) + \log \lambda_{n-1} + \log\log \lambda_n.$$

Also $\chi_n \to \infty$ as $n \to \infty$, we see that

$$(3.5) \qquad \frac{\log\log \mu(\chi_n)}{\chi_n} = 0(1) + \frac{1}{\chi_n}(\log \lambda_{n-1} + \log\log \lambda_n).$$

Now

$$\frac{\log \lambda_{n-1}}{\chi_n} = \frac{\log \lambda_{n-1}(\lambda_n - \lambda_{n-1})}{\lambda_n \log \lambda_n - \lambda_{n-1} \log \lambda_{n-1}}$$
$$= \frac{\lambda_n \log \lambda_{n-1} + 0(\lambda_n)}{\lambda_n\, \lambda_{n-1} \log \alpha + 0(\lambda_n)}$$
$$(3.6) \qquad = (1 + 0(1)) \frac{\log \lambda_{n-1}}{\lambda_{n-1} \log \alpha} \to 0 \quad (n \to \infty).$$

Also $\log\log \lambda_n = (1 + 0(1) \log \lambda_{n-1}$ and so the right-hand term in (3.5) $\to 0$ as $n \to \infty$ in view of (3.6). Therefore the lower order $\lambda$ of $f(s)$ is zero on account of (3.5). Hence from Theorem C

$$(3.7) \qquad \overline{\lim_{\sigma\to\infty}} \frac{\log \mu(\sigma)}{\sigma \lambda_{\nu(\sigma)}} \leqslant 1.$$

Also

$$\frac{\log \mu(\chi_{n+1})}{\chi_{n+1} \lambda_{\nu(\chi_{n+1})}} = 1 - \frac{\log \lambda_n}{\chi_{n+1}}$$
$$= 1 - \frac{(\lambda_{n+1} - \lambda_n) \log \lambda_n}{\lambda_{n+1} \log \lambda_{n+1} - \lambda_n \log \lambda_n} \to 1 \quad (n \to \infty),$$

for the above solution see the technique used in getting (3.6). Hence

$$(3.8) \qquad \overline{\lim_{\sigma\to\infty}} \frac{\log \mu(\sigma)}{\sigma \lambda_{\nu(\sigma)}} \geqslant 1.$$

Therefore from (3.7) and (3.8) one gets

$$\overline{\lim_{\sigma\to\infty}} \frac{\log \mu(\sigma)}{\sigma \lambda_{\nu(\sigma)}} = 1,$$

giving thereby again a best possible nature of Theorem C in case $\lambda = 0$ and $\rho < \infty$.

4. Results involving derivatives of $f(s)$:

I have already spoken in the article 1 about $\mu_{(m)}(\sigma)$ and $\lambda_{\nu^{(m)}(\sigma)}$. I first prove:

THEOREM D. — *For all $\sigma \geqslant \sigma_0$ ($\sigma_0$ is a fixed large number) one should have:*

$$\mu_{(m)}(\sigma) > \mu(\sigma) \left[\frac{\log \mu(\sigma)}{\sigma}\right]^m,$$

*$m$ is an integer $\geqslant 0$. This result I stated in a previous paper* ([6], p. 235) *without proof.*

*Proof.* — We have:

$$\lambda_{\nu^{(m)}(\sigma)} \leqslant \frac{\mu_{(m+1)}(\sigma)}{\mu_{(m)}(\sigma)} \leqslant \lambda_{\nu^{(m+1)}(\sigma)}, \qquad m = 0, 1, \dots \tag{4.1}$$

When $m = 0$ in (4.1), it reduces to a result which I have proved in ([3], p., Theorem 2) as follows

$$\mu_{(1)}(\sigma) = |a_{\nu^{(1)}(\sigma)}|\lambda_{\nu^{(1)}(\sigma)} \exp(\sigma\lambda_{\nu^{(1)}(\sigma)}) \leqslant \lambda_{\nu^{(1)}(\sigma)}\mu(\sigma);$$
$$\mu_{(1)}(\sigma) = |a_{\nu^{(1)}(\sigma)}|\lambda_{\nu^{(1)}(\sigma)} \exp(\sigma\lambda_{\nu^{(1)}(\sigma)}) \geqslant |a_{\nu(\sigma)}|\lambda_{\nu(\sigma)} \exp(\sigma\lambda_{\nu(\sigma)})$$
$$= \lambda_{\nu(\sigma)}\mu(\sigma).$$

The case $m \geqslant 1$ can aslo be treated by simple definitions, for let

$$f^{(m)}(S) = \Sigma A_n e^{s\lambda_n}, \qquad \lambda_{\nu^{(m)}(\sigma)} = \lambda_N; \qquad \lambda_{\nu^{(m+1)}(\sigma)} = \lambda_{N_1},$$

then

$$\mu_{(m+1)}(\sigma) = \lambda_{N_1}|A_{N_1}| \exp(\sigma\lambda_{N_1}) \leqslant \lambda_{N_1}\mu_{(m)}(\sigma),$$

and

$$\mu_{(m)}(\sigma) = \frac{1}{\lambda_N}(\lambda_N|A_N| \exp(\sigma\lambda_N)) \leqslant \frac{\mu_{(m+1)}(\sigma)}{\lambda_{\nu^{(m)}(\sigma)}},$$

and so these two inequalities complete (4.1) and from which we have:

$$\lambda_{\nu(\sigma)} \leqslant \frac{\mu_{(1)}(\sigma)}{\mu_{(\sigma)}} \leqslant \lambda_{\nu^{(1)}}(\sigma) \leqslant \frac{\mu_{(2)}(\sigma)}{\mu_{(1)}(\sigma)} \leqslant \cdots \leqslant \lambda_{\nu^{(m-1)}(\sigma)}$$
$$\leqslant \frac{\mu_{(m)}(\sigma)}{\mu_{(m-1)}(\sigma)} \leqslant \lambda_{\nu^{(m)}(\sigma)}.$$

Multiplying the ratios involving these $\mu$'s one finds that

$$\begin{aligned} \frac{\mu_{(m)}(\sigma)}{\mu_{(\sigma)}} &\geqslant \lambda_{\nu^{(m-1)}(\sigma)} \ldots \lambda_{\nu(\sigma)} \\ &\geqslant (\lambda_{\nu(\sigma)})^m. \end{aligned} \tag{4.2}$$

Now from (1.3)′ we get, for K to be sufficiently large,

$$\begin{aligned} \log |a_{\nu(\sigma)}| &< - K\lambda_{\nu(\sigma)}; \quad \sigma \geqslant \sigma_0 \\ |a_{\nu(\sigma)}| &< \exp(- k\lambda_{\nu(\sigma)}) < 1, \quad \sigma \geqslant \sigma_0. \end{aligned} \tag{4.3}$$

Again

$$\begin{aligned} \log \mu(\sigma) &= \log |a_{\nu(\sigma)}| + \sigma\lambda_{\nu(\sigma)} \\ &< \sigma\lambda_{\nu(\sigma)}, \qquad \sigma \geqslant \sigma_0 \end{aligned} \tag{4.4}$$

from (4.3). The inequalities (4.2) and (4.4) result in for $\sigma \geqslant \sigma_0$

$$\frac{\mu_{(m)}(\sigma)}{\mu(\sigma)} > \left(\frac{\log \mu(\sigma)}{\sigma}\right)^m.$$

The above theorem is useful in deducing the following interesting.

Theorem E. — *One has (with the terms involved in to be known):*

$$\overline{\lim_{\sigma\to\infty}} \frac{\log (\mu_{(m)}(\sigma)/\mu(\sigma))^{1/m}}{\sigma} = \frac{\rho}{\lambda};$$

*Proof.* — We have:

$$\begin{aligned} \frac{\mu_{(m)}(\sigma)}{\mu(\sigma)} &\leqslant \lambda_{\nu^{(1)}(\sigma)} \ldots \lambda_{\nu^{(m)}(\sigma)} \\ &\leqslant (\lambda_{\nu^{(m)}(\sigma)})^m. \end{aligned}$$

Now $f^{(m)}(s)$ also posses the same order $\rho$ and lower order $\lambda$ as $f(s)$ has, and so (cf. Theorem B)

$$\overline{\lim_{\sigma\to\infty}} \frac{\log \lambda_{\nu^{(m)}(\sigma)}}{\sigma} = \frac{\rho}{\lambda};$$

consequently

$$\overline{\lim_{\sigma\to\infty}} \frac{\log (\mu_{(m)}(\sigma)/\mu(\sigma))^{1/m}}{\sigma} \leqslant \frac{\rho}{\lambda}; \tag{4.5}$$

But Theorem D provides us the inequality (to be deduced with the help of Theorem B and (1.4) [1]

$$(4.6) \qquad \overline{\lim_{\sigma\to\infty}} \frac{\log (\mu_{(m)}(\sigma)/\mu(\sigma))^{1/m}}{\sigma} \geqslant \frac{\rho}{\lambda};$$

The inequalities (4.5) and (4.6) yield the desired result.

*Remark.* — Theorem D has been stated without any proof by Srivastav ([11], p. 89 (i)) and that too under the restrictive condition that $\lambda > 0$. The proof of Theorem D removes this superflous restriction which Srivastav asserts. Secondly, Srivastav claims to prove Theorem E but to the best my surprise there is no clue available to its proof in his paper wherever he mentions it. I whish to add that I have stated Theorem D without proof in a recent paper of mine ([6], Theorem 1).

**5.** Towards the end of this paper, I would like to add a new result on the mean values of entire Dirichlet functions. To the best of my knowledge I introduced these means and discovered their properties relating to the order and lower order of $f(S)$ in a recent paper [9]. I do here a little more. I define

$$A_k(\sigma) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(S)|^k \, dt,$$

where the sequence $\{\lambda_n\}$ satisfies (1.1)-(1.3); $0 < k < \infty$.

THEOREM F. — *If $f(S)$ satisfies the conditions stated in* § 5, *then we have:*

$$\overline{\lim_{\sigma\to\infty}} \frac{\log\log A_k(\sigma)}{\sigma} = \frac{\rho}{\lambda};$$

(1) From (1.4), (i)

$$\log \mu(\sigma) \leqslant (1 + 0(1))\sigma\lambda_{\nu(\sigma)} \quad \text{and so} \quad \log\log \mu(\sigma)/\sigma \leqslant 0(1) + \log \lambda_{\nu(\sigma)}/\sigma;$$

and (ii) for $k > 0$, $\log \mu(\sigma + k) \geqslant k\lambda_{\nu(\sigma)}$ and so

$$\log\log \mu(\sigma + k)/(1 + 0(1))\,(\sigma + k) \geqslant 0(1) + \log \lambda_{\nu(\sigma)}/\sigma.$$

From (i) et (ii) one deduces that

$$\overline{\lim_{\sigma\to\infty}} \log\log \mu(\sigma)/\sigma = \overline{\lim_{\sigma\to\infty}} \log \lambda_{\nu(\sigma)}/\sigma.$$

*Remark.* — If $k = 2$, I have got the above result in a recent paper ([9], Theorem 1) where I supposed further that $\chi_n$ was non-decreasing. Here we need not, as one will soon find, make this supposition.

*Proof of Theorem F.* — One does have

$$A_k(\sigma) \leqslant \{M_s(\sigma)\}^k,$$

where

$$M_s(\sigma) = \max_{|t| \leqslant T} |f(\sigma + it)|.$$

But (see for references [9] and also [10])

$$\overline{\lim_{\sigma \to \infty}} \frac{\log \log M_s(\sigma)}{\sigma} = \frac{\rho}{\lambda};$$

So we find that

$$(5.1) \qquad \overline{\lim_{\sigma \to \infty}} \frac{\log \log A_k(\sigma)}{\sigma} \leqslant \frac{\rho}{\lambda};$$

To get the other part, it is sufficient to consider $f(S)$ in the representation given by:

$$f(S) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n}.$$

Then, if $S' = \Delta + ix$; $a_n = \alpha_n + i\beta_n$, we have

$$f(\Delta + ix)$$
$$= \sum_{n=0}^{\infty} [(\alpha_n \cos \lambda_n x - \beta_n \sin \lambda_n x) + i(\alpha_n \sin \lambda_n x + \beta_n \cos \lambda_n x)] e^{\Delta\lambda_n};$$

$$Rl\{f(\Delta + ix)\} = \sum_{n=0}^{\infty} (\alpha_n \cos \lambda_n x - \beta_n \sin \lambda_n x) e^{\Delta\lambda_n}.$$

Therefore

$(_*)$

$$\alpha_m e^{\Delta\lambda_m} = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} Rl\{f(\Delta + ix)\} \cos \lambda_m x \, dx, \qquad m > 0.$$

$(_{**})$

$$- \beta_m e^{\Delta\lambda_m} = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} Rl\{f(\Delta + ix)\} \sin \lambda_m x \, dx, \qquad m > 0.$$

$$\alpha_0 = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} Rl\{f(\Delta + ix)\} \, dx.$$

Therefore from $(_*)$ and $(_{**})$

$$\begin{aligned}(5.2)\quad & Rl\{f(\sigma + it)\} = \sum_{n=0}^{\infty} (\alpha_n \cos \lambda_n t - \beta_n \sin \lambda_n t) e^{\sigma\lambda_n} \\ & = \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} Rl\{f(\Delta + ix)\} \left\{1 + \sum_{n=1}^{\infty} \cos\{(x-t)\lambda_n\} e^{(\sigma-\Delta)\lambda_n}\right\} dx.\end{aligned}$$

We can treat (5.2) as an analogue to Poisson's formula in power series. Therefore, if we start our series for $f(s)$ from $n = 1$ to $\infty$, then

$$|f(s)| \leqslant \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\Delta + ix)| 2 \sum_{n=1}^{\infty} \exp\{(\sigma - \Delta)\lambda_n\}\, dx,$$

and since the right-hand side is independent of $t$, one finds that

$$\begin{aligned}(5.3)\quad M(\sigma) & \leqslant 2A(\Delta) \left(\sum_{n=1}^{n_0-1} + \sum_{n=n_0}^{\infty}\right) \exp\{(\sigma - \Delta)\lambda_n\} \\ & < 2A(\Delta) \left[(n_0 - 1)\exp\{(\sigma - \Delta)\lambda_1\} + \sum_{n=n_0}^{\infty} \exp\{(\sigma - \Delta)\lambda_n\}\right].\end{aligned}$$

But

$$\sum_{n=n_0}^{\infty} \exp\{(\sigma - \Delta)\lambda_n\} < \exp\{(\sigma - \Delta)\lambda_1\} \{1 + \exp(\sigma - \Delta)L + \exp(\sigma - \Delta)2L + \cdots\}.$$

Therefore

$$M(\sigma) < 2A(\Delta) \left[(n_0 - 1)\exp(\sigma - \Delta)\lambda_1 + \frac{\exp\{(\sigma - \Delta)\lambda_1\}\exp(\Delta L)}{\exp(\Delta L) - \exp(\sigma L)}\right].$$

Let $\Delta = \sigma + \eta$, $\eta > 0$. Then on simplifications, one gets

$$(5.4)\qquad M(\sigma) < 0(1)\, A(\sigma + \eta).$$

Similarly taking $\{f(s)\}^k$ instead of $f(s)$, one can prove that

$$(5.5)\qquad (M(\sigma))^k < 0(1) A_k(\sigma + \eta),$$

where the constants $0(1)$ in (5.4) and (5.5) might not be the same, and so

$$(5.6)\qquad \overline{\lim_{\sigma\to\infty}} \frac{\log\log A_k(\sigma)}{\sigma} \geqslant \overline{\lim_{\sigma\to\infty}} \frac{\log\log M(\sigma)}{\sigma} = \frac{\rho}{\lambda};$$

The inequalities (5.1) and (5.6) yield the required result. I might like to discuss further results on the means defined by $A_k(\sigma)$ in a next sequel of my work.

Before I close up the discussion, I would like to express my warm thanks to the University Grants Commission, India about its partial support for the project undertaken by me.

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