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ON RESTRICTED MEASURABILITY

by A. K. MOOKHOPADHYAYA

1. Introduction and Definitions.

The purpose of the present paper is to study some properties of the restricted measurability [5] and to show that a Radon measure similar to that of [4] can be constructed with the help of the notion of the restricted measurability. Before we go into details, we write out, for the sake of completeness, a few definitions and notations some of which are borrowed from the above papers and the standard texts such as Halmos [1] and Kelley [2].

1.1. Definition. — \( \mu \) is a measure (Carathéodory) on \( X \) if \( \mu \) is a function on the family of all subsets of \( X \) to \( 0 \leq t \leq \infty \) such that

(i) \( \mu(0) = 0 \)

(ii) \( 0 \leq \mu A \leq \sum_{n=1}^{\infty} \mu B_n \), whenever \( A \subset \bigcup_{n=1}^{\infty} B_n \subset X \).

1.2. Definition. — \( A \subset X \) is \( \mu \)-measurable if for every \( T \subset X \)

\( \mu T = \mu (T \cap A) + \mu (T \sim A) \)

where \( \mu \) is a measure on \( X \).

1.3. Definition. — A partition is a finite or infinite disjoint sequence \( \{E_i\} \) of sets such that \( \bigcup_i E_i = X \).
1.4. Definition. — A partition \( \{E_i\} \) is called a \( \mu \)-partition if
\[
\mu A = \sum_{i=1}^{\infty} \mu(A \cap E_i)
\]
for every \( A \) in \( X \) and where \( \mu \) is a measure on \( X \).

1.5. Definition. — If \( \mu \) is a measure on \( X \), then a partition \( \{E_i\} \) is called a \( \mu \)-partition \( \mathcal{F} \) if for every \( E \) of \( \mathcal{F} \)
\[
\mu(TE) = \sum_{i=1}^{\infty} \mu(TE \cap E_i)
\]
whenever \( T \subset X \).

1.6. Definition. — If \( \{E_i\} \) and \( \{F_j\} \) are partitions, then \( \{E_i\} \) is called a subpartition of \( \{F_j\} \) if each \( E_i \) is contained in some \( F_j \).

1.7. Definition. — A set \( E \) is a \( \mu \)-set if the partition \( \{E, E'\} \) is a \( \mu \)-partition.

1.8. Definition. — A set \( D \) is a \( \mu \)-set \( \mathcal{F} \) if the partition \( \{D, D'\} \) is a \( \mu \)-partition \( \mathcal{F} \).

1.9. Definition. — \( A \) is \( \mu \)-measurable \( \mathcal{F} \) if \( \mu \) is a measure and, for each member \( E \) of \( \mathcal{F} \)
\[
\mu(TE) = \mu(TE \cap A) + \mu(TE \sim A)
\]
whenever \( T \subset X \).

1.10. Definition. — \( \mathcal{F} \) is \( \mu \)-convenient if \( \mu \) is a measure, \( \mathcal{F} \) is hereditary, and corresponding to each \( T \) of finite \( \mu \) measure there exists such a sequence \( C \) that \( \mu \left( T \sim \bigcup_{j=0}^{\infty} C_j \right) = 0 \) and for each integer \( n \), \( C_n \cap C_{n+1} \in \mathcal{F} \) and \( C_n \) is a \( \mu \)-set \( \mathcal{F} \).

1.11. Definition. — \( \text{Sect}(\mu, B) \) is the function \( f \) on the subsets of \( X \) such that \( f(\alpha) = \mu(\alpha B) \) for \( \alpha \subset X \).

1.12. Definition. — If \( \varphi \) metrizes \( X \), then
\[
\text{dist}(A, B) = \inf \{ \varphi(x, y) ; x \in A, y \in B \}.
\]
1.13. Definition. — If $X$ is a topological space, then $\mu$ is a Radon measure on $X$ if $\mu$ is a measure and

(i) open sets are $\mu$-measurable
(ii) if $C$ is compact, then $\mu C < \infty$
(iii) if $\alpha$ is open, then $\mu \alpha = \sup \{ \mu C ; C \text{ compact}, C \subset \alpha \}$
(iv) if $A \subset X$, then $\mu A = \inf \{ \mu \alpha, \alpha \text{ open}, A \subset \alpha \}$.

1.14. Definition. — $(D, \prec)$ is a directed set if $D \neq \emptyset$, $D$ is partially ordered by $\prec$, and for each $i, j \in D$, there exists $k \in D$ with $i < k, j < k$.

Let $X$ be a regular topological space; $\mathcal{B}$ be a base for the topology; $(D, \prec)$ be a directed set and for each $i \in D$, $\mu_i$ be a Radon measure on $X$.

For each $\alpha \in \mathcal{B}$, let

$$g(\alpha, E) = \frac{\lambda t}{i \in D} \text{ Sect } (\mu_i, E) \alpha \text{ where } E \text{ is a member of } F.$$

Let $\varphi(A, E) = \inf \{ \sum_{\alpha \in H} g(\alpha, E) ; H \text{ countable}, H \subset \mathcal{B}, A \subset \bigcup_{\alpha \in H} \alpha \}$

and $\varphi^*(A, E) = \inf \sup_{\alpha \text{ open, } G \text{ compact} \subset X} \varphi(C, E)$ where $A \subset X$.

Then $\varphi$ is a measure on $X$ generated by $g$ and $\mathcal{B}$ [3].

2. Theorems and Corollaries.

2.1. Theorem. — Product of two $\mu$-partitions $F$ is a $\mu$-partition $F$.

Proof. — Let $\{E_i\}$ and $\{F_i\}$ be two $\mu$-partitions $F$, then for every $E$ of $F$

$$\mu(TE) = \sum_{i=1}^{\infty} \mu(TE \cap E_i) \text{ and } \mu(TE) = \sum_{i=1}^{\infty} \mu(TE \cap F_i)$$

whenever $T \subset X$.

Since

$$\sum_{i,j} \mu(TE \cap E_i \cap F_j) = \sum_j \{ \sum_i \mu(TEF_j \cap E_i) \}$$

$$= \sum_j \mu(TE \cap F_j)$$

$$= \mu(TE),$$

the proof is complete.
2.2. Theorem. — If a subpartition \( \{ F_i \} \) of a partition \( \{ E_i \} \) is a \( \mu \)-partition \( F \), then \( \{ E_i \} \) is a \( \mu \)-partition \( F \).

Proof. — For \( T \subset X \) and any member \( E \) of \( F \), we have
\[
\sum_i \mu(TE \cap E_i) = \sum_i \mu(TE \cap \bigcup_j E_{ji})
\]
where \( \bigcup_j E_{ji} = E_i \) and \( E_{ji} \) is a member of \( \{ F_i \} \)
\[
\leq \sum_i \sum_j \mu(TE \cap E_{ji}) = \mu(TE),
\]
since \( \{ F_i \} \) is a \( \mu \)-partition \( F \). The reverse inequality is, however, clear. This proves the theorem.

2.3. Theorem. — A partition \( \{ E_i \} \) is a \( \mu \)-partition \( F \) if each \( E_i \) is a \( \mu \)-set \( F \).

Proof. — Suppose that each \( E_i \) is a \( \mu \)-set \( F \). Then for \( E \) in \( F \) and \( T \subset X \), we have
\[
\mu(TE) = \mu(TE \cap E_1) + \mu(TE \cap E_2) + \cdots
\]
And
\[
\mu(TE) = \mu(TE \cap E_1) + \mu(TE \cap \{ E_2 \cup E_3 \cup \cdots \})
\]
So,
\[
\mu(TE) = \mu(TE \cap E_1) + \mu(TE \cap E_2) + \cdots
\]
Proceeding in this way, we ultimately obtain
\[
\mu(TE) = \mu(TE \cap E_1) + \mu(TE \cap E_2) + \cdots = \sum_i \mu(TE \cap E_i).
\]
This proves that \( \{ E_i \} \) is a \( \mu \)-partition \( F \).

Conversely, suppose that \( \{ E_i \} \) is a \( \mu \)-partition \( F \). Then for every \( E \) of \( F \) and \( T \subset X \), we have \( \mu(TE) = \sum_i \mu(TE \cap E_i) \).

Replacing \( T \) by \( T \cap \{ E_2 \cup E_3 \cup \cdots \} \), we obtain
\[
\mu(TE \cap \{ E_2 \cup E_3 \cup \cdots \}) = \sum_i \mu(TE \cap \{ E_2 \cup E_3 \cup \cdots \} \cap E_i)
\]
\[
= \mu(TE \cap E_2) + \mu(TE \cap E_3) + \cdots
\]
So,
\[ \mu(TE) = \mu(TE \cap E_i) + \mu(TE \cap \{E_2 \cup E_3 \cup \cdots\}) \]
\[ = \mu(TE \cap E_i) + \mu(TE \cap E_i'). \]

This shows that \(E_i\) is a \(\mu\)-set \(F\). Similarly, it can be shown that each \(E_i\), \(i = 2, 3, \ldots\) is a \(\mu\)-set \(F\).

**Corollary.** — *If \(F\) is \(\mu\)-convenient, then any \(\mu\)-partition \(F\) is a \(\mu\)-partition.*

**Proof.** — Let the partition \(\{E_i\}\) be a \(\mu\)-partition \(F\), then each \(E_i\) is a \(\mu\)-set \(F\). Since \(F\) is \(\mu\)-convenient, by Theorem 3.4 [5], a \(\mu\)-set \(F\) is a \(\mu\)-set. So, each \(E_i\) is a \(\mu\)-set and consequently the partition \(\{E_i\}\) is a \(\mu\)-partition \(F\). (p. 48 [1]).

In the following two theorems, we shall suppose that \(\rho\) metrizes \(X\).

**2.4. Theorem.** — *If \(F\) is hereditary and \(\mu(A \cup B) = \mu A + \mu B\) whenever \(A\) and \(B\) are such members of \(F\) that \(d(A, B) > 0\), then each open set is a \(\mu\)-set \(F\).*

**Proof.** — This theorem is due to Trevor J. McMinn [5].

**2.5. Theorem.** — *If \(F\) is \(\mu\)-convenient and every open set is a \(\mu\)-set \(F\), then \(\mu\) is a metric outer measure.*

**Proof.** — It follows from Theorem 3.4 [5] that each open set is a \(\mu\)-set. Let \(A\) and \(B\) be two sets with \(d(A, B) > 0\). Let \(\alpha\) be an open set such that \(A \subset \alpha\) and \(\alpha \cap B = 0\). Then
\[ \mu(A \cup B) = \mu(\{A \cup B\} \cap \alpha) + \mu(\{A \cup B\} \sim \alpha) \]
\[ = \mu A + \mu B. \]

In the following theorems, we shall suppose that \(X\) is a regular topological space and \(\mathfrak{B}\) be a base for the topology.

**2.6. Theorem.** — *If \(A\) and \(B\) are disjoint, closed compact sets, then*
\[ \varphi(A \cup B, E) = \varphi(A, E) + \varphi(B, E) \quad \text{for each} \quad E \text{ of } F. \]

**Proof.** — Let \(\alpha\) and \(\beta\) be two open sets such that \(A \subset \alpha\), \(B \subset \beta\) and \(\alpha \cap \beta = 0\). This is possible, since \(X\) is regular. If \(\varepsilon > 0\) is arbitrary, there exists a sequence \(\{\gamma_n\}\) of open
sets such that
\[ A \cup B \subseteq \bigcup_n \gamma_n \quad \text{and} \quad \sum_n g(\gamma_n, E) \leq \varphi(A \cup B, E) + \varepsilon. \]

Let \( \gamma'_n = \gamma_n \cap \alpha \) and \( \gamma''_n = \gamma_n \cap \beta \), then \( \gamma'_n, \gamma''_n \) are open and
\[ A \subseteq \bigcup_n \gamma'_n, \quad B \subseteq \bigcup_n \gamma''_n. \]

So,
\[ \varphi(A, E) + \varphi(B, E) \leq \sum_n \{ g(\gamma'_n, E) + g(\gamma''_n, E) \} \]
\[ = \sum_n \{ g(\gamma_n \cap \alpha, E) + g(\gamma_n \cap \beta, E) \} \]
\[ = \sum_n \left\{ \frac{\lambda t}{i \in D} \text{Sect}(\mu_i, E)(\gamma_n \cap \alpha) \right\} \]
\[ + \sum_n \left\{ \frac{\lambda t}{i \in D} \text{Sect}(\mu_i, E)(\gamma_n \cap \beta) \right\} \]
\[ \leq \sum_n \left\{ \frac{\lambda t}{i \in D} \text{Sect}(\mu_i, E)\gamma_n \right\} \]
\[ = \sum_n g(\gamma_n, E) \]
\[ \leq \varphi(A \cup B, E) + \varepsilon. \]

Since \( \varepsilon > 0 \) is arbitrary, we have
\[ \varphi(A, E) + \varphi(B, E) \leq \varphi(A \cup B, E). \]

The reverse inequality is clear, because \( \varphi \) is a Carathéodory measure. This proves the theorem.

2.7. Theorem. — For each \( E \) of \( F \), \( \varphi^* \) is a Radon measure on \( X \).

Proof. — (i) If \( \alpha \) is any open set, by definition
\[ \varphi^*(\alpha, E) = \sup_{C \text{ compact}} \varphi(C, E) \leq \varphi(\alpha, E). \]

So, for any \( A \subseteq X \), we have
\[ \varphi^*(A, E) = \inf_{\alpha \text{ open}} \sup_{C \text{ compact}} \varphi(C, E) = \inf_{A \subseteq \alpha} \varphi^*(\alpha, E) \]
\[ \leq \inf_{\alpha \text{ open}} \varphi(\alpha, E) = \varphi(A, E). \]
If C is compact and α is open, C c α, we have

\[ \varphi(C, E) \leq \varphi^*(\alpha, E), \quad \text{so} \quad \varphi(C, E) \leq \varphi^*(C, E). \]

Therefore, if C is compact, \( \varphi(C, E) = \varphi^*(C, E) \). But, it is clear that for any compact C, \( \varphi(C, E) < \infty \) and hence \( \varphi^*(C, E) < \infty \).

(ii) Let α be an open set, \( T \subset X \) and \( \varepsilon > 0 \) arbitrary. Since for any \( A \subset X \), \( \varphi^*(A, \varepsilon) = \inf_{v \text{ open}} \varphi(v, \varepsilon) \), there exists open set \( T' \), \( T \subset T' \) and \( \varphi^*(T', E) < \varphi^*(T, E) + \varepsilon \).

Also, \( \varphi^*(\alpha, E) = \sup_{C \text{ compact}} \varphi(C, E) \).

Therefore, since X is regular, there exists a closed compact set \( C_1 \subset T' \cap \alpha \) such that \( \varphi^*(T' \cap \alpha, E) \leq \varphi(C_1, E) + \varepsilon \). Similarly, there exists a closed compact set \( C_2 \subset T' \sim C_1 \) such that \( \varphi^*(T' \sim C_1, E) \leq \varphi(C_2, E) + \varepsilon \).

So,

\[ \varphi^*(T \cap \alpha, E) + \varphi^*(T \sim \alpha, E) \leq \varphi^*(T' \cap \alpha, E) + \varphi^*(T' \sim C_1, E) \leq \varphi(C_1, E) + \varphi(C_2, E) + 2\varepsilon = \varphi^*(C_1 \cup C_2, E) + 2\varepsilon, \text{ by Theorem 2.6} \leq \varphi^*(T', E) + 2\varepsilon \leq \varphi^*(T, E) + 3\varepsilon. \]

Since \( \varepsilon > 0 \) is arbitrary, this shows that α is \( \varphi^* \)-measurable.

The other properties are evident. This proves the theorem.

2.8. Theorem. — If A and B are sets of which any one of them is open and \( A \cap B = \emptyset \), then

\[ \varphi^*(A \cup B, E) = \varphi^*(A, E) + \varphi^*(B, E) \quad \text{for each} \quad E \text{ of } F. \]

Proof. — Let A be open and so it is \( \varphi^* \)-measurable. Hence

\[ \varphi^*(A \cup B, E) = \varphi^*\{ (A \cup B) \cap A, E \} + \varphi^*\{ (A \cup B) \sim A, E \} = \varphi^*(A, E) + \varphi^*(B, E). \]

2.9. Theorem. — If X is a metric space, then \( \varphi^* \) is a metric outer measure.
Proof. — This is clear.

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