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On Φ-bounded harmonic functions


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ON $\Phi$-BOUNDED HARMONIC FUNCTIONS

by Mitsuru Nakai

1. Throughout this paper, we denote by $\Phi(t)$ a non-negative real-valued function defined on the half real line $[0, \infty) = (t : 0 \leq t < \infty)$. A harmonic function $u$ on a Riemann surface $R$ is called $\Phi$-bounded if the composite function $\Phi(|u|)$ admits a harmonic majorant on $R$, i.e. there exists a harmonic function $h$ such that $\Phi(|u|) \leq h$ on $R$. We denote by $H\Phi = H\Phi(R)$ the totality of $\Phi$-bounded harmonic functions on a Riemann surface $R$ and by $O_{\Phi}$ the class of all Riemann surfaces on which every $\Phi$-bounded harmonic function reduces to a constant. In our study, the following two quantities will play an important role:

$$d(\Phi) = \limsup_{t \to \infty} \Phi(t)/t$$
$$d(\Phi) = \liminf_{t \to \infty} \Phi(t)/t$$

The properties of $H\Phi$-functions on Riemann surfaces and the class $O_{\Phi}$ are first investigated by Parreau [3] for the special $\Phi(t)$ which is increasing and convex (1). In the present paper we shall investigate the same problem for general $\Phi(t)$. Our conclusion is, roughly speaking, that Parreau's result about $O_{\Phi}$ holds essentially for general $\Phi(t)$ and his result about properties of $H\Phi$-functions can be derived by assuming $d(\Phi) > 0$ instead of increasingness and convexity which is, in a sense, the weakest condition.

2. As for the class $O_{\Phi}$, Parreau [3] showed that the class $O_{\Phi}$ for

(1) For such a function, it is well-known that $\overline{d}(\Phi) = d(\Phi) > 0$. 
increasing and convex $\Phi(t)$ coincides with $O_{HP}$ or $O_{HB}$ (2) according to $\bar{d}(\Phi) < \infty$ or $\bar{d}(\Phi) = \infty$, respectively. Now we ask what can be said about $O_{\text{ns}}$ for general $\Phi(t)$. The answer is given by

**Theorem 1.** — If $\bar{d}(\Phi) < \infty$ (resp. $\bar{d}(\Phi) = \infty$), then $O_{\text{ns}} \subset O_{HP}$ (resp. $O_{\text{ns}} \supset O_{HB}$).

This was proved implicitly in our former paper [2] by using Wiener's compactification of Riemann surfaces. We shall again give an alternating elementary proof in § 1. In this theorem, we cannot replace the inclusion relation by the equality in general. But the function $\Phi(t)$, by which the equality does not hold in the above theorem, is very singular and trivial one from the view point of $H\Phi$-functions as the following shows:

**Theorem 2.** — (i) If $\Phi(t)$ is bounded on $[0, \infty)$, then $O_{\text{ns}}$ consists of all closed Riemann surfaces;

(ii) If $\Phi(t)$ is completely unbounded (3) on $[0, \infty)$, then $O_{\text{ns}}$ consists of all open Riemann surfaces;

(iii) If $\Phi(t)$ is not bounded and not completely unbounded, then $O_{\text{ns}} = O_{HP}$ or $O_{HB}$ according to $\bar{d}(\Phi) < \infty$ or $\bar{d}(\Phi) = \infty$, respectively.

This was proved in [2] and determines the class $O_{\text{ns}}$ completely for any possible $\Phi(t)$. This is easily proved by using Theorem 1. We will do this also in § 1.

Observing Theorem 2, we are tempted to conclude that $H\Phi$-property is closely related to positiveness or boundedness properties except trivial $\Phi$'s as in (i) or (ii). Next we consider this problem. To state the problem formally, let us recall three notions for harmonic functions: essentially positive, quasi-bounded and singular.

3. A harmonic function $u$ on a Riemann surface $R$ is called **essentially positive** if $u$ can be represented as a difference of two HP-functions on $R$, or equivalently, if $u$ admits a harmonic majorant on $R$. We denote the totality of essentially positive harmonic functions on $R$ by

$$HP' = HP'(R).$$

(2) As usual, $HP(R)$ (resp. $HB(R)$) denotes the totality of non-negative (resp. bounded) harmonic functions on $R$. The meaning of $O_{HP}$ and $O_{HB}$ is similar to that of $O_{\text{ns}}$.

(3) We say that $\Phi(t)$ is **completely unbounded** on $[0, \infty)$ if $\Phi(t)$ is not bounded at any neighbourhood of any point in $[0, \infty)$. 

Clearly $HP'(R) \supset HP(R)$. For two functions $u$ and $v$ in $HP'(R)$, there always exists the least harmonic majorant (resp. the greatest harmonic minorant) of $u$ and $v$, which we denote by $u \lor v$ (resp. $u \land v$). Then $HP'(R)$ forms a vector lattice with lattice operations $\lor$ and $\land$. For $u$ in $HP'(R)$, we denote by $Mu$ the function $u \lor 0 + (-u) \lor 0$, which is the least harmonic majorant of $|u|$. Next first for $u$ in $HP(R)$, we denote by $Bu$ the $HP$-function defined by $\sup (v(p); u \geq v \in HB(R))$ on $R$. Clearly $B$ is order-preserving, linear and $B^2 = B$ on $HP'(R)$ (see Ahlfors-Sario [1], p. 210). Next for $u$ in $HP'(R)$, we put $Bu = Bu_1 - Bu_2$, where $u = u_1 - u_2$ and $u_1$ and $u_2$ are in $HP(R)$. Here, by the linearity of $B$ on $HP(R)$, $Bu$ does not depend on the special decomposition of $u$ into $HP$-functions. Again the operator $B$ is order-preserving, linear and $B^2 = B$ on $HP'(R)$ and moreover $B$ commutes with $M$, $\lor$, and $\land$. This is clear on $HP(R)$ by definitions of $B$, $\lor$, $\land$ and $M$. For the general case, we have only to show that $B(u \lor 0) = (Bu) \lor 0$. Since

$$Bu = B(u \lor 0) - B((-u) \lor 0)$$

and

$$B(u \lor 0) \land B((-u) \lor 0) = B((u \lor 0) \land ((-u) \lor 0))$$

$$= BO = 0,$$

$B(u \lor 0)$ is the positive part of the Jordan decomposition of $Bu$.

An $HP'$-function $u$ is called quasi-bounded (resp. singular) if $Bu = u$ (resp. $Bu = 0$). These notions were introduced by Parreau [3]. We denote the totality of quasi-bounded harmonic functions on $R$ by

$$HB' = HB'(R).$$

Clearly $HB' \supset HB$. Since $B$ commutes with $M$, $\lor$ and $\land$, we see that $Bu = u$ is equivalent to $BMu = Mu$. Hence we can also define

$$HB'(R) = (u \in HP'(R) ; BMu = Mu).$$

4. Parreau [3] showed that, for increasing and convex function $\Phi(t)$, $H\Phi \subset HP'$ and if moreover $d(\Phi) = \infty$, then $H\Phi \subset HB'$. Our next problem is to investigate whether such relations hold or not for general $\Phi(t)$. The answer is negative in general: we shall single out in § 4 an increasing continuous unbounded function $\Phi(t)$ with $d(\Phi) < \infty$ and $d(\Phi) = 0$ and an $H\Phi$-function in the open unit disc which is not an $HP'$-function there (Example 2). This shows the invalidity of $H\Phi \subset HP'$.
in general. Only for this aim, we may take bounded $\Phi(t)$. But we are interested in unbounded $\Phi(t)$. We shall also construct in § 3 an increasing continuous function $\Phi(t)$ with $d(\Phi) = \infty$ and $d' (\Phi) = 0$ and an $H\Phi$-function in the open unit disc which is not an $H\Psi'$-function there (Example 1). This shows the invalidity of the relation $H\Phi \subset H\Psi'$ and so of the relation $H\Psi \subset H\Psi'$ even if $\bar{d}(\Phi) = \infty$.

Then there arises the question when can we conclude the relation $H\Phi \subset H\Psi'$ or $H\Psi'$. Both examples above show that unboundedness, not completely unboundedness, increasingness, continuity or all of them cannot give the condition. In both examples above, $d(\Phi) = 0$. This suggests us that the required condition may be $d(\Phi) > 0$. This is really the case. Firstly the answer for $H\Phi \subset H\Psi'$ is given completely by the following which includes Parreau’s case:

**Theorem 3.** — *In order that $H\Phi(R) \subset H\Phi'(R)$ for any Riemann surface $R$, it is necessary and sufficient that $d(\Phi) > 0$ (no matter whether $\bar{d}(\Phi)$ is finite or infinite).*

The proof of this will be given in § 5. Similarly we ask about the condition which assures the relation $H\Psi \subset H\Psi'$. In this case, even in the Parreau’s case, we must assume that $\bar{d}(\Psi) = \infty$ as the following simple example shows: $\Phi(t) = t$, $R = (z ; 0 < |z| < 1)$ and $u(z) = -\log |z|$. The best possible general conclusion is as follows:

**Theorem 4.** — *If $\bar{d}(\Phi) = \infty$, then $H\Phi(R) \cap H\Phi'(R) \subset H\Psi'(R)$.*

Here we cannot drop $H\Phi'(R)$ in the above relation as Example 1 shows. The above theorem will be proved in § 6. Now assume that $d(\Phi) > 0$, then by Theorems 3 and 4, $H\Phi(R) \subset H\Psi'(R)$. Conversely if $H\Phi(R) \subset H\Psi'(R)$ for any $R$, then $H\Phi(R) \subset H\Psi'(R)$ for any $R$ and by Theorem 3, $d(\Phi) > 0$. Thus we get the following which includes Parreau’s case:

**Theorem 5.** — *Assume that $d(\Phi) = \infty$. In order $H\Phi(R) \subset H\Psi'(R)$ for any Riemann surface $R$, it is necessary and sufficient that $d(\Phi) > 0$.*

### 1. Proofs of Theorems 1 and 2.

1. **Proof of Theorem 1.** — I. The case $\bar{d}(\Phi) = \infty$: Assume that there exists a non-constant $H\Phi$-function $u$ on $R$. By the definition of
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$\Phi$-boundedness, there exists an HP-function $h$ such that $\Phi (|u|) \leq h$ on $R$. We have to show that $R \not\in O_{HB}$. Contrary to the assertion, assume that $R \in O_{HB}$. Since $\tilde{d} (\Phi) = \infty$, we can find a strictly increasing sequence $(r_n)_{n=1}^{\infty}$ of positive numbers $r_n$ such that $\lim_{n \to \infty} r_n = \infty$, $\Phi (r_n) > 0$, $G_n = \{p \in R ; |u(p)| < r_n\} \neq \emptyset$ and $\lim_{n \to \infty} a_n = 0$, where $a_n = r_n/\Phi (r_n)$. Then clearly

$$G_1 \subset G_2 \subset \ldots \subset G_n \subset \ldots, \ R = \bigcup_{n=1}^{\infty} G_n.$$  

First we show that $G_n \not\in SO_{HB}$ for some $n$ on $(^4)$. If this is not the case, then $G_n \in SO_{HB}$ for all $n$'s. Then since $a_n h - |u|$ is superharmonic and bounded from below on $G_n$ and

$$a_n h - |u| \geq a_n \Phi (|u|) - |u| = a_n \Phi (r_n) - r_n = 0$$  

on $\partial G_n$, we can conclude that $a_n h - |u| \geq 0$ on $G_n$. Since $a_n \searrow 0$, we must have $u \equiv 0$ on $R$, which is clearly a contradiction. Hence we may assume that $G_n \not\in SO_{HB}$ ($n = 1, 2, 3, \ldots$) by choosing a suitable subsequence of $(r_n)$, if necessary.

Next we assert that $G_n - \bar{G}_1 \in SO_{HB}$ ($n = 1, 2, 3, \ldots$). For, if there exists a $G_n$ with $G_n - \bar{G}_1 \not\in SO_{HB}$, then there would exist two disjoint non-empty open sets $G_1$ and $G_n - \bar{G}_1$ not belonging to $SO_{HB}$. By the so called "two domains criterion", we must have that $R \not\in O_{HB}$ (see Ahlfors-Sario [1], p. 213). But this contradicts our assumption $R \in O_{HB}$.

Now consider the function $\omega_n = a_n h + r_1 - |u|$ on $G_n$, which is superharmonic and bounded from below on $G_n$ and so on $G_n - \bar{G}_1$. By the similar manner as before, we see that $\omega_n \geq a_n h - |u| \equiv 0$ on $\partial G_n$. Clearly $\omega_n \geq r_1 - |u| = 0$ on $\partial G_1$. Hence $\omega_n \equiv 0$ on $\partial (G_n - \bar{G}_1)$. Since $G_n - \bar{G}_1 \in SO_{HB}$, we can conclude that $\omega_n \geq 0$ on $G_n$ or $|u| \leq a_n h + r_1$ on $G_n$. Hence by the fact that $a_n \searrow 0$, we get that $|u| \leq r_1$ on $R$. This contradicts our assumption that $R \in O_{HB}$. Thus we must have $R \not\in O_{HB}$.

II. The case $\tilde{d} (\Phi) < \infty$ : Assume that there exists a non-constant HP-function $u$ on $R$. By $\tilde{d} (\Phi) < \infty$, we can find a point $s$ in $[0, \infty)$ such that there exists a finite positive constant $C$ with $\Phi (t) \leq Ct (s \leq t < \infty)$. Let $v = s + u$. Clearly $v$ is a non-constant HP-function on $R$ with

$(^4)$ An open subset $G$ of a Riemann surface $R$ with smooth relative boundary $\partial G$ is said to belong to $SO_{HB}$ if every HB-function on $G$ with continuous boundary value zero at $\partial G$ reduces to a constant zero.
2. Proof of Theorem 2. — Ad (i): If \( \Phi (t) \) is bounded, then every non-constant harmonic function is an \( \Phi \)-function. Thus \( O_{\Phi} \) consists of all Riemann surfaces carrying no non-constant harmonic function, which are closed Riemann surfaces.

Ad (ii): For any non-constant harmonic function \( u \) on \( R \), since \( u \) is open map of \( R \) into \([0, \infty)\) by the maximum principle, \( \Phi (|u|) \) is completely unbounded on \( R \) along with \( \Phi (t) \) and so \( u \) is not \( \Phi \)-function. Thus there exists no non-constant \( \Phi \)-function on any Riemann surface and \( O_{\Phi} \) consists of all Riemann surfaces.

Ad (iii): Assume that \( \overline{d} (\Phi) = \infty \) and that there exists a non-constant HB-function \( u \) on \( R \). As \( \Phi (t) \) is not completely unbounded, so there exists an interval \((a, b)\) in which \( \Phi (t) < c = \text{const} \). Let

\[
v = (a + b)/2 + ((b - a)/2) (\sup_R |u|)^{-1} u.
\]

Then \( v \) is a non-constant HB-function and \( \Phi (|v|) = \Phi (v) < c \) on \( R \). Thus \( O_{\Phi} \subset O_{HB} \). This with Theorem 1 gives \( O_{\Phi} = O_{HB} \).

Next assume that \( \overline{d} (\Phi) < \infty \). By Theorem 1, \( O_{HP} \supset O_{\Phi} \). Contrary to the assertion, assume that there exists an \( R \in O_{HP} - O_{\Phi} \). Let \( u \) be a non-constant \( \Phi \)-function on \( R \). Then \( \Phi (|u|) \leq c = \text{constant} \) on \( R \). Since \( \Phi (t) \) is unbounded and \( |u| (R) \) is connected in \([0, \infty)\) and contains 0, \( u \) must be bounded on \( R \). Then \( \sup_R |u| + u \) is a non-constant HP-function on \( R \), which contradicts the assumption that \( R \in O_{HP} \). Hence \( O_{\Phi} = O_{HP} \).

2. Preparations for Examples.

Let \( U = (z; |z| < 1) \) and \( A \) be an arc in \( \partial U = (z; |z| = 1) \). We denote by \( w(z; A) \) the harmonic measure of \( A \) calculated at \( z \) in \( U \) with respect to \( U \). It is well known that

\[
w(z; A) = (2\beta - \alpha)/2\pi,
\]

where \( \alpha \) is the length of \( A \) and \( \beta \) is the angle seeing the arc \( A \) from \( z \).
We denote by $L_A$ the line segment connecting both end points of $A$. Then from (1), we easily see that

$w(0; A) = \alpha$;\\
(3) $w(z; A) = 1 - \alpha/2\pi$ on $L_A$.

Next let $A_j$ be the arc in $\partial U = \{z; |z| = 1\}$ with end points 1 and $e^{i\alpha_j}$ ($j = 1, 2$) such that $0 < \alpha_1 < \alpha_2$, $\alpha_1 < \pi/2$, $\alpha_2 < \pi/2$. We denote by $A_j$ (resp. $A_j'$) the arc with end points 1 and $e^{-i\alpha_j}$ (resp. $A_j = A_j \cup \tilde{A}_j$). For simplicity, we set $L_2 = L_{A_2}$, i.e. $L_2$ is the line segment connecting two end points of $A_2$. Then we get the following inequality which plays an important role in our forth-coming examples: there exists a universal constant $s_0$ ($\leq 4^{-1} \pi^4$) such that

$|w(z; A_1) - w(z; \tilde{A}_1)| \leq s_0 \alpha_1^2 / (\alpha_2^2 - \alpha_1^2)^2$ on $L_2$.

**Proof.** — We denote the points $e^{i\alpha_1}$, $e^{-i\alpha_1}$, $(e^{i\alpha_1} + e^{-i\alpha_1})/2$, 1, $(e^{i\alpha_2} + e^{-i\alpha_2})/2$ and $z$ on $L_2$ with $\text{Im}(z) \geq 0$ by $D$, $E$, $F$, $G$, $H$ and $P$ respectively. We set $DF = FE = d$, $FH = k$, $DP = a$, $PF = b$ and $PE = c$. By (1), $w(z; A_1) - w(z; \tilde{A}_1) = \angle DPG - \angle GPE \pi$. Let $\angle DPF = \theta_1$ and $\angle FPE = \theta_2$. Then clearly $\angle DPG \leq \theta_1$ and $\angle GPE \geq \theta_2$. Hence we have $0 \leq w(z; A_1) - w(z; \tilde{A}_1) \leq (\theta_1 - \theta_2)/\pi$. Applying the cosine theorem to triangles $\Delta DPF$ and $\Delta GPE$ and then Pappos' identity to the triangle $\Delta DPE$, we have

$$\sin 2^{-1}(\theta_1 - \theta_2) = (c - a)(8abc \sin 2^{-1}(\theta_1 + \theta_2))^{-1}(4d^2 - (a - c)^2).$$

Here we have

$$ac \sin 2^{-1}(\theta_1 + \theta_2) \geq ac \sin 2^{-1}(\theta_1 + \theta_2) \cos 2^{-1}(\theta_1 + \theta_2)$$

$$= 2^{-1} ac \sin \angle DPE = \Delta DHE = dk.$$

By the triangle inequality applied for $\Delta DPE$, $c - a \leq 2d$. Thus by noticing $b \geq k$, we have $\sin 2^{-1}(\theta_1 - \theta_2) \leq d^2 k^{-2}$. As

$$\sin \theta \geq (2/\pi) \theta \quad (0 \leq \theta \leq 2^{-1} \pi),$$

so $\theta_1 - \theta_2 \leq \pi d^2 k^{-2}$. Now we have $d = \sin \alpha_1 \leq \alpha_1$ and

$$k = \cos \alpha_1 - \cos \alpha_2$$

$$= 2 \sin^{-1}(\alpha_1 + \alpha_2) \sin 2^{-1}(\alpha_2 - \alpha_1) \geq 2 \pi^{-2}(\alpha_2^2 - \alpha_1^2).$$

Hence

$$0 \leq w(z; A_1) - w(z; \tilde{A}_1) \leq 4^{-1} \pi^4 \alpha_1^2 / (\alpha_2^2 - \alpha_1^2)^2. \quad Q.E.D.$$
We shall use (4) in the particular case where $0 < \alpha_1 < \alpha_2/\sqrt{2}$. In this case, by using universal constant $s (\leq \pi^4)$, we get
\[
|w(z; A_1) - w(z; A_1')| \leq s(\alpha_1^2/\alpha_2^2) \quad \text{on } L_2.
\]

3. Example 1.

We are now able to construct an example of a function $\Phi$ which is continuous, increasing, $d(\Phi) = \infty$ and $d(\Phi) = 0$; and an $H^{\Phi}$-function $u$ in then open unit disc $U = \{z; |z| < 1\}$ which is not an $\text{HP}'$-function.

**Example 1.** Let $p$ be a constant such that $0 < p < \min(1/4, 1/4\pi)$, where $s$ is the constant in (5) in § 2, and $(p_n)_{n=1}^\infty$ be a sequence defined by $p_n = (p^n)^{2^n+\nu}$ for $n = 2^\nu + \mu$ ($\nu = 0, 1, 2, \ldots$; $\mu = 1, 2, 3, \ldots, 2^\nu$). Let $A_n$ and $\tilde{A}_n$ be arcs on the unit circumference such that
\[
A_n = \{e^{i\theta}; 0 \leq \theta \leq 2p_n \pi/n\}
\]
and
\[
\tilde{A}_n = \{e^{i\theta}; -2p_n \pi/n \leq \theta \leq 0\}.
\]
Let $(r_\nu)^{\nu=1}_\nu$ and $(b_\nu)^{\nu=1}_\nu$ be two sequences of positive numbers defined by $r_\nu = 2/(p^{\nu+1})^{2^\nu}$ and $b_\nu = 2^{2^\nu}/r_\nu$. Define the function $\Phi(t)$ on $[0, \infty]$ by
\[
\Phi(t) = \begin{cases}
0, & t \in [0, r_1]; \\
b_1(t - r_1), & t \in [r_1, r_1 + 1]; \\
b_\nu, & t \in [r_\nu + 1, r_{\nu+1}] (\nu = 1, 2, \ldots); \\
b_\nu + (b_{\nu+1} - b_\nu)(t - r_{\nu+1}), & t \in [r_{\nu+1}, r_{\nu+1} + 1] (\nu = 1, 2, \ldots)
\end{cases}
\]
and the function $u(z)$ in $U$ by
\[
\sum_{n=1}^{\infty} (w(z; A_n) - w(z; \tilde{A}_n))/p_n.
\]
Then the following hold:

(a) $\Phi(t)$ is continuous, increasing, $d(\Phi) = \infty$ and $d(\Phi) = 0$;
(b) $u(z)$ is well defined in $U$ and harmonic there;
(c) $u(z) \in H \Phi(U)$;
(d) $u(z) \notin \text{HP}'(U)$.

**Proof of (a).** — Is immediate by the definition of $\Phi(t)$. 


Proof of (b). — For each $n = 1, 2, \ldots$, set

$$v_n(z) = w(z; A_n) - w(z; \widetilde{A}_n), \quad u_n(z) = \sum_{k=1}^{n} v_k(z)/p_k.$$ 

Then $v_n$ and $u_n$ are harmonic in $U$, positive in the upper half of $U$ and $v_n(-z) = -v_n(z)$ and $u_n(-z) = -u_n(z)$ in $U$. Hence to show that the series defining $u(z)$ is convergent in $U$ and defines a harmonic function there, we have only to prove that $(u_n(i/2))_{n=1}^{\infty}$ is convergent. By (5) in § 2, we have that

$$0 < v_n(i/2) \leq s(2 p_n \pi/n)^2/(\pi/2)^4 \leq s' p_n^2,$$

where $s'$ is a constant independent of $n \geq 1$. Thus

$$0 < u_{n+m}(i/2) - u_n(i/2) = \sum_{k=n+1}^{n+m} v_k(i/2)/p_k \leq s' \sum_{k=n+1}^{n+m} p_k < s' p^n/(1 - p).$$

This shows that $(u_n(i/2))_{n=1}^{\infty}$ is convergent.

Proof of (c). — For each $\nu = 1, 2, \ldots$, we denote by $L_\nu$ the line segment $L_{A_\nu}$, i.e. the line segment connecting two end points of $A_{2^\nu} = A_{2^\nu} \cup \widetilde{A}_{2^\nu}$. Since $|v_k(z)| < 1$ in $U$, we have

$$|v_k(z)/p_k| \leq 1/p_k \leq 1/(p^{4^\nu-1}k) \quad (1 \leq k \leq 2^\nu)$$

on $U$ and so on $L_\nu$. Next for $k = 2^\nu + \mu (\mu = 1, 2, \ldots)$ and $z \in L_\nu$, by (5) in § 2, we have that

$$v_k(z)/p_k \leq s(2 p_{k} \pi/k)^2/(2 p_{2^\nu} \pi/2)^4 p_k = s(2^{4^\nu}/4 \pi^2 k^2) [p_k/p_1] \leq s(2^{4^\nu}/4 \pi^2 k^2) [(p^{4^\nu})^2/(p^{4^\nu-3}2^{2^\nu}2^{\nu})] = s(2^{4^\nu}/4 \pi^2 k^2) p^{4^\nu} \leq p^{4^\nu}(\mu - 1).$$

Hence for $z$ in $L_\nu$, we get that

$$|u(z)| \leq \sum_{k=1}^{2^\nu} |v_k(z)/p_k| + \sum_{k=2^{\nu+1}}^{\infty} |v_k(z)/p_k| \leq \sum_{k=1}^{2^\nu} 1/(p^{4^\nu-1}k) + \sum_{\mu=1}^{\infty} p^{4^\nu}(\mu - 1) \leq 2/(p^{4^\nu-2}2^{\nu}) = r_\nu.$$

Since $u(z)$ is quasi-bounded in the upper half of $U$ and in the lower half of $U$ respectively, we have, for $e^{i\theta}$ in $U - A_{2^\nu}$, that

$$|u(e^{i\theta})| = \sum_{k=1}^{2^\nu} |v_k(e^{i\theta})| \leq \sum_{k=1}^{2^\nu} 1/p_k$$
Hence by the maximum principle, $0 \leq u(z) \leq r_v$ in the intersection of the upper half of $U$ and the left side of $L_v$ in $U$. Hence $|u(z)| \leq r_v$ in the left side of $L_v$ in $U$. By (3) in § 2, we see that $w(z; A_2^v) \geq 1 - p_{2v}/2^v$ on $L_v$ and so on the right side of $L_v$ in $U$. Hence if $z$ lies between $L_v$ and $L_{v+1}$ in $U$, $b_v w(z; A_2^v) \geq b_v - 2^{-v/2+2} \geq \Phi (|u(z)|) - 2^{-v/2+2}$, or $\Phi (|u(z)|) \leq b_v w(z; A_2^v) + 2^{-v/2+2}$, since $\Phi (t) \leq b_v$ if $t \leq r_{v+1}$. On the other hand,

$$2\pi b_v w(0; A_{2v}^v) = b_v (4 p_{2v} \pi/2^v) = 8 \pi 2^{-v/2}.$$ 

Hence if we set $w(z) = \sum_{v=1}^{\infty} (b_v w(z; A_{2v}^v) + 2^{-v/2+2})$, then $w(0) = 8 \sum_{v=1}^{\infty} 2^{-v/2} < \infty$ and so $w(z) \in H\Phi (U)$. Thus $\Phi (|u(z)|) \leq b_v w(z; A_2^v) + 2^{-v/2+2} \leq w(z)$ between $L_v$ and $L_{v+1}$ in $U$. As $v$ is arbitrary, so $\Phi (|u(z)|) \leq w(z)$ in $U$ (5). This shows that $u \in H\Phi (U)$.

**Proof of (d).** — Contrary to the assertion, assume that $u \in H\Phi (U)$. Then $|u(z)|$ has a harmonic majorant $h(z)$ on $U$. As $u(z)$, $u_n(z)$ and $v_n(z)$ are positive in the upper half of $U$ and antisymmetric with respect to the real line (i.e. $u(z) = -u(-z)$ etc.), so $h(z) \geq |u(z)| \geq |u_n(z)|$ in $U$. Clearly $|u_n(z)| = \sum_{k=1}^{n} |w(z; A_k) - w(z; \tilde{A}_k)|/p_k$ and the least harmonic majorant of the subharmonic function $|u_n(z)|$ is $\sum_{k=1}^{n} w(z; A_k)/p_k$, where $A'_k = A_k \cup \tilde{A}_k$ as before. Hence

$$\sum_{k=1}^{n} w(z; A_k)/p_k \leq h(z)$$

on $U$ for any $n = 1, 2, \ldots$. Thus in particular, $\sum_{k=1}^{\infty} w(0; A'_k)/p_k \leq h(0)$, which gives the following contradiction:

$$\infty = 2 \sum_{k=1}^{\infty} 1/k = \frac{1}{2\pi} \sum_{k=1}^{\infty} (4 p_k \pi/k)/p_k \leq h(0).$$

(5) Notice that if $z$ lies in the left of $L_v$ in $U$, then $|u(z)| \leq r_1$ and so $0 = \Phi (|u(z)|) \leq w(z)$ there.
4. Example 2.

Consider functions
\[
\begin{align*}
\Phi(t) &= \log^+ t = \max(\log t, 0) \quad \text{on } [0, \infty); \\
u(z) &= r^{-1} \cos \theta \ (z = r e^{i\theta}) \quad \text{on } U_0 = \{z; 0 < |z| < 1\}.
\end{align*}
\]
Then \(\Phi(t)\) is unbounded, increasing, continuous and \(\bar{d}(\Phi) = d(\Phi) = 0\).

We can also easily see that \(u(z)\) is an \(H\)-function in \(U_0\) but not an \(HP'\)-function in \(U_0\). But this example deeply depends on the weakness of the special boundary point 0 of \(U_0\). However, without using such a special boundary property, we can construct such an example in the open unit disc \(U = \{z; |z| < 1\}\) by the aid of Example 1.

**Example 2.** Let \(\Phi(t)\) and \(u(z)\) be as in Example 1. Let
\[
\Phi_a (t) = \min(\Phi(t), at) \quad (0 < a < \infty).
\]
Then the followings hold:

(a) \(\Phi_a (t)\) is increasing, continuous, \(\bar{d}(\Phi_a) = a\) and \(d(\Phi_a) = 0\);

(b) \(u(z) \in H\Phi_a(U)\);

(c) \(u(z) \notin HP'(U)\).

5. Proof of Theorem 3.

First we prove that \(H\Phi(R) \subseteq HP'(R)\) for any \(R\) if \(d(\Phi) > 0\). Let \(u \in H\Phi(R)\) and \(d(\Phi) = 2c > 0\). Then there exists a point \(t_0\) in \([0, \infty)\) such that \(\Phi(t) > ct (t \geq t_0)\). Then for any \(t\) in \([0, \infty)\), \(\Phi(t) + ct_0 \geq ct\). As \(\Phi(|u|)\) possesses a harmonic majorant \(h\) on \(R\), so
\[
h + ct_0 \geq \Phi(|u|) + ct_0 > c |u|
\]
on \(R\). Thus \(u\) possesses a harmonic majorant \((h + ct_0)/c\), i.e. \(u \in HP'(R)\).

Conversely, if \(H\Phi(R) \subseteq HP'(R)\) for any \(R\), then Examples 1 and 2 show that \(d(\Phi) > 0\) no matter whether \(\bar{d}(\Phi)\) is finite or infinite.

Let \( u \in H\Phi(\mathbb{R}) \cap H'p(\mathbb{R}) \). We have to show that \( u \in HB'(\mathbb{R}) \). As \( u \in H\Phi(\mathbb{R}) \), so there exists an HP-function \( h \) such that \( \Phi(|u|) \leq h \) on \( \mathbb{R} \). Since \( u \in H'p(\mathbb{R}) \), we can consider \( Mu = u \vee 0 + (-u) \wedge 0 \geq |u| \) and \( Bu \). To show that \( u \in HB'(\mathbb{R}) \), we have to prove that \( Bu = u \) or equivalently, \( BMu = Mu \) (see 3 in the introductory part of this paper).

By the assumption that \( \delta(\Phi) = \infty \), we can find an increasing sequence \( (r_n)_{n=1}^{\infty} \) of positive numbers converging to \( \infty \) such that \( \Phi(r_n) > 0 \) and \( \lim_{n \to \infty} a_n = 0 \), \( a_n = r_n/\Phi(r_n) \). Let \( G_n = \{ p \in \mathbb{R} \mid |u(p)| < r_n \} \) \( (n = 1, 2, \ldots) \). Clearly

\[
G_1 \subset G_2 \subset \ldots \subset G_n \subset \ldots, \quad \mathbb{R} = \bigcup_{n=1}^{\infty} G_n.
\]

Let \( (R_m)_{m=1}^{\infty} \) be an exhaustion of \( \mathbb{R} \) and \( w_m \) be a harmonic function on \( R_m \cap G_n \) with the boundary value

\[
w_m = \begin{cases} 
\min (Mu - BMu, r_n) & \text{on } (\partial R_m) \cap G_n; \\
0 & \text{on } \partial G_n.
\end{cases}
\]

Since \( \min (Mu - BMu, r_n) \) is superharmonic on \( \mathbb{R} \), \( w_m \) is subharmonic on \( R_m \) if we define \( w_m = 0 \) in \( R_m - G_n \), and \( w_m \geq w_{m+1} \) on \( R_m \). Let \( w_m' \) be harmonic in \( R_m \) with the boundary value

\[
w_m' = \begin{cases} 
\min (Mu - BMu, r_n) & \text{on } (\partial R_m) \cap G_n; \\
0 & \text{on } \partial R_m - G_n.
\end{cases}
\]

Then clearly \( (w_m')_{m=1}^{\infty} \) is a bounded sequence and \( 0 \leq w_m' \leq Mu - BMu, \quad n = 1, 2, \ldots \). If \( w' \) is any limiting harmonic function of a convergent subsequence of \( (w_m')_{m=1}^{\infty} \), then \( 0 \leq w' \leq Mu - BMu \). By applying the operator \( B \), we get

\[
0 \leq Bw' \leq B(Mu - BMu) = BMu - B^2 Mu = BMu = BMu = 0.
\]

Since \( w' \) is bounded and positive, \( Bw' = w' \). Hence \( w' = 0 \) on \( \mathbb{R} \). Thus \( \lim_{m \to \infty} w_m' = 0 \) on \( \mathbb{R} \). As we have \( w_m' \geq w_m \geq 0 \) on \( R_m \), so we conclude that \( \lim_{m \to \infty} w_m = 0 \) on \( \mathbb{R} \).

On \( (\partial R_m) \cap G_n \), \( |u| \leq r_n \) and \( |u| \leq Mu = BMu + (Mu - BMu) \) or \( |u| = BMu \leq Mu - BMu \). Hence on \( (\partial R_m) \cap G_n \), \( |u| = BMu \) or \( |u| \leq BMu + w_m \). On \( \partial G_n \), we have \( |u| = a_n \Phi(|u|) \leq a_n h \). Thus we conclude that \( |u| \leq a_n h + \ldots \)
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+ BMu + $w_m$ on $\partial (R_m \cap G_n)$. Since $|u|$ is subharmonic and $a_n h + BMu + w_m$ is harmonic on $R_m \cap G_n$, we can conclude that

$$|u| \leq a_n h + BMu + w_m \text{ on } R_m \cap G_n.$$ 

By letting $m \nearrow \infty$ and then $n \nearrow \infty$, we conclude that $|u| \leq BMu$ on $R$. By the definition of $Mu$, we must have $Mu \leq BMu$ and hence $BMu = Mu$.

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