HANS WALLIN

Regularity properties of the equilibrium distribution


<http://www.numdam.org/item?id=AIF_1965__15_2_71_0>
1. Let $\mathbb{R}^m$, $m \geq 1$, be the $m$-dimensional Euclidean space with points $x = (x^1, \ldots, x^m)$. It is well known that the equilibrium distribution belonging to a compact set $F$ and the kernel $r^{2-m}$ if $m > 2$ and the kernel $-\log r$ if $m = 2$ is concentrated on the boundary of $F$. This is no longer true if the interior of $F$ is non-empty and if we, instead of $r^{2-m}$ or $-\log r$, consider the kernel $r^{-(m-\alpha)}$ where $0 < \alpha < 2$ if $m \geq 2$ and $0 < \alpha < 1$ if $m = 1$. In fact, since $|x|^{-(m-\alpha)}$ in this case is a strictly subharmonic function of $x$ when $x \neq 0$ it is easy to prove that the support of the equilibrium distribution $\mu_x^F$ contains every interior point of $F$, a fact which is also a consequence of the theorems below.

We shall here give some properties of $\mu_x^F$ in the interior of $F$ and examine its behaviour near the boundary $\partial F$ of $F$ when $0 < \alpha < 2$ if $m \geq 2$ and $0 < \alpha < 1$ if $m = 1$. We intend to prove that the restriction of $\mu_x^F$ to the interior of $F$ is absolutely continuous and has a density which is analytic and may be expressed by an explicit formula [the Theorems 1 and 2] and which, when we approach $\partial F$, tends to infinity in the same way as the distance to $\partial F$ raised to the power $-\frac{\alpha}{2}$, if certain conditions of regularity are satisfied [the Theorems 1, 3 and 4].

The methods of the proofs will be based on the sweeping-out process and a kind of inversion formula, [see the formula (11)
below]. The formula (6) for the energy integral is related to (11). A formula similar to (6) has been used by Beurling and Beurling-Deny on several occasions. Compare [2] and [5]. Beurling has also indicated the usefulness of an inversion formula of the type of (11) for the treatment of the problem considered in this paper. Finally it may be noted that some of the statements of Theorem 1 which we shall deduce by means of the sweeping-out process can be obtained from the formula (11) too.

2. We introduce some notations and definitions. For an arbitrary set E we denote the complement by \( E' \) and the interior by \( \mathring{E} \). F is a compact set with boundary \( \partial F \), \( \mu \) a positive measure with compact support \( S_\mu \) and \( \alpha \) a number satisfying \( 0 < \alpha < 2 \) if \( m \geq 2 \) and \( 0 < \alpha < 1 \) if \( m = 1 \). \( S(\mathring{x}_0, r) \) is the sphere determined by the set of points \( x \) which satisfy \( |x - x_0| \leq r \).

The \( \alpha \)-potential of \( \mu \) is denoted by \( u_\alpha^\mu \) and defined by

\[
    u_\alpha^\mu(x) = \int \frac{1}{|x - y|^{m-\alpha}} \, d\mu(y),
\]

and the energy integral of order \( \alpha \) of \( \mu \) is denoted by \( I_\alpha(\mu) \),

\[
    I_\alpha(\mu) = \iint \frac{1}{|x - y|^{m-\alpha}} \, d\mu(y) \, d\mu(x).
\]

Here and elsewhere, the integration is to be extended over the whole space, if no limits of integration are indicated. The \( \alpha \)-capacity of a bounded Borel set \( E \), \( C_\alpha(E) \), is defined as

\[
    C_\alpha(E) = \{ \inf I_\alpha(\nu) \}^{-1},
\]

where the infimum is taken over the class of all positive measures \( \nu \) with total mass 1 and \( S_\nu \subset E \).

If \( C_\alpha(F) > 0 \) we denote the equilibrium distribution of \( F \) belonging to the kernel \( r^{-(m-\alpha)} \) by \( \mu_\alpha^F \) and the equilibrium potential by \( u_\alpha^F \). We assume throughout the paper that \( \mu_\alpha^F \) is normalized so that \( \mu_\alpha^F(R^n) = 1 \). \( \nu_\alpha^F \) is the restriction of \( \mu_\alpha^F \) to \( \mathring{F} \). We shall prove below that \( \nu_\alpha^F \) is absolutely continuous; the density of \( \nu_\alpha^F \) is denoted by \( f_\alpha^F \). We put \( \nu_\alpha(F) = \{ C_\alpha(F) \}^{-1} \). M denotes different constants.
3. The following lemma is a consequence of the sweeping-out process.

**Lemma 1.** — Let $F_1$ and $F_2$ be two compact sets with $F_1 \subset F_2$, $C_\alpha(F_1) > 0$. Then, for every Borel set $E$ with $E \subset F_1$,

\[(1) \quad \nu_{\alpha}^F(E) \geq \nu_{\alpha}^{F_2}(E).\]

**Proof.** — Let $\tau_1$ and $\tau_2$ be the restrictions of $\nu_{\alpha}^{F_1}$ to $F_1$ and $F_2$ respectively. Then there exists a positive measure $\tau^*_2$ with $\tau^*_2(R^n) \leq \tau_2(R^n)$, $S_{\tau^*_2} \subset F_1$, such that $u_{\alpha}^{F_1}(x) = u_{\alpha}^{F_2}(x)$ for every $x \in F_1$ except on a subset of $F_1$ of $\alpha$-capacity zero and $u_{\alpha}^{F_1}(x) \leq u_{\alpha}^{F_2}(x)$ everywhere. Since the $\alpha$-potential of $\nu_{\alpha}^{F_1}$ is constant on $F_1$ except on a subset of $F_1$ of $\alpha$-capacity zero, we have

\[(2) \quad \nu_{\alpha}^{F_1}(F_1) = \tau_1(F_1) = \nu_{\alpha}^{F_2}(F_1),\]

which proves the lemma.

4. By means of Lemma 1 we prove the following theorem:

**Theorem 1.** — Let $F$ be a compact set such that $F \neq \emptyset$. Then $\nu_{\alpha}^F$ is absolutely continuous and $f_{\alpha}^F$ — properly defined on a set of Lebesgue measure zero — is bounded from below by a positive constant on $F$.

Let $x_0$ be a boundary point of $F$ belonging to the boundary of a sphere $S(x_1, r_1)$ such that $S(x_1, r_1) \subset F$ and let $V(x_0)$ be a bounded right circular cone with vertex at $x_0$, the line through $x_0$ and $x_1$ as axis and $V(x_0) \subset S(x_1, r_1)$. Then (1)

\[(3) \quad \limsup_{x \to x_0} f_{\alpha}^F(x,|x - x_0|^{\frac{\alpha}{2}}) \leq M < \infty, \quad x \to x_0, \quad x \in V(x_0),\]

where $M$ is a constant which may be chosen only depending on $\alpha$, $m$, $r_1$ and the generating angle of $V(x_0)$ and where $f_{\alpha}^F$ is properly defined on a set of Lebesgue measure zero.

**Proof.** — According to a result by M. Riesz ([7], p. 16) the equilibrium distribution of the sphere $S_2 = S(x_2, r_2)$ belonging

\[(1) \quad V(x_0) \text{ is a line segment when } m = 1.\]
to the kernel $r^{-(m-\alpha)}$ is absolutely continuous with density

$$(4) \quad f^a_2(x) = M \cdot (r_2^2 - |x - x_2|^2)^{-\frac{\alpha}{2}}, \quad |x - x_2| < r_2,$$

for a certain constant $M$ depending on $\alpha$, $m$ and $r_2$.

We choose $x_2$ and $r_2$ such that $S_2 \subset F$ and use Lemma 1 with $F_1 = S_2$ and $F_2 = F$. If $E$ is a Borel set, $E \subset S_2$, we obtain

$$0 \leq \mu^E_x(E) \leq \int_S f^a_2(x) \, dx,$$

which shows that the restriction of $\mu^E_x$ to $S_2$ is absolutely continuous. Hence $\nu^E_x$ is absolutely continuous. The inequality also proves that if $f^a_2$ is properly defined on a set of Lebesgue measure zero, then

$$(5) \quad f^a_2(x) \leq f^a_2(x) \quad \text{for every} \quad x \in S_2.$$

From (4) and (5) we conclude, by an elementary calculation, that (3) is true.

To show that $f^a_2$ is bounded from below by a positive constant in the interior of $F$ we choose a sphere $S_3$, $S_3 \supset F$, and use Lemma 1 with $F_1 = F$ and $F_2 = S_3$. This proves the theorem.

5. We now prove the following formula: if

$$A(\alpha, m) = 2^{-2\pi} \cdot (m+1) \cdot \alpha \cdot \sin \frac{\pi \alpha}{2} \cdot \Gamma \left( \frac{m-\alpha}{2} \right) \cdot \Gamma \left( \frac{m+\alpha}{2} \right),$$

then

$$(6) \quad I_\alpha(\mu) = A(\alpha, m) \int \int \frac{|u^\mu_2(x+y) - u^\mu_2(x)|^2}{|y|^{m+\alpha}} \, dy \, dx,$$

in the sense that if one member is finite then the other member is finite too and the equality holds true (2).

We shall prove (6) by means of the Fourier transformation. Let $\hat{T} = \mathcal{F}T$ denote the Fourier transform of a tempered distribution $T$ normed so that

$$\hat{f}(\xi) = \int e^{-2\pi i \langle x, \xi \rangle} f(x) \, dx, \quad \langle x, \xi \rangle = \sum_{1}^{m} x^j \xi^j,$$

(2) The formula (6) is proved for the sake of completeness and due to its independent interest in spite of the fact that it is not indispensable for our purpose.
if $f$ is in the Lebesgue class $L^1(R^n)$. Since $u_\alpha^\mu$ is a convolution,

$$ u_\alpha^\mu = |x|^{-(m-\alpha)} * \mu, $$

and (Schwartz [8], p. 113)

$$ \mathcal{F}[x]^{-(m-\alpha)} = A_1(\alpha, m)|x|^{-\alpha}, \quad A_1(\alpha, m) = \frac{\pi^{\frac{m-\alpha}{2}} \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{m-\alpha}{2}\right)}, $$

we have, in the distribution sense,

$$ u_\alpha^\mu = A_1(\alpha, m) \cdot |x|^{-\alpha} \cdot \hat{\mu}, $$

which shows that $u_\alpha^\mu$ is a function which is absolutely integrable over every compact set. We also have (cf. Cartan-Deny [4] and Deny [6]):

$$ I_\alpha(\mu) = A_1(\alpha, m) \int |\xi|^{-\alpha} |\hat{\mu}(\xi)|^2 d\xi. $$

According to the Parseval relation we get

$$ \int \frac{dy}{|y|^{m+\alpha}} \int |u_\alpha^\mu(x + y) - u_\alpha^\mu(x)|^2 dx \\
= \int \frac{dy}{|y|^{m+\alpha}} \int |\hat{u}_\alpha^\mu(\xi)|^2 \cdot |e^{2\pi i \langle y, \xi \rangle} - 1|^2 d\xi \\
= \int |\hat{u}_\alpha^\mu(\xi)|^2 \left\{ \int \frac{|e^{2\pi i \langle y, \xi \rangle} - 1|^2}{|y|^{m+\alpha}} dy \right\} d\xi. $$

A substitution in the integral shows that

$$ \int \frac{|e^{2\pi i \langle y, \xi \rangle} - 1|^2}{|y|^{m+\alpha}} dy = A_2(\alpha, m) |\xi|^\alpha, $$

where $A_2(\alpha, m)$ is a constant which can be calculated explicitly (cf. [1], p. 402):

$$ A_2(\alpha, m) = \frac{2\pi^{(m+2+2\alpha)/2}}{\Gamma\left(\frac{\alpha+2}{2}\right) \Gamma\left(\frac{m+\alpha}{2}\right) \sin \frac{\pi \alpha}{2}}. $$

(3) It may be noted that an easy calculation proves that the left member of (9) is infinite if $\alpha \geqslant 2$ or $\alpha \leqslant 0$. 

REGULARITY OF THE EQUILIBRIUM DISTRIBUTION 75
An application of (7) now yields
\[
\int \int \frac{|u^v_2(x + y) - u^v_2(x)|^2}{|y|^{m+\alpha}} \, dy \, dx
= A_2(\alpha, m) \cdot \{ A_1(\alpha, m) \}^2 \cdot \int |\xi|^{-2} |\hat{\mu}(\xi)|^2 \, d\xi.
\]
From this formula and (8) we conclude that (6) is valid.

6. Let \( \varphi \) be a function, defined everywhere in \( \mathbb{R}^n \), which is infinitely differentiable and has a compact support and suppose that \( I_{\alpha}(\mu) < \infty \). By means of Schwarz's inequality and (6) we get

\[
\int \int \frac{|u^v_2(x + y) - u^v_2(x)| \cdot |\varphi(x + y) - \varphi(x)|}{|y|^{m+\alpha}} \, dy \, dx < \infty.
\]

We are going to prove that

\[
\int \varphi(x) \, d\mu(x) = A(\alpha, m) \int \int \frac{(u^v_2(x + y) - u^v_2(x)) \cdot (\varphi(x + y) - \varphi(x))}{|y|^{m+\alpha}} \, dy \, dx,
\]

where \( A(\alpha, m) \) is the constant occurring in (6). We observe that the right member of (11) is absolutely convergent according to (10).

For a fixed \( y \) we introduce the functions \( \nu_y \) and \( \psi_y \) by

\[
\nu_y(x) = u^v_2(x + y) - u^v_2(x), \quad \psi_y(x) = \varphi(x + y) - \varphi(x).
\]

Since \( \nu_y \) defines a tempered distribution we obtain from the definition of the Fourier transform:

\[
\int \nu_y(x) \psi_y(x) \, dx = \int \hat{\nu}_y(\xi) \hat{\psi}_y(\xi) \, d\xi = \int \hat{u}^v_2(\xi) e^{2\pi i \langle \varphi, \xi \rangle} |e^{2\pi i \langle \varphi, \xi \rangle} - 1|^2 \, d\xi.
\]

If we divide the first and the last members of this formula by \( |y|^{m+\alpha} \) and integrate over \( y \) the first member is — except for the constant \( A(\alpha, m) \) — transformed into the right member of (11); the last member becomes by means of (7) and (9), after simplification,

\[
A_1(\alpha, m) A_2(\alpha, m) \int \frac{\varphi(\xi) \hat{\mu}(\xi)}{\hat{\varphi}(\xi)} \, d\xi,
\]
where $A^a(x, m)$ and $A^a(x, m)$ are the constants occurring in (7) and (9). By using the fact that

$$\int f(x) \, d\mu(x) = \int f(\xi) \, \hat{\mu}(\xi) \, d\xi,$$

we finally obtain that (11) holds true.

7. We now use (11) to study the equilibrium distribution $\mu_x$ of a compact set $F$ with non-empty interior.

Suppose that $x_0 \in \hat{F}$. Choose $\rho$ such that $S(x_0, \rho)$ is a subset of $\hat{F}$ and let $\psi$ be the characteristic function of $S(x_0, \rho)$. Let $\{\varphi_n\}$ be a uniformly bounded sequence of real-valued infinitely differentiable functions with supports in a prescribed neighborhood $G$ of $S(x_0, \rho)$ such that

$$\lim_{n \to \infty} \varphi_n(x) = \psi(x), \quad \text{for every } x.$$

If we choose the closure of $G$ as a subset of $\hat{F}$ and use (11) with $\varphi$ replaced by $\varphi_n$ and $\mu$ by $\mu_x$ we obtain from Lebesgue's theorem of dominated convergence, when $n$ tends to infinity, that (11) is true also with $\varphi$ replaced by $\psi$ and $\mu$ by $\mu_x$. By a substitution in the integral this can be written

$$\mu_x = \int_{S(x_0, \rho)} f_{\varphi_n}(x) \, dx = \int_{S(x_0, \rho)} \left( \frac{u_{\varphi_n}(y) - u_{\varphi_n}(x)}{|y - x|^{m + \alpha}} \right) \, dx \, dy,$$

where $f_{\varphi_n}$ as usual denotes the density of the restriction of $\mu_x$ to $\hat{F}$. Since $\psi$ is the characteristic function of $S(x_0, \rho)$ we can simplify this formula and get

$$(12) \quad \int_{S(x_0, \rho)} f_{\psi}(x) \, dx = 2A(\alpha, m) \int_{S(x_0, \rho)} dy \int_{S(x_0, \rho)} \frac{u_{\psi}(y) - u_{\psi}(x)}{|x - y|^{m + \alpha}} \, dx.$$ 

The fact that $u_{\psi}$ is constant in $\hat{F}$ means that the function $g$ defined by

$$g(y) = \int \frac{u_{\psi}(y) - u_{\psi}(x)}{|y - x|^{m + \alpha}} \, dx$$

is continuous and even analytic in $\hat{F}$. By dividing both members of (12) by the Lebesgue measure of $S(x_0, \rho)$ and letting $\rho$ tend to zero we consequently obtain, if $f_{\psi}$ is suitably defined
on a set of Lebesgue measure zero,

\[ f^{F}_{a}(x_0) = 2A(a, m) \int \frac{u^{F}_{a}(x_0) - u^{F}_{a}(x)}{|x_0 - x|^{m+a}} \, dx. \]

This formula shows that \( f^{F}_{a} \) is analytic in \( \hat{F} \). As \( u^{F}_{a}(x_0) = \nu_{a}(F) \)
where \( \nu_{a}(F) = \left\{ C_{a}(F) \right\}^{-1} \), we may sum up the results in the following way:

**Theorem 2.** — Let \( F \) be a compact set with interior points. If the density \( f^{F}_{a} \) of the restriction to \( \hat{F} \) of the equilibrium distribution of order \( a \) of \( F \) is properly defined on a set of Lebesgue measure zero, then \( f^{F}_{a} \) is an analytic function in \( \hat{F} \) which is given by

\[ f^{F}_{a}(x) = 2A(a, m) \int \frac{\nu_{a}(F) - u^{F}_{a}(y)}{|x - y|^{m+a}} \, dy \]
for every \( x \in \hat{F} \).

From now on we assume that \( f^{F}_{a} \) is defined and satisfies (13) for all \( x \in \hat{F} \).

8. In order to be able to use (13) to get an estimate of \( f^{F}_{a} \) near \( \nu_{a}(F) \) we have to study \( \nu_{a}(F) - u^{F}_{a}(y) \) when \( y \) is a point of \( \hat{F} \) which is situated near \( \nu_{a}(F) \).

Suppose that the interior of the sphere \( S(x_1, r_1) \) belongs to \( \hat{F} \) and that \( y \) is an interior point of \( S(x_1, r_1) \). The result of sweeping the measure consisting of the mass 1 at the point \( y \) onto the closure \( S^{*} \) of \( S(x_1, r_1) \) is, according to a result by M. Riesz, ([7], p. 17) the measure \( \lambda_{y}(x) \, dx \), where

\[ \lambda_{y}(x) = A_{3}(a, m) (r_1^{a} - |y - x_1|^{a})^{\frac{a}{2}} \cdot (|x - x_1|^{2} - r_1^{a})^{-\frac{a}{2}} |y - x|^{-m}, \]

\[ |y - x_1| < r_1, \quad x \in S^{*}, \]

and

\[ A_{3}(a, m) = \pi^{-(\frac{m+1}{2})} \cdot \Gamma\left( \frac{m}{2} \right) \sin \frac{\pi a}{2}. \]

We have

\[ \int_{S} \lambda_{y}(x) \, dx = 1, \]
which may, for instance, be proved in the following way: if \( y = x_1 \) (15) easily follows by the introduction of polar coordinates and a direct integration. If \( y \neq x_1 \) we introduce the sphere \( S(y, r_0) \) and choose \( r_0 \) so that \( S(y, r_0) \supset S(x_1, r_1) \). The measure \( \tau_y(x) \) which we obtain by sweeping the measure consisting of the mass 1 at the point \( y \) onto the closure of \( \bigcap S(y, r_0) \) has total mass 1, according to the above, and it may be obtained from \( \lambda_y(x) \) by sweeping to the closure of \( \bigcap S(y, r_0) \) the restriction to \( S(y, r_0) \) of \( \lambda_y(x) \) dx.

As the total mass does not increase by the sweeping-out process, we get

\[
1 = \int_{S(y, r_0)} \tau_y(x) \, dx \leq \int_{S^*} \lambda_y(x) \, dx \leq 1,
\]

which proves (15) when \( y \neq x_1 \).

The measure \( \lambda_y(x) \) dx can be used to express the value of the potential \( u^y \) at a point of the interior of \( S(x_1, r_1) \) by means of the values of \( u^y \) in \( S^* \) ([7], p. 17):

\[
u^y(y) = \int_{S^*} u^y(x) \lambda_y(x) \, dx \quad \text{if} \quad |y - x_1| < r_1.
\]

This formula and (15) give together

\[
u^y(F) - u^y(y) = A_a(a, m) \left( r_1^2 - |y - x_1|^2 \right)^{\frac{a}{2}} \int_{S^*} (\nu^y(F) - u^y(x)) (|x - x_1|^2 - r_1^2)^{-\frac{a}{2}} |y - x|^{-m} \, dx
\]

if \( |y - x_1| < r_1 \).

9. By means of (13) and (16) it is easy to prove the following theorem.

**Theorem 3.** — Let \( x_0 \) belong to \( \partial F \) and the closure of \( \hat{F} \). Suppose that there exists a closed sphere \( S \) with \( S \subset \bigcap F \) such that \( x_0 \) is a boundary point of \( S \). Then

\[
\liminf f^y_a(x). |x - x_0|^\frac{a}{2} > 0, \quad x \to x_0, \, x \in \hat{F}.
\]

**Proof.** — Let \( V \) be a bounded right circular cone having vertex at \( x_0 \), altitude \( r_0 \), axis along the normal at \( x_0 \) of the
boundary of $S$ and being contained, except the point $x_0$, in $\hat{S}$. Suppose that $S$ has center at the origin and radius $r_1$, $S = S(0, r_1)$. There exists a constant $M > 0$, [cf. the formula (24) below], such that

$$r_1 - |y| \geq M \cdot |y - x_0| \quad \text{for} \quad y \in V.$$ 

This and (16) give, with a new constant $M > 0$ which depends on $r_1$,

$$\phi_x(F) - \psi_x(y) \geq M \cdot |y - x_0|^\frac{a}{2}, \quad y \in V.$$ 

Remembering (13) this gives, when $x \in \hat{F}$, with constants $M > 0$,

$$f_\phi^x(x) > M \int_V \frac{|y - x_0|^\frac{a}{2}}{|x - x_0| + |x_0 - y|} dy$$

$$\geq |x - x_0|^{-\frac{a}{2}} \cdot M \int_0^a t^{\frac{a}{2}} \cdot (1 + t)^{m+\alpha} \cdot t^{m-1} dt,$$

where $a = r_1 \cdot |x - x_0|^{-1}$. This proves the theorem.

10. If we suppose that $F$ satisfies certain conditions of regularity in a neighborhood of a fix boundary point $x_0$, then the Theorems 1 and 3 show that the expression

$$(17) \quad f_\phi^x(x)|x - x_0|^\frac{a}{2},$$

takes values between two strictly positive constants when $x$ tends to $x_0$ inside some cone contained in $F$ and having vertex at $x_0$. We shall treat the problem to examine under what conditions the limit of (17) exists when $x$ tends to $x_0$. To keep the calculations comparatively simple we shall be content with the following theorem:

**Theorem 4.** — Let $x_0$ be a boundary point of $F$. Suppose that there exist positive numbers $r_0$ and $\delta_0$ such that for every $t_0 \in S(x_0, r_0) \cap \partial F$ we can find two closed spheres with radii $\delta_0$ — spheres which we denote by $S^l(t_0, \delta_0)$ and $S^r(t_0, \delta_0)$ — which have $t_0$ as a boundary-point, the same tangent plane at $t_0$ and are such that $S^l(t_0, \delta_0)$ is contained in $F$ and the interior of $S^r(t_0, \delta_0)$ in $\hat{F}$. Let $N(x_0)$ be the common normal of $S^l(x_0, \delta_0)$ and $S^r(x_0, \delta_0)$. 
Then

\begin{equation}
\lim_{x \to x_0} f^r(x), x \in \mathbb{R}, x \to x_0, x \in N(x_0) \cap F,
\end{equation}

exists and is strictly positive and finite.

The fact that the limit is strictly positive and finite if it exists, is an immediate consequence of the Theorems 1 and 3. The limit depends on \( m, \alpha, F \) and the position of \( x_0 \) in a way that will appear from the proof [cf. the end of § 14]. When \( m = 1 \) the assumption shall be interpreted to mean that \( r_0 \) and \( \delta_0 \) may be chosen such that \( S(x_0, r_0) \cap \partial F = \{ x_0 \} \), \( S'(x_0, \delta_0) \cap S''(x_0, \delta_0) = \{ x_0 \} \), \( S'(x_0, \delta_0) \) is a subset of \( F \) and the interior of \( S''(x_0, \delta_0) \) of \( \int F \).

We start the proof of Theorem 4 with some preliminary considerations after which the proof is completed in §§ 11-14 using the same notations in all the paragraphs.

Let \( \nu_0 \) be a number satisfying \( 0 < \nu_0 < \frac{\pi}{2} \) and let \( K \) be the infinite, two-sided cone of revolution — including the interior of the cone — with vertex at \( x_0 \) axis \( N(x_0) \) and generating angle \( \nu_0 \) [see Figure 1]. The plan of the proof is as follows. Using (13) we shall estimate \( f^r(x), x \in N(x_0) \cap F \). By means of the results of § 11 we show in § 12 that \( u^r \) satisfies a Lipschitz condition at \( x_0 \) which is then used to prove that the contribution to the integral occurring in (13), coming from the integration over \( \int K \) may be neglected if \( \nu_0 \) is chosen near \( \frac{\pi}{2} \).

In §§ 13-14 we estimate the contribution to (13) coming from the integration over \( K \), a contribution which consequently determines the limit (18).

We carry through the proof of the theorem only for the case \( m \geq 2 \). However, when \( m = 1 \) a proof follows almost immediately from (13) and (16) depending on the fact that in this case the complement of \( F \cup S'(x_0, \delta_0) \) is situated at a positive distance from \( x_0 \). Thus, from now on we assume that \( m \geq 2 \). We also suppose that \( \delta_0 < r_0/2 \) which we clearly may without limitation. This will guarantee that all points from \( \partial F \) with which we shall be concerned are situated in \( S(x_0, r_0) \).

The part of \( \partial F \) which is situated in \( S(x_0, r_0) \) has the following properties:
1° There exists a constant $c$ such that if $y_0 \in \partial F \cap S(x_0, r_0)$ and $\varphi$ is the angle between $N(x_0)$ and the common normal of $S'(y_0, \delta_0)$ and $S'(y_0, \delta_0)$, then

$$\varphi < c|x_0 - y_0|.$$ 

2° If $r_1$ is strictly positive and less than a certain number — which may be chosen equal to $\min\{\delta_0/2, \pi/4c\}$ where $c$ is the constant in 1° — then, for every $t \in S(x_0, r_1/2)$, the intersection between $\partial F \cap S(x_0, r_1)$ and the line through $t$ parallel to $N(x_0)$ consists of exactly one point, $t_0$.

1° is proved in [9], p. 112, for $m = 3$ and a certain class of surfaces without being stated explicitly. However, the same proof is valid for a general $m$ and with the assumptions we have made. To prove 2° we suppose that $r_1 < \min\{\delta_0/2, \pi/4c\}$ and let $l$ be the line through $t$, $t \in S(x_0, r_1/2)$, which is parallel to $N(x_0)$. $l \cap S(x_0, r_1)$ contains points both from the interior of $S'(x_0, \delta_0)$ and the interior of $S'(x_0, \delta_0)$, i.e. points both from the interior of $F$ and from $\int F$ and, consequently, also at least one point $t_0$ from $\partial F$. But the angle between $l$ and the common normal at $t_0$ of $S'(t_0, \delta_0)$ and $S'(t_0, \delta_0)$ is, according to 1°, less than $\pi/4$ as $c r_1 < \pi/4$. This means that at least every point from $l \cap S(x_0, \delta_0)$, except $t_0$, is situated in the interior of $S'(t_0, \delta_0)$ or the interior of $S'(t_0, \delta_0)$. Hence

$$l \cap \partial F \cap S(t_0, \delta_0) = \{t_0\}.$$ 

The distance from $x_0$ to a point from $l \cap \partial F$, different from $t_0$, is thus larger than

$$\delta_0 - |t_0 - x_0| > \delta_0 - r_1 > r_1,$$ 

which shows that

$$l \cap \partial F \cap S(x_0, r_1) = \{t_0\}$$

and so 2° is proved.

It is clear that $\partial F$ has a unique tangent plane at every point of $\partial F \cap S(x_0, r_0)$. $N(x_0)$ is the normal of $\partial F$ at $x_0$.

11. We start the proper proof of Theorem 4 by deducing an upper bound of $f_*^u$ in a neighborhood of $x_0$. Let $t$ be a fixed
point belonging to $\hat{F} \cap S(x_0, \delta_0)$. Let $t_0' \in \partial F$ be such that $|t - t_0'| = d(t, \partial F)$, where $d(t, \partial F)$ denotes the distance between $t$ and $\partial F$. The facts that $|t - x_0| \leq \delta_0$ and $\delta_0 < r_0/2$ imply that $t_0' \in \partial F \cap S(x_0, r_0)$ and consequently that $S'(t_0', \delta_0) \subset F$. As the line through $t_0'$ and $t$ contains the center of $S'(t_0', \delta_0)$ we may use § 4 to conclude that there exists a constant $M$, only depending on $m$, $\alpha$ and $\delta_0$, such that

$$f^F_\alpha(t) \leq M \{d(t, \partial F)\}^{\frac{\alpha}{2}} \quad \text{for all} \quad t \in S(x_0, \delta_0) \cap \hat{F}.$$  

(19) will be used to prove the existence of a constant $M$ such that if $\mu^F_\alpha(x_0, r)$ is the value of $\mu^F_\alpha$ for the sphere $S(x_0, r)$, then

$$\mu^F_\alpha(x_0, r) \leq Mr^{\frac{2m-\alpha}{2}}, \quad \text{for all} \quad r > 0.$$  

(20)

It is enough to prove (20) for all $r$ less than an arbitrarily chosen positive number. We suppose that $2r < \min\{\delta_0/2, \pi/4c\}$, where $c$ is the constant in 1°. Let $t \in F \cap S(x_0, r)$. According to 2° the intersection between $\partial F \cap S(x_0, 2r)$ and the line $l$ through $t$ parallel to $N(x_0)$ consists of exactly one point $t_0$. The angle between $l$ and the normal of $\partial F$ at $t_0$ is less than $\pi/4$ according to 1°. Combined with

$$|t - t_0| \leq |t - x_0| + |x_0 - t_0| \leq r + 2r \leq \frac{3\delta_0}{4},$$

this gives that $t$ belongs to the right circular cone which is contained in $S'(t_0, \delta_0)$, has axis along the normal to $\partial F$ at $t_0$, vertex at $t_0$, altitude $\frac{3\delta_0}{4}$ and generating angle $\pi/4$. As the distance from $t$ to the boundary of $S'(t_0, \delta_0)$ is less than or equal to $d(t, \partial F)$, we conclude, [cf. the formula (24) below], that there exists a number $M > 0$, only depending on $\delta_0$, such that

$$d(t, \partial F) \geq M|t - t_0|, \quad t \in F \cap S(x_0, r).$$  

(21)

In order to estimate $\mu^F_\alpha(x_0, r)$ we suppose for a moment that the coordinate-system is chosen with the origin at $x_0$ and the $x^1$-axis along $N(x_0)$. (19) and (21) give then

$$\nu^F_\alpha(x_0, r) = \int_{F \cap S(x_0, r)} f^F_\alpha(t) \, dt \leq M \int_{S(x_0, r)} |t - t_0|^\frac{\alpha}{2} \, dt$$

$$= M \int_{S(x_0, r)} |t - t_0|^\frac{\alpha}{2} \, dt $$
and if we evaluate the last integral by means of repeated one-dimensional integration, we obtain
\[ \nu_\alpha(x_0, r) \leq M r^{m-1} \int_{-2r}^{2r} |x|^\alpha \, dx = M r^{\frac{2m-\alpha}{2}}. \]

(20) now follows from this estimate and the following lemma:

**Lemma 2.** Let the assumptions in Theorem 4 be satisfied. Then
\[ \mu_\alpha^F(\partial F \cap S(x_0, r)) = 0, \quad \text{for some} \quad r > 0. \]

**Proof of Lemma 2.** If \( r \) is small enough the following discussion is valid for all sufficiently small \( \varepsilon \). Let \( t_0 \in \partial F \cap S(x_0, r) \).

For a fixed \( \delta_1 \), \( 0 < \delta_1 < \delta_0 \), we introduce the closed sphere \( S'(t_0, \delta_1) \) which is a subset of \( S(t_0, \delta_0) \), has radius \( \delta_1 \) and \( t_0 \) as a boundary point. Let \( S_\varepsilon(t_0, \delta_1), \varepsilon > 0, \) be the translation of \( S'(t_0, \delta_1) \) along the outer normal of \( \partial F \) at \( t_0 \).

The union of the interior \( S_\varepsilon(t_0, \delta_1) \) of all the spheres \( S_\varepsilon(t_0, \delta_1), \varepsilon > 0, \) covers \( \partial F \cap S(x_0, r) \). We choose a finite subcover of open spheres \( S_\varepsilon(t_0, \delta_1) \) and let \( K(\varepsilon) \) be the closure of the union of these spheres. Let \( F(\varepsilon) = K(\varepsilon) \cup F \). Then \( \mu_\alpha^F(\varepsilon) \) does not distribute any mass on \( \partial F \cap S(x_0, r) \). As we, except for a constant factor, obtain \( \mu_\alpha^F \) from \( \mu_\alpha^F(\varepsilon) \) by sweeping to \( F \) the restriction to \( \int F \) of \( \mu_\alpha^F(\varepsilon) \), it is clearly enough to prove that
\[ \mu_\alpha^F(\int F) \to 0 \quad \text{when} \quad \varepsilon \to 0. \]

However, by another application of the sweeping-out process we can realize that \( \mu_\alpha^F(\varepsilon)(\partial F(\varepsilon) \setminus \partial F) = 0 \). In fact, if this was not the case we would, due to the construction of \( K(\varepsilon) \), find a closed sphere \( S \subseteq K(\varepsilon) \) with \( \mu_\alpha^F(\varepsilon)(\partial S) > 0 \) and by sweeping to \( S \) the restriction to \( \int S \) of \( \mu_\alpha^F(\varepsilon) \) we would get \( \mu_\alpha^S(\partial S) > 0 \) which is wrong [cf. the formula (4)]. By combining the above facts we obtain that if \( G(\varepsilon) = F(\varepsilon) \setminus F \) it is enough to prove that
\[ \mu_\alpha^F(\varepsilon)(G(\varepsilon)) \to 0 \quad \text{when} \quad \varepsilon \to 0. \]

Let \( l \) be a line parallel to \( N(x_0) \) such that \( l \cap G(\varepsilon) \) is non-empty. We observe that if \( S_\varepsilon(t_0, \delta_1) \subseteq K(\varepsilon) \) then we have, as \( \delta_1 < \delta_0 \).
and \( S(\zeta_0, \delta_0) \subset F \), that the part of \( S(\zeta_0, \delta_1) \) which belongs to \( F \) is situated inside any chosen, fixed neighborhood of \( \zeta_0 \). This means that if \( r_1 \) is a fixed number chosen as indicated in 2° in § 10 then \( l \cap S(x_0, r_1/2) \) is non-empty and, according to 2°, that the intersection between \( l \) and \( \delta F \cap S(x_0, r_1) \) consists of exactly one point \( \eta_0 \). It also means that there exists at least one point \( \eta_1 \in l \cap (\delta F(\varepsilon) \setminus \delta F) \). \( \eta_1 \) is situated on the boundary of at least one sphere \( S(\varepsilon(\zeta_1), \delta_1) \subset K(\varepsilon) \), \( \zeta_1 \in \delta F \cap S(x_0, r) \). Finally it means that the angle between the normal \( n(\eta_1) \) of \( S(\zeta_1, \delta_1) \) at \( \eta_1 \) and the normal of \( \delta F \) at \( \zeta_1 \) is arbitrarily small and hence, according to the property 1° in § 10, also that the angle between \( n(\eta_1) \) and \( \zeta_1 \) is less than any prescribed, fixed positive number. Accordingly we get that \( \eta_0 \in V(\eta_1) \) where \( V(\eta_1) \subset S(\varepsilon(\zeta_1), \delta_1) \) is a bounded right circular cone with vertex at \( \eta_1 \) and axis \( n(\eta_1) \) having an altitude only depending on \( \delta_1 \) and a generating angle which is an absolute constant. Simple arguments now prove that \( \eta_1 \) is the only intersection between \( l \) and \( \delta F(\varepsilon) \setminus \delta F \), that there exists a function \( h \), defined on the positive numbers, such that \( h(\varepsilon) \) tends to zero when \( \varepsilon \) tends to zero and

\[
|\eta_0 - \eta_1| \leq h(\varepsilon),
\]

and, finally, that there exists a number \( M \) not depending on \( \eta_0, \eta_1 \) and \( \varepsilon \) such that (cf. § 4) for all \( y \) lying on the line-segment between \( \eta_0 \) and \( \eta_1 \),

\[
f^{F(\varepsilon)}_2(y) \leq M|y - \eta_1|^{-a/2}.
\]

If for a moment we suppose that the coordinate-system is chosen with the \( x^1 \)-axis along \( N(x_0) \) and use the estimates obtained we get by repeated one-dimensional integration

\[
\mu^{F(\varepsilon)}_2(G(\varepsilon)) \leq M\varepsilon^{m-1} \cdot \int_0^{h(\varepsilon)} |x^1|^{-a/2} \, dx^1,
\]

which tends to zero when \( \varepsilon \) tends to zero. The proof of the lemma is complete.

Remark. — Lemma 2 gives us a general class of sets \( F \) such that \( \mu^{F(\varepsilon)}_2(\delta F) = 0 \).

12. We need the following lemma:

**Lemma 3.** — \( \mu \) is a positive measure with compact support.
Suppose that, for some point \( x^0 \) in \( \mathbb{R}^m \) and for some \( \beta, 0 < \beta < 1 \), \[
\mu(x_1, r) \leq \text{const. } r^{m-a+\beta}, \quad \text{for all } \quad r > 0.
\]

Then \[
\nu^\mu_{\alpha}(x^0) - \nu^\mu_{\alpha}(x) \leq \text{const. } |x_1 - x|^\beta, \quad \text{for all } \quad x \in \mathbb{R}^m.
\]

**Remark.** — We have in this paper throughout assumed that \( 0 < \alpha < 2 \), but the lemma is true for all \( \alpha \) satisfying \( 0 < \alpha \leq m \). It is also true for \( m = 1 \).

The lemma has been proved by Carleson ([3], pp. 15-16) for \( \alpha = 2 \) in a somewhat different form. It is, however, possible to use his method of proof also for a general \( \alpha \) and in the form we have formulated the lemma.

The formula (20) and Lemma 3, used with \( x_1 = x_0 \) and \( \mu = \mu^\mu_{\alpha} \), give, as \( \nu^\mu_{\alpha}(x_0) = \nu_{\alpha}(F) \),

\[(22) \quad \nu_{\alpha}(F) - u^\mu_{\alpha}(y) \leq \text{const. } |x_0 - y|^{\beta} \quad \text{for all } y.
\]

Put

\[B(\nu_0) = \limsup_{x \to x_0} |x - x_0|^{\frac{\alpha}{2}} \cdot \int_{B(x_0)} \frac{\nu_{\alpha}(F) - u^\mu_{\alpha}(y)}{|x - y|^{m+\alpha}} \, dy,
\]

\[x \to x_0, \quad x \in N(x_0) \cap F.
\]

Using (22) we shall prove that

\[(23) \quad \lim_{\nu_0 \to \pi/2} B(\nu_0) = 0.
\]

To have a suitable reference in § 14 we give a detailed proof of (23). For an arbitrary \( y \neq x_0 \), let \( \theta \) be the angle between a vector directed along the outer normal to \( \partial F \) at \( x_0 \) and the vector from \( x_0 \) to \( y \) [see figure 1]. If \( x \in S'(x_0, \delta_0) \cap N(x_0) \) we have

\[|x - y|^2 = |x - x_0|^2 + 2|x - x_0|.|x_0 - y|\cos \theta + |x_0 - y|^2.
\]

Using (22) and introducing polar coordinates \((r, \theta, \theta_1, \ldots, \theta_{m-2})\), \( r = |x_0 - y| \), in the integral occurring in the expression \( B(\nu_0) \) we obtain, if \( x \in S'(x_0, \delta_0) \cap N(x_0) \),

\[0 \leq B(\nu_0) \leq M \limsup_{x \to x_0} |x - x_0|^{\frac{\alpha}{2}} \int_0^{\pi - \nu_0} \int_{\nu_0}^{\pi} \frac{r^{\alpha} \cdot \frac{r^2}{2}}{r^2} \cdot r^{m-1} \sin^{m-2} \theta \, dr \, d\theta,
\]
and putting \( r = |x - x_0| \cdot t \) we get the majorant \( M(\pi - 2\nu_0) \) when \( \nu_0 \to \pi/2 \). This proves (23).

13. The object of this paragraph is to put \( \nu_\alpha(F) - u_\alpha^r(y) \), \( y \in K \cap F \), in a form [see (26)] which is suitable for the final estimation of \( f_\alpha^r \).

We suppose from now on that the coordinate system is chosen so that the origin is the center of the sphere \( S'(x_0, \delta_0) \), i.e. \( S'(x_0, \delta_0) = S(0, \delta_0) \) and \( |x_0| = \delta_0 \). Simple geometric considerations show that, for a fixed \( \nu_0 \),

\[
\lim_{|y| \to 0} \frac{|x_0| - |y|}{|x_0 - y| \cdot \cos \theta} = 1, \quad y \to x_0, \quad y \in K \cap S(0, \delta_0),
\]

where \( \theta \) is defined immediately after the formula (23).

If we introduce the notation

\[
J(y) = \int_{|x| > \delta_0} (\nu_\alpha(F) - u_\alpha^r(x)) \left( |x|^2 - \delta_0^2 \right)^{-\frac{\alpha}{2}} |y - x|^{-m} \, dx,
\]

for all \( y \), then we have, according to (16),

\[
\nu_\alpha(F) - u_\alpha^r(y) = A_3(\alpha, m) \left( |x_0|^2 - |y|^2 \right)^{-\frac{\alpha}{2}} \cdot J(y) \quad \text{for} \quad |y| < \delta_0.
\]

Using (25), (24) and (22) we find

\[
\lim \sup J(y) < \infty, \quad y \to x_0, \quad y \in K \cap S(0, \delta_0).
\]
However, by a standard argument,
\[
\limsup J(y) \geq J(x_0), \quad y \to x_0, \quad y \in K \cap S(0, \delta_0)
\]
and accordingly the integral \( J(x_0) \) is convergent. From (24) it follows that the integrand of \( J(y) \) is majorized by a constant times the integrand of the convergent integral \( J(x_0) \) when \( y \in K \cap S(0, \delta_0) \) and \(|y - x_0|\) is small enough. Hence, by Lebesgue's theorem of dominated convergence,
\[
\lim J(y) = J(x_0), \quad y \to x_0, \quad y \in K \cap S(0, \delta_0).
\]
This, combined with (25) and (24), proves the existence of a function \( \eta \) such that \( \eta(y) \to 0 \) when \( y \to x_0, y \in K \cap S(0, \delta_0) \) and
\[
(26) \quad \nu_a(F) - u^F_a(y) = A_a(x, m) \cdot (2\delta_0)^{\frac{\alpha}{2}} \cdot (|x_0 - y| \cos \theta)^{\frac{\alpha}{2}} \cdot J(x_0) + |x_0 - y|^{\frac{\alpha}{2}} \cdot \eta(y), \quad y \in K \cap S(0, \delta_0).
\]
14. We finish the proof of Theorem 4 by means of (13), (23) and (26). Suppose that \( x \in S(x_0, \delta_0) \cap N(x_0) \) and let \( \rho_0 \) be a positive number.

\[
\nu_a(F) \cdot |x - x_0|^{\frac{\alpha}{2}} = 2A(x, m) \cdot |x - x_0|^{\frac{\alpha}{2}} \int \frac{\nu_a(F) - u^F_a(y)}{|x - y|^{m+\alpha}} dy
\]

\[
= 2A(x, m)|x - x_0|^{\frac{\alpha}{2}} \left\{ \int_{x(x_0, \rho_0)} + \int_{K \cap S(x_0, \rho_0)} + \int_{K \cap \{x \in S(0, \rho_0)\}} \right\} = I + II + III.
\]

According to (23) I becomes arbitrarily small when \( x \) tends to \( x_0 \) if we choose \( \nu_0 \) sufficiently near \( \pi/2 \). When \( \nu_0 \) has been fixed we choose \( \rho_0 \) so small that

\[
\{ K \cap S(x_0, \rho_0) \} \setminus F \in S(0, \delta_0).
\]

We observe that we only have to integrate over \( \int F \) in the integrals above. When we use (26) the second term of the right member of (26) gives a contribution to II which has the following majorant: \( M \cdot \sup \eta(y) \) where the supremum is taken over all \( y \in \{ K \cap S(x_0, \rho_0) \} \setminus F \) and where the constant \( M \) does not depend on \( \nu_0 \). This is proved by calculations which are analogous to those of the proof of (23). Hence, the contribution to II which comes from the second term of the right member of (26) is small, independently of \( x \), if \( \rho_0 \) is small. For a fixed \( \rho_0 \), III tends to
zero when $x$ tends to $x_0$ and accordingly it only remains to estimate the contribution to $\Pi$ which, using (26), comes from the first term of the right member of (26). This contribution is, if

$$A_4 = 2A(\alpha, m) \cdot A_3(\alpha, m) \cdot (2\delta_0)^{\frac{\alpha}{2}} \cdot J(x_0)$$

and

$$R(\rho_0) = \left\{ K \cap S(x_0, \rho_0) \right\} \setminus F,$$

$$A_4 \cdot |x - x_0|^\frac{\alpha}{2} \cdot \int_{R(\rho_0)} \frac{|x_0 - y|^\frac{\alpha}{2} \cdot \cos \frac{\alpha}{2} \cdot \theta}{|x - y|^{m+\alpha}} \, dy.$$  

This becomes after simplification if we introduce polar coordinates in the same way as in the proof of (23) and if

$$A_5 = 2A_4 \cdot \pi^\frac{m-1}{2} \cdot \left\{ \Gamma \left( \frac{m-1}{2} \right) \right\}^{-1} \quad \text{and} \quad \rho = \rho_0 \cdot |x - x_0|^{-1},$$

$$A_5 \int_0^\phi \int_0^{\nu_0} \frac{t^\frac{\alpha}{2} \cos \frac{\alpha}{2} \theta}{(t^2 + 2t \cos \theta + 1)^{\frac{m+\alpha}{2}}} \cdot t^{m-1} \sin^{m-2} \theta \, dt \, d\theta.$$  

If $\nu_0$ was chosen sufficiently near $\pi/2$ this expression is, when $|x - x_0|$ is small enough, arbitrarily near a certain constant. Together with the other estimates in this paragraph this shows that the limit (18) exists.

Added 8/10/65. Prof. L. Hörmander has informed me that the fact that $f^r_\alpha$ is infinitely differentiable in $\tilde{F}$ also may be obtained as a consequence of results for certain general classes of operators.

**BIBLIOGRAPHY**


Manuscrit reçu en juillet 1964.

Hans Wallin,
Department of Mathematics,
Sysslomansgatan 8,
Uppsala (Suède).