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PETER LOEB
BERTRAM WALSH

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THE EQUIVALENCE OF HARNACK'S PRINCIPLE AND HARNACK'S INEQUALITY IN THE AXIOMATIC SYSTEM OF BRELOT

by PETER A. LOEB (1) AND BERTRAM WALSH (2)

During the last ten years, Marcel Brelot [2] and others have investigated elliptic differential equations in an abstract setting, a setting in which the Harnack principle is assumed to be valid. When necessary, the Harnack principle has been replaced by another axiom which establishes a form of the Harnack inequality. In 1964, Gabriel Mokobodzki showed that the two axioms are equivalent when the underlying space has a countable base for its topology (see [1], pp. 16-18). We shall show that this restriction is unnecessary. First we recall some basic definitions.

Let W be a locally compact Hausdorff space which is connected and locally connected but not compact. Let $\mathfrak{H}$ be a class of real-valued continuous functions with open domains in W such that for each open set $\Omega \subseteq W$ the set $\mathfrak{H}_\Omega$, (consisting of all functions in $\mathfrak{H}$) with domains equal to Ω, is a real vector space. An open subset Ω of W is said to be *regular for* $\mathfrak{H}$ or *regular* iff its closure in W is compact and for every continuous real-valued function f defined on $\partial\Omega$ there is a *unique* continuous function h defined on $\overline{\Omega}$ such that

$$h|\partial\Omega = f, \quad h|\Omega \in \mathfrak{H}, \qquad \text{and} \qquad h \geqslant 0 \quad \text{if} \quad f \geqslant 0.$$

Moreover, the class $\mathfrak{H}$ is called a *harmonic class* on W if it satisfies the following three axioms which are due to Brelot [2] :

Axiom I. — *A function g with an open domain $\Omega \subseteq W$ is an element of $\mathfrak{H}$ if for every point $x \in \Omega$ there is a function $h \in \mathfrak{H}$ and an open set ω with $x \in \omega \subseteq \Omega$ such that $g|\omega = h|\omega$.*

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Axiom II. — *There is a base for the topology of* W *such that each set in the base is a regular region (non empty connected open set).*

Axiom III. — *If* $\mathfrak{F}$ *is a subset of* $\mathfrak{H}_\Omega$, *where* Ω *is a region in* W, *and* $\mathfrak{F}$ *is directed by increasing order on* Ω, *then the upper envelope of* $\mathfrak{F}$ *is either identically* $+\infty$ *or is a function in* $\mathfrak{H}_\Omega$.

It follows immediately from Axiom I that if h is in $\mathfrak{H}_\Omega$, then the restriction of h to any nonempty open subset of its domain is again in $\mathfrak{H}$. Given Axioms I and II, Constantinescu and Cornea ([3], p. 344 and p. 378) have shown that the following axioms are equivalent to Axiom III:

Axiom III$_1$. — *If* Ω *is a region in* W *and* $\{h_n\}$ *is an increasing sequence of functions in* $\mathfrak{H}_\Omega$, *then either* $\lim_n h_n$ *is identically* $+\infty$ *or* $\lim_n h_n$ *is in* $\mathfrak{H}_\Omega$.

Axiom III$_2$. — *If* Ω *is a region in* W, K *a compact subset of* Ω, *and* x_0 *a point in* K, *then there is a constant* $M \geqslant 1$ *such that every nonnegative function* $h \in \mathfrak{H}_\Omega$ *satisfies the inequality*

$$h(x) \leqslant M \cdot h(x_0)$$

at every point $x \in K$.

Given Axioms I and II, we shall show that the following axiom is equivalent to Axiom III.

Axiom III$_3$. — *If* Ω *is a region in* W *then every nonnegative function in* $\mathfrak{H}_\Omega$ *is either identically* **0** *or has no zeros in* Ω. *Furthermore, for any point* $x_0 \in \Omega$ *the set*

$$\Phi_{x_0} = \{h \in \mathfrak{H}_\Omega : h \geqslant \mathbf{0} \quad \text{and} \quad h(x_0) = 1\}$$

is equicontinuous at x_0.

Axiom III$_1$ is, of course, just the Harnack principle, and Axiom III$_2$ gives a « weak » Harnack inequality for $\mathfrak{H}_\Omega$. On the other hand, a consequence of Axiom III$_3$ is the fact that for any region Ω and any compact subset $K \subset \Omega$ there is a constant $M \geqslant 1$ such that for every nonnegative $h \in \mathfrak{H}_\Omega$ and every pair of points x_1 and x_2 in K the relation

$$(1) \qquad \frac{1}{M} \cdot h(x_1) \leqslant h(x_2) \leqslant M \cdot h(x_1)$$

holds. Moreover, for any point x in Ω and any constant $M > 1$ there is a compact neighborhood K of x in which (1) holds. Thus Axiom III_3 establishes a strong Harnack inequality for $\mathfrak{H}_\Omega$. Mokobodzki has established the equivalence of III_3 and III for the case in which the topology of W has a countable base; it is this restriction which we shall now remove.

That Axioms III and III_3 are equivalent follows from the

THEOREM. — *Let $\mathfrak{H}$ be a harmonic class and Ω be a region in* W. *Let x_0 be a point in Ω, and set $\Phi = \{h \in \mathfrak{H}_\Omega : h \geqslant 0$ and $h(x_0) = 1\}$. Then Φ is equicontinuous at x_0.*

Proof. — Let ω be a regular region and K a compact neighborhood of x_0 such that $x_0 \in K \subset \omega \subset \overline{\omega} \subset \Omega$. Each continuous function f on $\partial\omega$ has a unique extension $H(f) \in \mathfrak{H}_\omega$, and for each $x \in \omega$ the mapping $f \to H(f)(x)$ from $C(\partial\omega)$ into the reals is a nonnegative Radon measure on $\partial\omega$, which we denote by ρ_x. Axiom III_2 (which follows from Axiom III) gives for each pair of points x_1 and x_2 in ω a constant M (depending on those points) for which $H(f)(x_1) \leqslant M.H(f)(x_2)$, i.e.

$$\rho_{x_1} \leqslant M \cdot \rho_{x_2}$$

in the usual ordering of measures on $\partial\omega$. Hence all the measures $\{\rho_x\}_{x\in\omega}$ are absolutely continuous with respect to one another, and the Radon-Nikodym density of any one with respect to any other is essentially bounded (« essentially » being unambiguous because all the measures have the same null sets). Following an idea of Mokobodzki's, we now consider for each $x \in \omega$ the Radon-Nikodym density of ρ_x with respect to ρ_{x_0}, which we denote by g_x; each g_x is essentially bounded, and $d\rho_x = g_x \cdot d\rho_{x_0}$.

Let $A = \{h|\partial\omega : h \in \Phi\}$. Axiom III_2 states that the functions in A are uniformly bounded on $\partial\omega$, and of course they are continuous there. Thus, if S is any countably infinite subset of A, there is a function $f \in L^\infty(\rho_{x_0})$ which is an accumulation point of S with respect to the weak* topology of $L^\infty(\rho_{x_0})$ (i.e. the topology determined by $L^1(\rho_{x_0})$; see [4], p. 424). Since $L^\infty(\rho_{x_0}) \subset L^1(\rho_{x_0})$, f is also an accumulation point of S with respect to the weak topology of $L^1(\rho_{x_0})$ (i.e. the topology determined by $L^\infty(\rho_{x_0})$.) Thus by the Eberlein-Šmulian theorem

([4], p. 430), any sequence in A has a subsequence which converges weakly to an element of $L^1(\rho_{x_0})$. Since each

$$g_x \in L^\infty(\rho_{x_0}) = L^1(\rho_{x_0})^*,$$

it follows that any sequence $\{h_n\}$ in Φ has a subsequence (which we may also denote by $\{h_n\}$) for which

$$h_n(x) = \int_{\partial\omega} h_n(y) g_x(y) d\rho_{x_0}(y)$$

converges for each $x \in \omega$; the pointwise limit function h on ω belongs to $\mathfrak{H}_\omega$ since it is the extension in $\mathfrak{H}_\omega$ of the weak limit (in $L^1(\rho_{x_0})$) of the sequence $\{h_n|\partial\omega\}$. By a result of R.-M. Hervé ([5], p. 432)

$$h = \sup_n (\widehat{\inf_{k>n} h_n})$$

where $\hat{f}(x) = \sup_\delta (\inf_{y\in\delta} f(y))$ as δ ranges over the neighborhood system of x. Thus h is the limit of the increasing sequence of lower-semicontinuous functions $\widehat{\inf_{k>n} h_n}$, and that limit is attained uniformly on the compact set K. It follows that $h_n \to h$ uniformly on K, and thus $\Phi|K$ is relatively sequentially compact, hence relatively compact, in the uniform norm topology of C(K). So $\Phi|K$ is equicontinuous (Arzelà; see [4], p. 266), whence Φ is equicontinuous at the interior points of K, and in particular at x_0.

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Peter A. Loeb and B. Walsh,
Department of Mathematics,
University of California,
Los Angeles, Calif. (U.S.A.).