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ON SPECTRAL REPRESENTATION FOR SELFADJOINT OPERATORS. EXPANSION IN GENERALIZED EIGENELEMENTS

by Eberhard GERLACH

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Introduction.

The principal aim of the present paper is to give a supplement to the theory of spectral representation for selfadjoint operators in a separable Hilbert space. This theory was begun by Hellinger [7] and Hahn [6]; among the further contributors we have to mention mainly M. H. Stone [16] (Chapter VII) and J. von Neumann [15]. A detailed account can also be found in Dunford and Schwartz [3] (Sections X-5 and XII-3).

If μ is a spectral measure for the selfadjoint operator A in a Hilbert space $\mathcal{H}$ then the « classical » theory of spectral representation sets up an isometric isomorphism mapping $\mathcal{H}$ onto a suitable space $\mathfrak{L}^2(d\mu)$ of vectorvalued functions on the real line; $\mathfrak{L}^2(d\mu)$ is the direct integral (rel. the measure μ) of Hilbert spaces whose elements can be considered as generalized eigenelements of the operator A. Various results in the classical theory hold μ-almost everywhere, but the exceptional sets and the relations between them were not specified. We shall give a precise description of the exceptional sets; the consequences of this description will be useful for questions of expansion in generalized eigenelements of A. The feasibility of this study was recognized after reading papers of Yu. M. Berezanskiĭ [2], C. Foiaş [4], [5], G. I. Kac [9], [10], and K. Maurin [12], [13]. In the remainder of the present paper we apply the general results to the problem of eigenfunction expansions in proper functional Hilbert spaces. We have divided our investigation into three chapters.

Chapter I deals with spectral representation. In § I.1 we give a résumé of the classical theory, as far as we need it. In § I.2 we start with a generating system $e = \{e_n\}$ in $\mathcal{H}$ rel. the operator A, and with the « minimal » μ-null set Λ_e. The exceptional sets $\Lambda_{u,(e)}$ are introduced (for every $u \in \mathcal{H}$) and the generalized eigenspaces $\mathcal{H}_e^{(\lambda)}$ (defined for every $\lambda \notin \Lambda_e$)

are investigated. The domain $\mathfrak{D}_e^{(\lambda)}$ of the « infinitesimal » projection $P_e^{(\lambda)} : \mathcal{H} \to \mathcal{H}_e^{(\lambda)}$ consists of all u for which $\lambda \notin \Lambda_{u,(e)}$; it is a subspace of $\mathcal{H}$; $P_e^{(\lambda)}$ maps onto all of $\mathcal{H}_e^{(\lambda)}$. For every $\lambda \notin \Lambda_e$ a closed subspace $\hat{\mathcal{H}}_e^{(\lambda)}$ of $\mathcal{H}$ is constructed so that the mapping $P_e^{(\lambda)} : \hat{\mathcal{H}}_e^{(\lambda)} \to \mathcal{H}_e^{(\lambda)}$ is an isometric isomorphism. The spaces $\mathcal{H}_e^{(\lambda)}$ are found to be essentially independent of the choice of e. In § I.3 we introduce the notion of expansibility. A subset $S \subset \mathcal{H}$ is called A-expansible if $\mu\left(\bigcup_{u \in S} \Lambda_{u,(e)}\right) = 0$ for some generating system e. We give some examples of expansible Banach-subspaces $\mathcal{B}$ of $\mathcal{H}$, based on the fact that if $\mathcal{B}$ is contained in $\mathfrak{D}^{(\lambda)}$ then the mapping $P^{(\lambda)} : \mathcal{B} \to \mathcal{H}^{(\lambda)}$ is bounded. The range of any Hilbert-Schmidt operator in $\mathcal{H}$ is expansible for every selfadjoint operator A ([2]); a set contained in such a range is called Hilbert-Schmidt expansible. If a set S is A-expansible then for every fixed $\lambda \notin \bigcup_{x \in S} \Lambda_{x,(e)}$ the components $\xi_n(x; \lambda)$ of the generalized eigenelements are functions defined everywhere on S, they appear as generalized eigenfunctions of A. In § I.4 we consider operators T in $\mathcal{H}$ which are closed, have dense domain, and commute with the resolution of identity E(.) of the selfadjoint operator A. The spectral decomposition of these into operators $T^{(\lambda)}$ in $\mathcal{H}^{(\lambda)}$ is obtained and described completely.

Chapter II gives applications to proper functional Hilbert spaces $\{\mathcal{F}, \mathcal{E}\}$, i.e. Hilbert spaces $\mathcal{F}$ consisting of functions f defined everywhere on a basic set $\mathcal{E}$ such that the evaluation $f(x)$ for fixed $x \in \mathcal{E}$ represents a bounded linear functional on $\mathcal{F}$: $f(x) = (f, K_x)$. The reproducing kernel is given by $K(x, y) = (K_y, K_x)$. In § II.1, for every λ outside the minimal exceptional set $\Lambda_{\mathcal{E}}$, the generalized eigenspaces $\{\mathcal{F}_{\mathcal{E}}^{(\lambda)}, \mathcal{E}\}$ are defined directly; they are determined by their reproducing kernels $K(x, y; \lambda) = d(E(\lambda)K_y, E(\lambda)K_x)/d\mu(\lambda)$. The p.f. Hilbert space $\{\mathcal{F}, \mathcal{E}\}$ is called expansible if the set $\{K_x | x \in \mathcal{E}\}$ is expansible in $\mathcal{F}$; in that case $\mu(\Lambda_{\mathcal{E}}) = 0$, and the canonical isometric isomorphism between $\mathcal{F}_e^{(\lambda)}$ as given by the abstract theory and $\mathcal{F}_{\mathcal{E}}^{(\lambda)}$ is established. In § II.2 we employ the properties of

([2]) This result can be interpreted as a slight improvement of a corresponding result of G. I. Kac [9].

Hilbert-subspaces of $\{\mathcal{F}, \mathcal{E}\}$ to investigate Hilbert-Schmidt expansibility.

In Chapter III we consider p.f. Hilbert $\{\mathcal{F}, \mathrm{D}\}$ consisting of functions analytic in some domain D in complex space C^n. These spaces are all Hilbert-Schmidt expansible, and the generalized eigenfunctions in $\mathcal{F}_{\mathrm{D}}^{(\lambda)}$ (for μ-almost all λ) are analytic in D.

We have not touched upon the theory of spectral representation for normal operators. Also, in § I.4 we have not tried to obtain a spectral decomposition for general T not necessarily commuting with E(.) into operators $\mathrm{T}^{(\lambda, k)} : \mathcal{H}^{(k)} \to \mathcal{H}^{(\lambda)}$.

The notation and terminology of this paper will be explained in the course of the development.

Formulas as well as items (such as « Definition », « Theorem », etc.) are numbered through in each chapter. When referring to formula (3) (or « 1. Theorem ») of Chap. I we shall in other chapters write (3.I) (or Theorem 1.I), and in the same chapter write simply (3) (or Theorem 1).

CHAPTER I

SPECTRAL REPRESENTATION FOR HILBERT SPACES

I.1. Résumé of the classical theory of spectral representation.

Let $\mathcal{H}$ be a separable Hilbert space. Let A be a fixed selfadjoint operator in $\mathcal{H}$ with E(.) its resolution identity. An interval (open, halfopen or closed) will be denoted by Δ. For a Borel subset S of the real axis E(S) denotes the corresponding projection.

Definition 1. — *A sequence* $\{e_n\} \subset \mathcal{H}$ *is called a generating system rel.* E(.) *if* 1° *the elements* $E(\Delta)e_n$ *generate the space* $\mathcal{H}$ *and* 2° $(E(\Delta)e_n, E(\Delta')e_m) = 0$ *for* $n \neq m$ *and any* Δ, Δ'. *We assume the system normalized by* $\|e_n\| = 1$ *for all* n. *The generating system* $\{e_n\}$ *defines a system of measures* μ_n *given by* $\mu_n(\Delta) = \|E(\Delta)e_n\|^2$ (3).

For our selfadjoint operator A we choose a fixed *spectral measure* as follows. Consider the class T of all measures ν such that, for every generating system $\{e_n\}$, $\nu \succ \mu_n$ (4) for all n. In this class we consider the equivalence class S of measures μ satisfying $\mu \prec \nu$ for every $\nu \in$ T. Let us choose a fixed measure $\mu \in$ S of total mass 1. This μ will be our spectral measure.

From now on we shall keep $\mathcal{H}$, A, E(.), and μ fixed.

Let $e = \{e_n\}$ be a generating system.

(3) We do not insist on any canonical choice of a generating system $\{e_n\}$. Neither are we interested in relations between a generating system and a (canonical) sequence of measures $\nu_1, \nu_2, \ldots$ (where ν_{k+1} is absolutely continuous with respect to ν_k) describing the operator A up to unitary equivalence (cf. M. H. Stone, [16], Chap. VII).

(4) $\nu \succ \mu_n$ (or equivalent $\mu_n \prec \nu$) means that the measure μ_n is absolutely continuous with respect to the measure ν.

Let Λ_e be the minimal set such that

$$(1) \qquad 0 \leqslant \lim_{\Delta \downarrow \lambda} \frac{\|E(\Delta)e_n\|^2}{\mu(\Delta)} = \frac{d\mu_n(\lambda)}{d\mu(\lambda)} \overset{\text{def}}{=} \theta_n(\lambda)$$

exists and is finite for every $\lambda \notin \Lambda_e$. Clearly $\mu(\Lambda_e) = 0$. For $\lambda \notin \Lambda_e$ we define the following sequential Hilbert space $\mathcal{H}_e^{(\lambda)}$:

We introduce the abstract vectors $e_n(\lambda)$. Then $\mathcal{H}_e^{(\lambda)}$ consists of all elements $u^{(\lambda)} = \Sigma \xi_n e_n(\lambda)$ where $\xi_n = 0$ whenever $\theta_n(\lambda) = 0$ and for which the norm defined by

$$(2) \qquad \|u^{(\lambda)}\|^2 = \sum_n |\xi_n|^2 \theta_n(\lambda)$$

is finite.

The direct integral $\int \mathcal{H}_e^{(\lambda)}\, d\mu(\lambda)$ of these spaces is well-defined. It consists of all sequences of μ-measurable functions $\{\varphi_n(\lambda)\}$ such that $\Sigma\varphi_n(\lambda)e_n(\lambda) \in \mathcal{H}_e^{(\lambda)}$ for μ-almost every all λ (this implies $\varphi_n(\lambda) = 0$ if $\theta_n(\lambda) = 0$) and

$$\int \|\Sigma\varphi_n(\lambda)e_n(\lambda)\|^2\, d\mu(\lambda) = \int \Sigma|\varphi_n(\lambda)|^2\theta_n(\lambda)\, d\mu(\lambda)$$
$$= \sum_n \int |\varphi_n(\lambda)|^2\, d\mu_n(\lambda) < \infty.$$

For every $u \in \mathcal{H}$ the limits $\lim\limits_{\Delta \downarrow \lambda} \dfrac{(u, E(\Delta)e_n)}{\mu(\Delta)} = \dfrac{d(E(\lambda)u, E(\lambda)e_n)}{d\mu(\lambda)}$ exist, are finite, and equal zero if $\theta_n(\lambda) = 0$ for μ-almost all λ. We set

$$(3) \qquad \xi_n(u;\lambda) = \begin{cases} \dfrac{1}{\theta_n(\lambda)} \dfrac{d(E(\lambda)u, E(\lambda)e_n)}{d\mu(\lambda)} & \text{if } \theta_n(\lambda) > 0 \\ 0 & \text{if } \theta_n(\lambda) = 0 \end{cases}$$

Furthermore, for $u, v \in \mathcal{H}$ and any Δ,

$$(4) \qquad (E(\Delta)u, E(\Delta)v) = \sum_n \int_\Delta \xi_n(u;\lambda)\overline{\xi_n(v;\lambda)}\, d\mu_n(\lambda)$$
$$= \int_\Delta \sum_n \xi_n(u;\lambda)\overline{\xi_n(v;\lambda)}\theta_n(\lambda)\, d\mu(\lambda).$$

The correspondence $u \to \{\xi_n(u;\lambda)\}$ establishes the canonical isometric isomorphism between $\mathcal{H}$ and the direct integral $\int \mathcal{H}_e^{(\lambda)}\, d\mu(\lambda)$. Under this isometry any function $F(A)$ of the operator A in $\mathcal{H}$ corresponds to multiplication by $F(\lambda)$ in the direct integral: $F(A)u \to \{F(\lambda)\xi_n(u;\lambda)\}$ for μ-almost all λ, and

$$(F(A)u, E(\Delta)v) = \sum_n \int_\Delta F(\lambda)\xi_n(u;\lambda)\overline{\xi_n(v;\lambda)}\, d\mu_n(\lambda).$$

Thus $\{\xi_n(u;\lambda)\}$ can be considered as *generalized eigenelements* of the operator A. If λ is an eigenvalue of A then the measure μ has a point mass at λ, and $\mathcal{H}_e^{(\lambda)}$ is identified with the corresponding eigensubspace of $\mathcal{H}$.

We are now interested in a precise description of the exceptional sets (all of μ-measure zero) outside of which the various results of the classical theory hold, and in some of the consequences of such a description.

I.2. Generalized eigenspaces.

DEFINITION 2. — *For a fixed generating system e and an element $u \in \mathcal{H}$ we define $\Lambda_{u,(e)}$ as the smallest set such that $\Lambda_{u,(e)} \supset \Lambda_e$ and for every $\lambda \notin \Lambda_{u,(e)}$:*

(5)
$$\begin{cases} a) \ \dfrac{d(E(\lambda)u, E(\lambda)e_n)}{d\mu(\lambda)} = \xi_n(u;\lambda)\theta_n(\lambda) \ \textit{exists, is finite, and} \\ \quad \textit{equals zero whenever } \theta_n(\lambda) = 0 \ (\textit{then we set } \xi_n(u;\lambda) = 0), \\ b) \ \dfrac{d\|E(\lambda)u\|^2}{d\mu(\lambda)} \ \textit{exists}, \ \sum_n |\xi_n(u;\lambda)|^2\theta_n(\lambda) \ \textit{converges, and} \\ \quad \textit{the two are equal.} \end{cases}$$

Clearly $\mu(\Lambda_{u,(e)}) = 0$.

For fixed n let P_n be the projection onto the closed subspace generated by all $E(\Delta)e_n$. Then we denote $P_nE(.) = E(.)P_n$ by $E_n(.)$.

LEMMA 3. — *Let $\lambda \notin \Lambda_{u,(e)}$. For each k and arbitrary $\varepsilon > 0$ there is an interval $\Delta_0 = \Delta_0(k,\varepsilon)$, containing λ in its interior, such that*

(6)
$$\left|\sum_{n=k}^{\infty} \frac{\|E_n(\Delta)u\|^2}{\mu(\Delta)} - \sum_{n=k}^{\infty} |\xi_n(u;\lambda)|^2\theta_n(\lambda)\right| < \varepsilon \ \textit{for all } \Delta \textit{ with } \lambda \in \Delta \subset \Delta_0.$$

Furthermore, for every k

$$(7) \qquad \frac{d\|E_n(\lambda)u\|^2}{d\mu(\lambda)} \quad \textit{exists and equals} \quad |\xi_k(u;\lambda)|^2\,\theta_k(\lambda).$$

Proof. — We shall proceed by induction on k. For $k = 1$, (6) follows from condition $b)$ in (5). For every n we obtain from (4) and Schwarz's inequality

$$\frac{\|E_n(\Delta)u\|^2}{\mu(\Delta)} = \frac{1}{\mu(\Delta)}\int_\Delta |\xi_n(u;\rho)|^2\,\theta_n(\rho)\,d\mu(\rho)$$
$$\geqslant \frac{1}{\mu(\Delta)\mu_n(\Delta)}\left|\int_\Delta \xi_n(u;\rho)\,\theta_n(\rho)\,d\mu(\rho)\right|^2.$$

For any sequence of intervals $\Delta_j \downarrow \lambda$, $\lambda \notin \Lambda_{u,(e)}$ and (5) $a)$ then yield

$$\liminf_{\Delta_j \downarrow \lambda} \frac{\|E_n(\Delta_j)u\|^2}{\mu(\Delta_j)} \geqslant |\xi_n(u;\lambda)|^2\,\theta_n(\lambda).$$

Now suppose (6) is valid for the index k. We shall prove (7) for k. For any $\lambda \in \Delta \subset \Delta_0(k, \varepsilon)$ we have

$$-\varepsilon < \frac{\|E_k(\Delta)u\|^2}{\mu(\Delta)} - |\xi_k(u;\lambda)|^2\,\theta_k(\lambda) + \sum_{k+1}^{\infty} \frac{\|E_n(\Delta)u\|^2}{\mu(\Delta)}$$
$$- \sum_{k+1}^{\infty} |\xi_n(u;\lambda)|^2\,\theta_n(\lambda) < \varepsilon$$

and for an arbitrary sequence $\Delta_0 \supset \Delta_j \downarrow \lambda$

$$0 \leqslant \left(\liminf_{\Delta_j \downarrow \lambda} \frac{\|E_k(\Delta_j)u\|^2}{\mu(\Delta)_j} - |\xi_k(u;\lambda)|^2\,\theta_k(\lambda)\right)$$
$$+ \left(\sum_{k+1}^{\infty} \liminf_{\Delta_j \downarrow \lambda} \frac{\|E_n(\Delta_j)u\|^2}{\mu(\Delta_j)} - \sum_{k+1}^{\infty} |\xi_n(u;\lambda)|^2\,\theta_n(\lambda)\right) < \varepsilon$$

where each of the two terms is non-negative. Consequently

$$(7') \qquad 0 \leqslant \liminf_{\Delta_j \downarrow \lambda} \frac{\|E_k(\Delta_j)u\|^2}{\mu(\Delta_j)} - |\xi_k(u;\lambda)|^2\,\theta_k(\lambda) < \varepsilon$$

for any ε and every $\Delta_j \downarrow \lambda$, and so (7) holds for k. Applying (6) for k and $\frac{\varepsilon}{2}$ and (7) for k:

$$\left|\frac{\|E_k(\Delta)u\|^2}{\mu(\Delta)} - |\xi_k(u;\lambda)|^2\,\theta_k(\lambda)\right| < \frac{\varepsilon}{2} \quad \textit{for} \quad \lambda \in \Delta \subset \Delta_1\left(k, \frac{\varepsilon}{2}\right)$$

we obtain (6) for $k+1$ and $\Delta_0(k+1, \varepsilon) = \Delta_0\left(k, \frac{\varepsilon}{2}\right) \cap \Delta_1\left(k, \frac{\varepsilon}{2}\right)$.

The assignment $u \to \{\xi_n(u; \lambda)\}$ defines a mapping

$$P_e^{(\lambda)} : \mathcal{H} \to \mathcal{H}_e^{(\lambda)}, \quad P_e^{(\lambda)} u = \Sigma\, \xi_n(u; \lambda) e_n(\lambda),$$
$$\|P_e^{(\lambda)} u\|^2 = \frac{d\|E(\lambda)u\|^2}{d\mu(\lambda)}$$

for $\lambda \notin \Lambda_{u,(e)}$. $P_e^{(\lambda)}$ may be considered as an « infinitesimal projection ».

Theorem 4. — *The domain $\mathfrak{D}_e^{(\lambda)}$ of $P_e^{(\lambda)}$ is a linear subspace of $\mathcal{H}$. Furthermore, for $u, v \in \mathcal{H}$ and every $\lambda \notin \Lambda_{u,(e)} \cup \Lambda_{v,(e)}$:*

$$\text{(8)} \qquad \frac{d(E(\lambda)u, E(\lambda)v)}{d\mu(\lambda)} = \sum_n \xi_n(u; \lambda)\overline{\xi_n(v, \lambda)}\theta_n(\lambda).$$

Proof. — Clearly $u \in \mathfrak{D}_e^{(\lambda)}$ implies $\alpha u \in \mathfrak{D}_e^{(\lambda)}$ for any complex number α. For $u, v \in \mathfrak{D}_e^{(\lambda)}$:

$$\frac{d(E(\lambda)\,(u+v), E(\lambda)e_n)}{d\mu(\lambda)} = (\xi_n(u, \lambda) + \xi_n(v, \lambda))\theta_n(\lambda).$$

Now we consider condition *b*). We have

$$\|E(\Delta)\,(u+v)\|^2 = \|E(\Delta)u\|^2 + \|E(\Delta)v\|^2 + (E(\Delta)u, E(\Delta)v) + (E(\Delta)v, E(\Delta)u).$$

If (8) holds then we can apply it to the right hand side here and obtain $u + v \in \mathfrak{D}_e^{(\lambda)}$. It remains to prove (8). By (4) we have

$$\text{(9)} \qquad \frac{1}{\mu(\Delta)}(E(\Delta)u, E(\Delta)v) = \frac{1}{\mu(\Delta)} \sum_n (E_n(\Delta)u, E_n(\Delta)v)$$
$$= \frac{1}{\mu(\Delta)} \sum_n \int_\Delta \xi_n(u; \rho)\overline{\xi_n(v; \rho)}\,\theta_n(\rho)\, d\mu(\rho).$$

Furthermore

$$\text{(9')} \qquad \lim_{\Delta \downarrow \lambda} \frac{1}{\mu(\Delta)} \int_\Delta \xi_n(u; \rho)\overline{\xi_n(v; \rho)}\,\theta_n(\rho)\, d\mu(\rho) = \xi_n(u; \lambda)\,\xi_n(v; \lambda)\,\theta_n(\lambda) \text{ [5]}$$

[5] To prove this we establish

$$\lim_{\Delta \downarrow \lambda} \frac{1}{\mu_n(\Delta)} \int_\Delta \xi_n(u; \rho)\overline{\xi_n(v; \rho)}\, d\mu_n(\rho) = \xi_n(u, \lambda)\overline{\xi_n(v, \lambda)} \quad \textit{for every } n$$

by use of Lemma 3 and the following elementary

Lemma. — *Let ν be a measure on the real line and let $\varphi(\tau)$ and $\psi(\tau)$ be two ν-measu-*

for every n. For arbitrary $\varepsilon > 0$ choose k sufficiently large so that $\left|\sum_{n=k+1}^{\infty} \xi_n(u;\lambda)\overline{\xi_n(v;\lambda)}\theta_n(\lambda)\right|$, $\sum_{k+1}^{\infty} |\xi_n(u;\lambda)|^2\,\theta_n(\lambda)$, and $\sum_{k+1}^{\infty} |\xi_n(v;\lambda)|^2\theta_n(\lambda)$ are each less than ε. We keep k fixed now. By (9′), the first k terms on the right hand side of (9) will approximate $\sum_{n=1}^{k} \xi_n(u;\lambda)\overline{\xi_n(v;\lambda)}\,\theta_n(\lambda)$ as close as we please provided that Δ is sufficiently small. It remains to evaluate the remainder in (9). We have

$$\left|\frac{1}{\mu(\Delta)}\sum_{n=k+1}^{\infty}\int_\Delta \xi_n(u;\rho)\overline{\xi_n(v;\rho)}\,\theta_n(\rho)\,d\mu(\rho)\right|$$
$$\leqslant \frac{2}{\mu(\Delta)}\left\{\sum_{k+1}^{\infty}\int_\Delta |\xi_n(u;\rho)|^2\theta_n(\rho)\,d\mu(\rho)+\sum_{k+1}^{\infty}\int_\Delta |\xi_n(v;\rho)|^2\theta_n(\rho)\,d\mu(\rho)\right\}$$
$$= 2\left\{\sum_{k+1}^{\infty}\frac{\|E_n(\Delta)u\|^2}{\mu(\Delta)}+\sum_{k+1}^{\infty}\frac{\|E_n(\Delta)v\|^2}{\mu(\Delta)}\right\}.$$

Since $\lambda \notin \Lambda_{u,(e)} \cup \Lambda_{v,(e)}$ we may apply Lemma 3 and find that the right hand side differs from

$$2\left\{\sum_{k+1}^{\infty}|\xi_n(u;\lambda)|^2\theta_n(\lambda)+\sum_{k+1}^{\infty}|\xi_n(v;\lambda)|^2\theta_n(\lambda)\right\}$$

by less than 2ε for $\lambda \in \Delta \subset \Delta_0(k,\varepsilon)$, and $2\{\ldots\} < 2\varepsilon$ by the choice of k.

Since the subspace $\mathfrak{D}_e^{(\lambda)}$ contains all elements $E(\Delta)e_n$, where Δ is an arbitrary open interval, it is dense in $\mathcal{H}$.

THEOREM 5. — *The range of* $P_e^{(\lambda)}$ *is all of* $\mathcal{H}_e^{(\lambda)}$.

Proof. — If $\lambda \notin \Lambda_e$ and $\theta_n(\lambda) > 0$ then $P_e^{(\lambda)}e_n = e_n(\lambda)$. Let $\lambda \notin \Lambda_e$ be fixed. For every n we choose an open interval Δ_n such that $\lambda \in \Delta_n$, $\Delta_n \downarrow \lambda$, and (for $\theta_n(\lambda) > 0$)

(10)
$$\frac{1}{2}\,\theta_n(\lambda)\mu(\Delta') < \|E(\Delta')e_n\|^2 < 2\theta_n(\lambda)\mu(\Delta') \textit{ for all } \Delta' \textit{ with } \lambda \in \Delta' \subset \Delta_n$$

rable functions such that the four functions $\Phi_1(\rho) = \int_a^\rho \varphi(\tau)\,d\nu(\tau)$, $\Phi_2(\rho) = \int_a^\rho |\varphi(\tau)|^2\,d\nu(\tau)$, $\Psi_1(\rho) = \int_a^\rho \psi(\tau)\,d\nu(\tau)$, $\Psi_2(\rho) = \int_a^\rho |\psi(\tau)|^2\,d\nu(\tau)$ *are differentiable with respect to* ν *at the point* λ *and their derivatives there equal to the corresponding integrands* $\varphi(\lambda)$, $|\varphi(\lambda)|^2$, $\psi(\lambda)$, $|\psi(\lambda)|^2$. *Then the function* $\chi(\rho) = \int_a^\rho \varphi(\tau)\psi(\tau)\,d\nu(\tau)$ *is differentiable at* λ *and* $\left.\dfrac{d\chi(\rho)}{d\nu(\rho)}\right|_{\rho=\lambda} = \varphi(\lambda)\psi(\lambda)$.

Now let $u^{(\lambda)} = \Sigma\xi_n e_n(\lambda) \in \mathcal{H}_e^{(\lambda)}$, i.e., $\Sigma|\xi_n|^2\theta_n(\lambda) < \infty$ and $\xi_n = 0$ whenever $\theta_n(\lambda) = 0$. Then $u = \Sigma\xi_n \mathrm{E}(\Delta_n)e_n$ belongs to $\mathcal{H}$ since $||u||^2 = \Sigma|\xi_n|^2||\mathrm{E}(\Delta_n)e_n||^2 < \infty$ by (10). We show that $u \in \mathfrak{D}_e^{(\lambda)}$. By construction we obtain condition *a*) with $\xi_n(u;\lambda) = \xi_n$. It remains to establish *b*). We shall show existence (and equality to $\Sigma|\xi_n|^2\theta_n(\lambda)$) of both the left and right hand derivatives. By $\Delta_n^+(\Delta_n^-)$ we denote the part of Δ_n lying to the right (left) of λ and including λ. Let Δ denote intervals to the right (left) of λ now: $\Delta = [\lambda, \lambda'[$. Then for fixed Δ and $n' = n'(\Delta)$ we have

$$\mathrm{E}(\Delta)u = \sum_{n=1}^{n'} \xi_n \mathrm{E}(\Delta)e_n + \sum_{n=n'+1}^{\infty} \xi_n \mathrm{E}(\Delta_n^+)e_n.$$

For $\varepsilon > 0$, fix k large enough so that $\sum\limits_{k+1}^{\infty} |\xi_n|^2\theta_n(\lambda) < \varepsilon$. Then for Δ sufficiently small $n'(\Delta) > k$, and

$$\frac{||\mathrm{E}(\Delta)u||^2}{\mu(\Delta)} = \frac{1}{\mu(\Delta)}\sum_{n=1}^{k}|\xi_n|^2||\mathrm{E}(\Delta)e_n||^2 + \frac{1}{\mu(\Delta)}\sum_{k+1}^{n'(\Delta)}|\xi_n|^2||\mathrm{E}(\Delta)e_n||^2$$
$$+ \frac{1}{\mu(\Delta)}\sum_{n'(\Delta)+1}^{\infty}|\xi_n|^2||\mathrm{E}(\Delta_n^+)e_n||^2.$$

The first term approximates $\sum\limits_{1}^{k}|\xi_n|^2\theta_n(\lambda)$ as $\Delta \downarrow \lambda$. In the second term we have $\Delta \subset \Delta_n^+ \subset \Delta_n$, and so by (10) it is bounded by $2\sum\limits_{k+1}^{n'(\Delta)}|\xi_n|^2\theta_n(\lambda)$. In the remainder term we have $\Delta_n^+ \subset \Delta$, $\frac{\mu(\Delta_n^+)}{\mu(\Delta)} \leqslant 1$, and so it is bounded by $2\sum\limits_{n'(\Delta)+1}^{\infty}|\xi_n|^2\theta_n(\lambda)$. Thus the second and third terms together are bounded by 2ε (by the choice of k) for every Δ small enough so that $n'(\Delta) > k$. This proves that the right-hand derivative of $||\mathrm{E}(\lambda)u||^2$ exists and equals $\sum\limits_n|\xi_n|^2\theta_n(\lambda)$. The left-hand derivative is treated analogously.

Proposition 6. — *Let $\lambda \notin \Lambda_e$. For a suitable choice of elements $g_n \in \mathcal{H}$ satisfying $\mathrm{P}_e^{(\lambda)}g_n = e_n(\lambda)$, the elements $\Sigma\xi_n g_n$ for which $\Sigma|\xi_n|^2\theta_n(\lambda) < \infty$ ($\xi_n = 0$ for $\theta_n(\lambda) = 0$) form a closed subspace $\hat{\mathcal{H}}_e^{(\lambda)}$ of $\mathcal{H}$ which by $\mathrm{P}_e^{(\lambda)}$ is isometrically isomorphic to $\mathcal{H}_e^{(\lambda)}$.*

Proof. — If λ is an eigenvalue of A, then $\mathrm{P}_e^{(\lambda)} = \mathrm{E}(\{\lambda\})$ is the projection of $\mathcal{H}$ onto the eigenspace corresponding to λ, and we take $g_n = \mathrm{E}(\{\lambda\})e_n$ (then $||g_n||^2 = \theta_n(\lambda)$).

Suppose now that λ is not an eigenvalue. Let A_n be the restriction of A to the closed subspace generated by $E(\Delta)e_n$, for Δ arbitrary. If now $\theta_n(\lambda) > 0$, then λ is in the continuous spectrum of A_n and is not an eigenvalue of infinite multiplicity. Hence every open interval containing λ must contain points of the spectrum of A_n different from λ. We choose such an interval, Δ_n say, and take it so small that it satisfies (10) and also $\|E(\Delta_n)e_n\|^2 < \theta_n(\lambda)$. Then we can find an open subinterval Δ'_n containing λ such that both Δ'_n and $\Delta_n - \Delta'_n \overset{\text{def}}{=} \Delta''_n$ meet the spectrum of A_n, and thus $E(\Delta'_n)e_n \neq 0 \neq E(\Delta''_n)e_n$. Clearly $P_e^{(\lambda)}E(\Delta''_n)e_n = 0$ and $P_e^{(\lambda)}E(\Delta'_n)e_n = e_n(\lambda)$. We determine the constant $c_n > 0$ such that:

$$\|(c_nE(\Delta''_n) + E(\Delta'_n))e_n\|^2 = c_n^2\|E(\Delta''_n)e_n\|^2 + \|E(\Delta'_n)e_n\|^2 = \theta_n(\lambda):$$
$$c_n = (\theta_n(\lambda) - \|E(\Delta'_n)e_n\|^2)^{1/2}/\|E(\Delta''_n)e_n\|$$

and set

$$g_n = (c_nE(\Delta''_n) + E(\Delta'_n))e_n.$$

Then we have $P_e^{(\lambda)}g_n = e_n(\lambda)$ and $\|g_n\|^2 = \theta_n(\lambda)$. The elements $u = \Sigma\xi_n g_n$ are by $P_e^{(\lambda)}$ in one-to-one correspondence with the elements of the complete space $\mathcal{H}_e^{(\lambda)}$, and this correspondence is isometric.

THEOREM 7. — *Define* $\Lambda_{(e,\,e')} = \bigcup_1^\infty \Lambda_{e'_k,\,(e)} \cup \bigcup_1^\infty \Lambda_{e_n,\,(e')}$. *Clearly* $\mu(\Lambda_{(e,\,e')}) = 0$. *Then for every* $\lambda \notin \Lambda_{(e,\,e')}$, *there exists a canonical isometric isomorphism* $J_{e,\,e'}^{(\lambda)}$ *of* $\mathcal{H}_{e'}^{(\lambda)}$ *onto* $\mathcal{H}_e^{(\lambda)}$ *such that for* $u \in \mathfrak{D}_{e'}^{(\lambda)} \cap \mathfrak{D}_e^{(\lambda)}$

$$J_{e,\,e'}^{(\lambda)}P_{e'}^{(\lambda)}u = P_e^{(\lambda)}u.$$

Furthermore for every $u \in \mathcal{H}$

$$\Lambda_{u,\,(e')} \subset \Lambda_{u,\,(e)} \cup \Lambda_{(e,\,e')}. \tag{11}$$

Proof. — Let $\lambda \notin \Lambda_{(e,\,e')}$. We introduce $\chi_{k,\,n}(\lambda) = \xi_n(e'_k;\lambda)$ and $\tau_{n,\,k}(\lambda) = \xi'_k(e_n;\lambda)$, that is

$$\left\{\begin{array}{l} \chi_{k,\,n}(\lambda) = \dfrac{1}{\theta_n(\lambda)}\dfrac{d(E(\lambda)e'_k,\,E(\lambda)e_n)}{d\mu(\lambda)} \\ and \\ \tau_{n,\,k}(\lambda) = \dfrac{1}{\theta'_k(\lambda)}\dfrac{d(E(\lambda)e_n,\,E(\lambda)e'_k)}{d\mu(\lambda)} = \dfrac{\theta_n(\lambda)}{\theta'_k(\lambda)}\overline{\chi_{k,\,n}(\lambda)} \\ for\ \theta_n(\lambda) > 0\ and\ \theta'_k(\lambda) > 0, \\ and\ \chi_{k,\,n}(\lambda) = \tau_{n,\,k}(\lambda) = 0\ for\ \theta_n(\lambda) = 0\ or\ \theta'_k(\lambda) = 0. \end{array}\right. \tag{12}$$

Then we define the linear transformation $J^{(\lambda)}_{e,e'}: \mathcal{H}^{(\lambda)}_{e'} \to \mathcal{H}^{(\lambda)}_{e}$ by

$$J^{(\lambda)}_{e,e'}e'_k(\lambda) = \sum_n \chi_{k,n}(\lambda)e_n(\lambda), \tag{13}$$

and similarly $J^{(\lambda)}_{e',e}: \mathcal{H}^{(\lambda)}_{e} \to \mathcal{H}^{(\lambda)}_{e'}$ by

$$J^{(\lambda)}_{e',e}e_n(\lambda) = \sum_k \tau_{n,k}(\lambda)e'_k(\lambda). \tag{13'}$$

We have for any k and l:

$$\begin{aligned}\delta_{kl}\theta'_k(\lambda) = (e'_k(\lambda), e'_l(\lambda))_{\mathcal{H}^{(\lambda)}_{e'}} &= \frac{d(E(\lambda)e'_k, E(\lambda)e'_l)}{d\mu(\lambda)} \\ &= \sum_n \chi_{k,n}(\lambda)\overline{\chi_{l,n}(\lambda)}\,\theta_n(\lambda) = (J^{(\lambda)}_{e,e'}e'_k(\lambda), J^{(\lambda)}_{e,e'}e'_l(\lambda))_{\mathcal{H}^{(\lambda)}_{e}}\end{aligned}$$

which shows that the mapping $J^{(\lambda)}_{e,e'}$ is an isometry and thus can be extended to all of $\mathcal{H}^{(\lambda)}_{e'}$. Similarly $J^{(\lambda)}_{e',e}$ is an isometry since for any m and n:

$$\begin{aligned}\delta_{mn}\theta_m(\lambda) = (e_m(\lambda), e_n(\lambda))_{\mathcal{H}^{(\lambda)}_{e}} &= \frac{d(E(\lambda)e_m, E(\lambda)e_n)}{d\mu(\lambda)} \\ &= \sum_k \tau_{mk}(\lambda)\overline{\tau_{nk}(\lambda)}\,\theta'_k(\lambda) = (J^{(\lambda)}_{e',e}e_m(\lambda), J^{(\lambda)}_{e',e}e_n(\lambda))_{\mathcal{H}^{(\lambda)}_{e'}}.\end{aligned}$$

Using these formulas and (12) we find for any n and l

$$\sum_k \tau_{n,k}(\lambda)\chi_{k,l}(\lambda) = \delta_{nl} = \Sigma\chi_{n,k}(\lambda)\tau_{k,l}(\lambda)$$

which means that the matrices $\chi(\lambda) = (\chi_{k,n}(\lambda))$ and $\tau(\lambda) = (\tau_{l,n}(\lambda))$ are inverse to each other. Consequently $J^{(\lambda)}_{e',e}J^{(\lambda)}_{e,e'}$ acts as identity in $\mathcal{H}^{(\lambda)}_{e'}$, and $J^{(\lambda)}_{e,e'}J^{(\lambda)}_{e',e}$ acts as identity in $\mathcal{H}^{(\lambda)}_{e}$. This also shows that the ranges of $J^{(\lambda)}_{e,e'}$ and $J^{(\lambda)}_{e',e}$ are all of $\mathcal{H}^{(\lambda)}_{e}$ and $\mathcal{H}^{(\lambda)}_{e'}$, respectively.

From our constructions we see that for $u \in \mathfrak{D}^{(\lambda)}_{e'} \cap \mathfrak{D}^{(\lambda)}_{e}$

$$J^{(\lambda)}_{e,e'}P^{(\lambda)}_{e'}u = P^{(\lambda)}_{e}u.$$

It remains to prove (11). Let $u \in \mathcal{H}$ and $\lambda \notin \Lambda_{u,(e)} \cup \Lambda_{(e,e')}$. We want to show $\lambda \notin \Lambda_{u,(e')}$, that is $u \in \mathfrak{D}^{(\lambda)}_{e'}$. By the hypothesis on λ

$$\frac{d(E(\lambda)u, E(\lambda)e'_k)}{d\mu(\lambda)} = \sum_n \theta_n(u;\lambda)\overline{\chi_{k,n}(\lambda)}\,\theta_n(\lambda).$$

This is zero if $\theta'_k(\lambda) = 0$ since then all $\chi_{k,n}$ vanish. It remains to establish condition b) rel. the system e'. We have (for $\theta'_k(\lambda) > 0$),

$$\xi'_k(u;\lambda) = \frac{1}{\theta'_k(\lambda)} \sum_n \xi_n(u;\lambda)\overline{\chi_{k,n}(\lambda)}\,\theta_n(\lambda) = \frac{(J^{(\lambda)}_{e',e}P^{(\lambda)}_e u,\ e'_k(\lambda))}{\|e'_k(\lambda)\|^2}.$$

This means that $\xi'_k(u;\lambda)$ are the components of the element $J^{(\lambda)}_{e',e}P^{(\lambda)}_e u$ in $\mathcal{H}^{(\lambda)}_{e'}$, and because of the isometric isomorphism

$$\sum_k |\xi'_k(u;\lambda)|^2\,\theta'_k(\lambda) = \|J^{(\lambda)}_{e',e}P^{(\lambda)}_e u\|^2 = \|P^{(\lambda)}_e u\|^2 = \frac{d\|E(\lambda)u\|^2}{d\mu(\lambda)}$$

which gives condition b).

Remark. — Concerning the generalized eigenelements $\Sigma\xi_n(u;\lambda)e_n(\lambda)$ we note that $u \in \mathfrak{D}^{(\lambda)}_e$ implies $Au \in \mathfrak{D}^{(\lambda)}_e$ and, for $\lambda \neq 0$, also $Au \in \mathfrak{D}^{(\lambda)}_e$ implies $u \in \mathfrak{D}^{(\lambda)}_e$. If $F(A)$ is a function of the operator A then for every $\lambda \notin \Lambda_{u,(e)} \cup \Lambda_{F(A)u,(e)}$ one has

$$P^{(\lambda)}_e F(A)u = \Sigma F(\lambda)\xi_n(u;\lambda)e_n(\lambda),$$

or in other words

$$\{\xi_n(F(A)u;\lambda)\} = \{F(\lambda)\xi_n(u;\lambda)\}.$$

THEOREM 8. — *If the sequence $\{x_k\}$ is a total* [6] *subset of $\mathcal{H}$ then a generating system e can be obtained from it. Let e' be another generating system. Then for every $\lambda \notin \Lambda_{(e,e')} \cup \bigcup_k \Lambda_{x_k,(e')}$ the set $\{P^{(\lambda)}_{e'}x_k\}$ is total in $\mathcal{H}^{(\lambda)}_{e'}$. If an particular we set $e' = e$ we find that $\{P^{(\lambda)}_e x_k\}$ is total in $\mathcal{H}^{(\lambda)}_e$ for $\lambda \notin \bigcup_k \Lambda_{x_k,(e)}$.*

Proof. — The generating system $\{e_k\}$ is obtained from $\{x_k\}$ as follows. Let $\mathcal{H}_n$ be the closed subspace of $\mathcal{H}$ generated by all elements $E(\Delta)x_k$ where Δ is arbitrary and $k = 1, \ldots, n$. Let y_k be the projection of x_k onto $\mathcal{H}_{k-1}$ and set $z_k = x_k - y_k$. If $z_k \neq 0$ we define $e_k = \dfrac{z_k}{\|z_k\|}\cdot$ Then $\{e_k\}$ obviously forms a generating system.

On the other hand, for fixed $\lambda \notin \Lambda_{(e,e')} \cup \bigcup_{k=1}^{\infty} \Lambda_{x_k,(e')}$ consider the sequence $\{P^{(\lambda)}_{e'}x_k\}$ in $\mathcal{H}^{(\lambda)}_{e'}$. We orthogonalize it (without insisting on any normalization). Let us write $P^{(\lambda)}_{e'}x_k = \xi^{(k)}(\lambda)$

[6] A subset $S \subset \mathcal{H}$ is total if $(u, s) = 0$ for all $s \in S$ implies $u = 0$.

and define $\eta^{(k)}(\lambda) \in \mathcal{H}_{e'}^{(\lambda)}$ successively by $\eta^{(1)}(\lambda) = \xi^{(1)}(\lambda)$ and

$$\eta^{(k)}(\lambda) = \xi^{(k)}(\lambda) - \sum_{j=1}^{k-1} \frac{1}{\|\eta^{(j)}(\lambda)\|^2} (\xi^{(k)}(\lambda), \eta^{(j)}(\lambda))\eta^{(j)}(\lambda)$$

where in the sum only terms with $\eta^{(j)}(\lambda) \neq 0$ are taken. Inspection now shows that

$$P_{e'}^{(\lambda)} e_k = a_k \eta^{(k)}(\lambda) \quad \textit{for all } k$$

where the a_k are positive constants. These vectors form a basis for $\mathcal{H}_{e'}^{(\lambda)} = J_{e',e}^{(\lambda)} \mathcal{H}_e^{(\lambda)}$ since e is a generating system. Consequently the sequence $\{P_{e'}^{(\lambda)} x_k\}$ is total in $\mathcal{H}_{e'}^{(\lambda)}$. The particular case is clear since $\Lambda_{(e,e)} = \Lambda_e \subset \bigcup_k \Lambda_{x_k,(e)}$.

I.3. Banach-Subspaces. Expansibility.

Let the generating system e and $\lambda \notin \Lambda_e$ be fixed. Frequently we shall omit the subscript e and simply write $P^{(\lambda)}$, $\mathfrak{D}^{(\lambda)}$, $\mathcal{H}^{(\lambda)}$.

Proposition 9. — *Let* B *be a Banach-subspace of* $\mathcal{H}$ *with norm* $\| \ \|_B : \|u\|_B \geq \|u\|$ *for* $u \in B$. *Suppose that* $B \subset \mathfrak{D}^{(\lambda)}$. *Then the mapping*

$$(14) \qquad P^{(\lambda)} : \{B, \| \ \|_B\} \to \mathcal{H}^{(\lambda)}$$

is bounded.

Proof. — We use the uniform boundedness theorem. For fixed $\Delta \ni \lambda$, $(E(\Delta)u, E(\Delta)e_n)/\mu(\Delta)$ represents a bounded linear functional on B. The expression converges (as $\Delta \downarrow \lambda$) for every $u \in B$ since $B \subset \mathfrak{D}^{(\lambda)}$. Consequently the limit also represents a bounded linear functional:

$$|\xi_n(u; \lambda)\theta_n(\lambda)| = \left| \lim_{\Delta \downarrow \lambda} \frac{(E(\Delta)u, E(\Delta)e_n)}{\mu(\Delta)} \right| \leq M_n \|u\|_B.$$

Next, for each k the linear operator $P_k : B \to \mathcal{H}^{(\lambda)}$ defined by

$$P_k u = \sum_{n=1}^{k} \xi_n(u; \lambda) e_n(\lambda)$$

is bounded since

$$\left(\sum_{n=1}^{k} |\xi_n(u; \lambda)|^2 \theta_n(\lambda) \right)^{1/2} \leq \left(\sum_{\substack{n=1 \\ \theta_n(\lambda) > 0}}^{k} \frac{M_n^2}{\theta_n(\lambda)} \right)^{1/2} \|u\|_B.$$

For every fixed $u \in \mathrm{B}$, the $\mathrm{P}_k u$ converge to $\mathrm{P}^{(\lambda)}u$ in $\mathcal{H}^{(\lambda)}$. Consequently $\mathrm{P}^{(\lambda)} = \lim \mathrm{P}_k$ is a bounded operator of B into $\mathcal{H}^{(\lambda)}$.

Definition 10. — *A subset* S *of* $\mathcal{H}$ *is called* A-*expansible if*

$$\mu\Big(\bigcup_{u \in \mathrm{S}} \Lambda_{u,(e)}\Big) = 0 \tag{15}$$

for some generating system e (then by Theorem 7, the same condition holds for any other generating system). The set S *is called totally expansible if it is* A-*expansible for every selfadjoint operator* A *in* $\mathcal{H}$. *A countable union of* A-*expansible (totally expansible) sets is also* A-*expansible (totally expansible). Let* [S] *be the linear span of* S *(that is, all finite linear combinations of elements in* S*). Then* (15) *implies* $\mu\Big(\bigcup_{u\in[\mathrm{S}]} \Lambda_{u,(e)}\Big) = 0$.

As an application of Proposition 9 we shall give some examples of expansible subspaces.

Let a sequence $\{u_k\} \subset \mathcal{H}$ and an index p, $1 \leqslant p \leqslant \infty$, be given such that

$$\begin{cases} \sum_n \int_{-\infty}^{\infty} \Big(\sum_k |\xi_n(u_k;\rho)|^p\Big)^{2/p} \theta_n(\rho)\, d\mu(\rho) < \infty & \textit{if } p < \infty, \\ \sum_n \int_{-\infty}^{\infty} \sup_k |\xi_n(u_k;\rho)|^2 \theta_n(\rho)\, d\mu(\rho) < \infty & \textit{if } p = \infty, \end{cases} \tag{16}$$

where $\sum_n \xi_n(u_k;\lambda)e_n(\lambda) = \mathrm{P}_e^{(\lambda)}u_k$ (rel. some fixed generating system e). We denote the expression in (16) by $\|\{u_k\}\|^2_{(p)}$. Let $\zeta = \{\zeta_k\} \in l^{p'}$ now. Then $\sum_k \zeta_k u_k = \lim_{\mathrm{N}\to\infty} \sum_1^{\mathrm{N}} \xi_k u_k$ belongs to $\mathcal{H}$ since

$$\Big\|\sum_{\mathrm{M}}^{\mathrm{N}} \zeta_k u_k\Big\|^2 = \sum_n \int_{-\infty}^{\infty} \Big|\sum_{\mathrm{M}}^{\mathrm{N}} \zeta_k \xi_n(u_k;\rho)\Big|^2 \theta_n(\rho)\, d\mu(\rho)$$
$$\leqslant \Big(\sum_{\mathrm{M}}^{\mathrm{N}} |\zeta_k|^{p'}\Big)^{2/p'} \sum_n \int_{-\infty}^{\infty} \Big(\sum_1^{\infty} |\xi_n(u_k;\rho)|^p\Big)^{2/p} \theta_n(\rho)\, d\mu(\rho)$$

where the second factor is finite by (16), and the first tends to zero as M, N $\to \infty$.

Consequently all elements of X which can be represented in the form $\Sigma\zeta_k u_k$ as just described form a subspace

$$\Phi = \Phi_p = \Phi_p(\{u_k\}).$$

By giving a norm to Φ it can be made into a Banach subspace of $\mathcal{H}$ in two equivalent ways: Either we make the direct definition

$$||\Sigma\zeta_k u_k||_\Phi = \inf_{\substack{\Sigma\eta_k u_k = \Sigma\zeta_k u_k \\ \eta = \{\eta_k\} \in l^{p'}}} ||\eta||_{l^{p'}} ||\{u_k\}||_{(p)}.$$

Or we consider the linear mapping

$$U : l^{p'} \ni \zeta \to \Sigma\zeta_k u_k \in \mathcal{H}$$

with null space $\mathfrak{N}(U)$. Then Φ provided with the topology of $l^{p'}/\mathfrak{N}(U)$ will be a B-subspace of $\mathcal{H}$.

THEOREM 11. — *For every Banach-subspace* $\Phi = \Phi_p(\{u_k\})$ *of* $\mathcal{H}$ *there exists a* μ*-null set* Λ_Φ *such that* $\Phi \subset \mathfrak{D}_e^{(\lambda)}$ *for every* $\lambda \notin \Lambda_\Phi$.

Proof. — The set Λ_Φ is defined as follows. The complement $R_1 - \Lambda_\Phi$ consists of all $\lambda \notin \bigcup_k \Lambda_{u_k(e)}$ for which the derivative of

$$(17) \qquad \int_{-\infty}^{\lambda} \sum_n \left(\sum_{k=1}^{\infty} |\xi_n(u_k;\rho)|^p\right)^{2/p} \theta_n(\rho)\, d\mu(\rho)$$

exists, is finite, and equal to the integrand. Clearly $\mu(\Lambda_\Phi) = 0$. Let $\lambda \notin \Lambda_\Phi$ be fixed; we show that $\Phi \subset \mathfrak{D}_e^{(\lambda)}$.

To establish condition *a*) of (5) we consider

$$\frac{1}{\mu(\Delta)} \sum_k \zeta_k(E(\Delta)u_k, E(\Delta)e_n) = \sum_k \frac{1}{\mu(\Delta)} \zeta_k \int_\Delta \xi_n(u_k;\rho)\theta_n(\rho)\, d\mu(\rho)$$

as $\Delta \downarrow \lambda$. In the series on the right, each term converges individually as $\Delta \downarrow \lambda$. We split the series into $\sum_{k=1}^{N} \ldots + \sum_{k=N+1}^{\infty} \ldots$ and evaluate the remainder. Let us write

$$(0, \ldots, 0, \zeta_{N+1}, \zeta_{N+2}, \ldots) = \zeta^{(N)} \quad \text{and} \quad (\zeta_1, \ldots, \zeta_N, 0, \ldots) = \zeta_N.$$

$$\begin{aligned}
\left|\sum_{N+1}^{\infty}\cdots\right| &\leq \frac{1}{\mu(\Delta)} \int_\Delta \left|\sum_{k=N+1}^{\infty} \zeta_k\xi_n(u_k;\rho)\right| \theta_n(\rho)\, d\mu(\rho) \\
&\leq \frac{1}{\mu(\Delta)} \left(\sum_{N+1}^{\infty} |\zeta_k|^{p'}\right)^{1/p'} \int_\Delta \left(\sum_{k=N+1}^{\infty} |\xi_n(u_k;\rho)|^p\right)^{1/p} \theta_n(\rho)\, d\mu(\rho) \\
&\leq ||\xi^{(N)}||_{l^{p'}} \left(\frac{\mu_n(\Delta)}{\mu(\Delta)}\right)^{1/2} \left(\frac{1}{\mu(\Delta)} \int_\Delta \left(\sum_{k=1}^{\infty} |\xi_n(u_k;\rho)|^p\right)^{2/p} \theta_n(\rho)\, d\mu(\rho)\right)^{1/2}.
\end{aligned}$$

The third factor in this evaluation remains bounded as $\Delta \downarrow \lambda$ (actually it converges to a finite limit, as can be seen from the assumptions about (17), using arguments similar to those of Lemma 3). The second factor tends to $\sqrt{\theta_n(\lambda)}$ as $\Delta \downarrow \lambda$, and the first to zero as $N \to \infty$. Thus we found

$$\frac{d}{d\mu(\lambda)} \sum_k \zeta_k(E(\lambda)u_k, E(\lambda)e_n) = \sum_k \zeta_k \xi_n(u_k; \lambda)\, \theta_n(\lambda).$$

We turn to condition *b*).

$$\begin{aligned}
&\frac{1}{\mu(\Delta)}\left\|\sum_k \zeta_k E(\Delta)u_k\right\|^2 \\
&= \frac{1}{\mu(\Delta)}\left\{\left\|\sum_{k=1}^{N} \cdots\right\|^2 + \left\|\sum_{k=N+1}^{\infty} \cdots\right\|^2 + \left(\sum_1^N \cdots, \sum_{N+1}^{\infty} \cdots\right) + \left(\sum_{N+1}^{\infty} \cdots, \sum_1^N \cdots\right)\right\} \\
&= \frac{1}{\mu(\Delta)}\left\|\sum_{k=1}^{N} \zeta_k E(\Delta)u_k\right\|^2 + R.
\end{aligned}$$

The first term will converge to $\sum_n \left|\sum_{k=1}^{N} \zeta_k \xi_n(u_k; \lambda)\right|^2 \theta_n(\lambda)$ as $\Delta \downarrow \lambda$. We evaluate R.

$$\begin{aligned}
|R| \leq \frac{1}{\mu(\Delta)}\Bigg\{ &\int_\Delta \sum_n \left|\sum_{k=N+1}^{\infty} \zeta_k \xi_n(u_k; \rho)\right|^2 \theta_n(\rho)\, d\mu(\rho) + \\
&2\left(\int_\Delta \sum_n \left|\sum_{k=1}^{N} \zeta_k \xi_n(u_k; \rho)\right|^2 \theta_n(\rho)\, d\mu(\rho)\right)^{1/2} \\
&\qquad \left(\int_\Delta \sum_n \left|\sum_{N+1}^{\infty} \zeta_k \xi_n(u_k; \rho)\right|^2 \theta_n(\rho)\, d\mu(\rho)\right)^{1/2}\Bigg\} \\
\leq\ & \|\xi^{(N)}\|_{l^{p'}}^2 \frac{1}{\mu(\Delta)} \int_\Delta \sum_n \left(\sum_{k=N+1}^{\infty} |\xi_n(u_k; \rho)|^p\right)^{2/p} \theta_n(\rho)\, d\mu(\rho) \\
&+ 2\|\zeta_N\|_{l^{p'}} \|\zeta^{(N)}\|_{l^{p'}} \left(\frac{1}{\mu(\Delta)} \int_\Delta \sum_n \left(\sum_{k=1}^{N} |\xi_n(u_k; \rho)|^p\right)^{2/p} \theta_n(\rho)\, d\mu(\rho)\right)^{1/2} \\
&\left(\frac{1}{\mu(\Delta)} \int_\Delta \sum_n \left(\sum_{N+1}^{\infty} \cdots\right)^{2/p} \cdots\right)^{1/2} \\
\leq\ & \|\zeta^{(N)}\|_{l^{p'}} \{\|\zeta^{(N)}\|_{l^{p'}} + 2\|\zeta_N\|_{l^{p'}}\} \frac{1}{\mu(\Delta)} \\
&\qquad \int_\Delta \sum_n \left(\sum_{k=1}^{\infty} \xi_n|(u_k; \rho)|^p\right)^{2/p} \theta_n(\rho)\, d\mu(\rho).
\end{aligned}$$

The third factor remains bounded (actually by the conditions on (17) it converges) as $\Delta \downarrow \lambda$, the second is bounded by $3\|\zeta\|_{l^{p'}}$, and the first tends to zero as $N \to \infty$. Thus we find that

$$\frac{d}{d\mu(\lambda)} \|\Sigma\zeta_k E(\lambda) u_k\|^2 = \sum_n \Big|\sum_k \zeta_k \xi_n(u_k;\lambda)\Big|^2 \theta_n(\lambda).$$

Proposition 9 and Theorem 11 give

COROLLARY 12. — *The Banach subspace* $\Phi_p(\{u_k\})$ *of* $\mathcal{H}$ *is* A-*expansible.*

Remark. — The question arises whether there exist such spaces $\Phi_p(\{u_k\})$ with $p \neq 2$ which are not contained in any other space $\Phi_2(\{v_k\})$ of this kind. It is conjectured that for $p > 2$ there do exist such spaces.

Consider a sequence $\{u_k\} \subset \mathcal{H}$ for which $\sum_k \|u_k\|^2 < \infty$; the corresponding space $\Phi_2(\{u_k\})$ is the range of a Hilbert-Schmidt operator T in $\mathcal{H}$ defined by $T\varphi_k = u_k$ where $\{\varphi_k\}$ is an orthonormal system. Then for every arbitrary selfadjoint operator A in $\mathcal{H}$ with resolution of identity E(.), spectral measure μ, and generating system e we obtain

$$\infty > \sum_k \|u_k\|^2 = \sum_k \sum_n \int_{-\infty}^{\infty} |\xi_n(u_k;\rho)|^2 \theta_n(\rho)\, d\mu(\rho)$$
$$= \sum_n \int_{-\infty}^{\infty} \sum_k |\xi_n(u_k;\rho)|^2 \theta_n(\rho)\, d\mu(\rho),$$

that is, condition (16). Thus we have

COROLLARY 13. — *The space* $\Phi_2(\{u_k\})$ *is totally expansible.*

DEFINITION 14. — *A subset* S *of* $\mathcal{H}$ *is called* H.S.-*expansible (Hilbert-Schmidt expansible) if there exist* $\{u_k\}$, $\sum_k \|u_k\|^2 < \infty$ *such that* $S \subset \Phi_2(\{u_k\})$. *A set which is contained in the union of countably many* H.S.-*expansible sets is also* H.S.-*expansible.*

The results of Section 2 and the definition of expansibility give

THEOREM 15. — *Let* $\mathcal{E} \subset \mathcal{H}$ *be* A-*expansible. Then* 1° *For every* $x \in \mathcal{E}$

$$\lim_{\Delta \downarrow \lambda} \frac{(E(\Delta)x,\, E(\Delta)e_n)}{\mu(\Delta)} = \xi_n(x;\lambda)\,\theta_n(\lambda) \tag{18}$$

exists for all n and every $\lambda \notin \Lambda_{\mathcal{E}} = \bigcup_{x\in\mathcal{E}} \Lambda_{x,(e)}$. 2° *The expansion*

$$x = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \xi_n(x;\lambda)e_n(\lambda)\, d\mu(\lambda)$$

holds. 3° *For fixed* $\lambda \notin \Lambda_{\mathcal{E}}$ *the* $\xi_n(x;\lambda)$ *defined by* (18) *are functions of* x *defined everywhere on* $\mathcal{E}$; *they appear as generalized eigenfunctions of the operator* A.

I.4. Spectral decomposition of operators commuting with E(.).

Throughout this section we keep A, E(.) and μ fixed and consider operators T in $\mathcal{H}$ which are closed, have dense domain, and commute with E(.):

$E(\lambda)\mathfrak{D}(T) \subset \mathfrak{D}(T)$ *and* $TE(\lambda)u = E(\lambda)Tu$ *for* $u \in \mathfrak{D}(T)$, *and every* λ.

An equivalent way of saying that T commutes with every $E(\lambda)$ is

$$(19) \qquad E(\lambda)T \subset TE(\lambda) \quad \textit{for all } \lambda,$$

that is, $TE(\lambda)$ is an extension of $E(\lambda)T$. Taking adjoints in (19) we see that then also T^* commutes with E(.). Furthermore T and T^* commute with every $E(\Delta)$ where Δ is an interval.

We shall need the following well known

Lemma 16. — *Let* T *be a closed operator with dense domain in* $\mathcal{H}$. *Suppose that the bounded operators* B_n *converge strongly to an operator* B *and that* T *commutes with every* B_n. *Then* T *commutes also with* B.

Theorem 17. — *Let* T *be closed, with dense domain, and commuting with* E(.). *Then one can construct a generating system* $\{e_n\}$ *which is contained in* $\mathfrak{D}(T)$.

Proof. — *a*) Suppose first that A has no eigenvalues (i.e., μ has no point masses). We make use of the one-to-one correspondence between pairs of non-negative integers (k, l), $k, l = 0, 1, 2, \ldots$, and positive integers $n = 1, 2, 3, \ldots$ given by the formula $(2k+1)2^l = n$. Let the sequence $\{x_k^{(0)}\} \subset \mathfrak{D}(T)$ be total in $\mathcal{H}$. To obtain the desired generating system $\{e_n\}$ we construct elements $y_n \in \mathfrak{D}(T)$ and $x_k^{(l)} \in \mathfrak{D}(T)$ by induction on n and set $e_n = \frac{y_n}{\|y_n\|}$ for $y_n \neq 0$. Our induction hypothesis for the step n is the following:

$$(20) \quad \begin{cases} y_{n'} \textit{ and } x_{k'}^{(l')} \textit{ have been constructed and belong to } \mathfrak{D}(\mathrm{T}) \\ \quad \textit{for all } n' < n \textit{ and } (2k'+1)2^{l'} \leqq n. \end{cases}$$

We already have the sequence $\{x_k^{(0)}\} \subset \mathfrak{D}(\mathrm{T})$. We set $y_1 = x_0^{(0)}$ and $x_0^{(l)} = 0$ for all $l > 0$. Then (20) is satisfied for $n = 2$.

We proceed to construct y_n from hypothesis (20). Let Y_{n-1} be the closed subspace generated by all elements $\mathrm{E}(\Delta)y_{n'}$ for which $n' < n$. Let k and l be determined by $n = (2k+1)2^l$. Then $x_k^{(l)} \in \mathfrak{D}(\mathrm{T})$ is known by (20). We shall decompose:

$$(21) \qquad x_k^{(l)} = h_n + z_n + x_k^{(l+1)}$$

where $y_n \perp \mathrm{Y}_{n-1}$, $z_n \in \mathrm{Y}_{n-1}$, $\|x_k^{(l+1)}\| \leqq \frac{1}{n}$, and all three belong to $\mathfrak{D}(\mathrm{T})$. If $x_k^{(l)} \in \mathrm{Y}_{n-1}$ we choose $z_n = x_k^{(l)}$ and $y_n = x_k^{(l+1)} = 0$. Consider $x_k^{(l)} \notin \mathrm{Y}_{n-1}$. Let P_{n-1} be the projection onto Y_{n-1}. Then

$$\mathrm{P}_{n-1}x_k^{(l)} = \sum_{n'=1}^{n-1} z'_{n'},$$

where $\quad z'_{n'} = \int \varphi_{n'}(\lambda)\, d\mathrm{E}(\lambda)y_{n'} \quad$ *and* $\quad \varphi_{n'}(\lambda) \in \mathrm{L}^2(d\|\mathrm{E}(\lambda)y_{n'}\|^2)$,

since by construction the $e_{n'}$ form part of a generating system. Since μ has no point masses, the measures $\|\mathrm{E}(\Delta)y_{n'}\|^2$ do not have point masses either. We may take $\varphi_{n'}(\lambda) = 0$ whenever $\frac{d\|\mathrm{E}(\lambda)y_{n'}\|^2}{d\mu(\lambda)} = 0$; then $\varphi_{n'}(\lambda)$ is also measurable rel. μ. Since for every $n' \leqq n-1$ the elements $\int_{\{|\varphi_{n'}(\lambda)|<\mathrm{M}\}} \varphi_{n'}(\lambda)\, d\mathrm{E}(\lambda)y_{n'}$ approximate $z'_{n'}$ as $\mathrm{M} \to \infty$ we can find a set S_{n-1} whose complement has a measure as small as we please and so that all $\varphi_{n'}(\lambda)$, $n' = 1, 2, \ldots, n-1$ are uniformly bounded on S_{n-1}. The choice of S_{n-1} will be specified later; let χ_{n-1} denote the characteristic function of S_{n-1}. We define

$$(21') \qquad \begin{cases} z_n = \sum_{n'=1}^{n-1} \int \varphi_{n'}(\lambda)\chi_{n-1}(\lambda)\, d\mathrm{E}(\lambda)y_{n'}, \\ y_n = \int \chi_{n-1}(\lambda)\, d\mathrm{E}(\lambda)x_k^{(l)} - z_n \\ x_k^{(l+1)} = \int (1 - \chi_{n-1}(\lambda))\, d\mathrm{E}(\lambda)x_k^{(l)} \end{cases}$$

which will give the decomposition (21). We have to prove the desired properties. By construction $y_n \perp Y_{n-1}$ and $z_n \in Y_{n-1}$. The operators $\int \chi_{n-1}(\lambda)\, dE(\lambda)$ and $\int \chi_{n-1}(\lambda)\varphi_{n'}(\lambda)\, dE(\lambda)$ are strong limits of finite linear combinations of operators $E(\Delta)$, hence they commute with T by Lemma 16. Since $y_{n'}$, $x_k^{(l)} \in \mathfrak{D}(T)$ then also z_n, $y_n \in \mathfrak{D}(T)$, and finally $x_k^{(l+1)} \in \mathfrak{D}(T)$. The set S_{n-1} is chosen so large that $y_n \neq 0$ and

$$||x_k^{(l+1)}||^2 = \int |1 - \chi_{n-1}(\lambda)^2|d||E(\lambda)x_k^{(l)}||^2 \leqq \frac{1}{n^2}.$$

To complete the induction we have to show that (20) holds for $n+1$. We only have to check on $x_{k_1}^{(l_1)}$ where $(2k_1+1)2^{l_1} = n+1$. Then $(2k_1+1)2^{l_1-1} = n_1 \leqq n$, $x_{k_1}^{(l_1-1)}$ are given by (20) for n and $x_{k_1}^{(l_1)}$ was obtained together with y_{n_1}.

It remains to show that the set $\{E(\Delta)y_n\}_{\Delta, n}$ is total in $\mathcal{H}$, i.e. that every $u \in \mathcal{H}$ is a limit of finite linear combinations of elements $E(\Delta)y_n$. So let $u \in \mathcal{H}$ be arbitrary. Since $\{x_k^{(0)}\}$ is total in $\mathcal{H}$, for every $\varepsilon > 0$ there are numbers $\xi_1, \ldots, \xi_N$ such that

$$\left\| u - \sum_{k=1}^{N} \xi_k x_k^{(0)} \right\| < \varepsilon.$$

For each fixed k ($k = 1, \ldots, N$) we decompose

$$\begin{aligned} x_k^{(0)} &= y_{n_1} + z_{n_1} + x_k^{(1)}, & n_1 &= (2k+1) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & \\ &= y_{n_1} + z_{n_1} + \cdots + y_{n_l} + z_{n_l} + x_k^{(l)}, & n_l &= (2k+1)2^{l-1}, \end{aligned}$$

where l is sufficiently large so that $x_k^{(l)} = 0$ or $||x_k^{(l)}|| < \dfrac{\varepsilon}{N \max |\xi_k|}$ which can be achieved by the condition on (21). Then $\sum_k |\xi_k|\, ||x_k^{(l)}|| < \varepsilon$. In these decompositions we obtained finitely many elements z_i and y_i. The z_i can be approximated by finite linear combinations of elements $E(\Delta)y_n$ as closely as we wish. So finally u can be approximated by such linear combinations as closely as we please.

b) If the operator A has eigenvalues let $\mathcal{H}(\lambda)$ denote the eigenspace corresponding to the eigenvalue λ. The projection $E(\{\lambda\})$ onto $\mathcal{H}(\lambda)$ is the strong limit of operators $E(\Delta)$, $\Delta \downarrow \lambda$; by Lemma 1 then T and T* commute with $E(\{\lambda\})$. Let Λ

be the set of eigenvalues of A (there are at most countably many) and let $\mathcal{H}(\Lambda)$ be the orthogonal direct sum of the eigenspaces $\mathcal{H}(\lambda)$. T and T^* commute also with the projection $E(\Lambda)$ onto $\mathcal{H}(\Lambda)$. The operator A has no eigenvalues in the complement $\mathcal{H}_1 = \mathcal{H} \ominus \mathcal{H}(\Lambda)$. From part a) we now see that we can find a generating system $e = e' \cup e''(\Lambda)$ in $\mathcal{H}$ where $e' \subset \mathcal{H}_1 \cap \mathfrak{D}(T)$ is a generating system in $\mathcal{H}_1$ for the restriction $A|_{\mathcal{H}_1}$ and $e''(\Lambda)$ is a union of orthonormal complete systems $\{e^{(n)}(\lambda)\}$ in $\mathcal{H}(\lambda)$ for each $\lambda \in \Lambda$ such that $\{e^{(n)}(\lambda)\} \subset \mathfrak{D}(T)$.

Remark 17'. — The same construction shows that for an arbitrary class of operators T_i, closed, with dense domains, commuting with A and such that $\bigcap_i \mathfrak{D}(T_i)$ is dense in $\mathcal{H}$, there exists a generating system $\{e_n\} \subset \bigcap_i \mathfrak{D}(T_i)$.

DEFINITION 18. — *For* $\lambda \in \Lambda$ *we define the operator* $T^{(\lambda)}$ *in* $\mathcal{H}(\lambda)$ *as the restriction of* T *to* $\mathcal{H}(\lambda)$. *Clearly* $T^{(\lambda)}$ *is closed with dense domain and it is bounded if* T *is bounded. Consider now a generating system* $e^* = \{e_n^*\} \subset \mathfrak{D}(T^*)$. *For every*

$$\lambda \notin \Lambda \cup \bigcup_n \Lambda_{T^*e_n^*,\,(e^*)}$$

we define the operator $T_{e^*}^{(\lambda)}$ *in* $\mathcal{H}_{e^*}^{(\lambda)}$ *as follows:*

$u^{(\lambda)} \in \mathfrak{D}(T_{e^*}^{(\lambda)})$ *if there exists* $u \in \mathfrak{D}_{e^*}^{(\lambda)} \cap \mathfrak{D}(T)$ *such that* $P_{e^*}^{(\lambda)}u = u^{(\lambda)}$ *and* $Tu \in \mathfrak{D}_{e^*}^{(\lambda)}$; *we set* $T_{e^*}^{(\lambda)}u^{(\lambda)} = P_{e^*}^{(\lambda)}Tu$.

Then we have $u^{(\lambda)} = \Sigma\xi_n(u;\lambda)e_n^*(\lambda)$ and $T_{e^*}^{(\lambda)}u^{(\lambda)} = \Sigma\xi_n(Tu;\lambda)e_n^*(\lambda)$. Consequently

$$(22)\quad \xi_n(Tu;\lambda)\,\theta_n^*(\lambda) = \frac{d(E(\lambda)Tu,\,E(\lambda)e_n^*)}{d\mu(\lambda)} = \frac{d(E(\lambda)u,\,E(\lambda)T^*e_n^*)}{d\mu(\lambda)} = \sum_k \xi_k(u;\lambda)\,\overline{\xi_k(T^*e_n^*;\lambda)}\,\theta_k^*(\lambda),$$

and the operator $T_{e^*}^{(\lambda)}$ is realized by the assignment

$$(23)\qquad \{\xi_n\} \to \left\{\frac{1}{\theta_n^*(\lambda)} \sum_k \xi_k\overline{\xi_k(T^*e_n^*;\lambda)}\,\theta_k^*(\lambda)\right\}.$$

From (22) we see that u, $v \in \mathfrak{D}_{e^*}^{(\lambda)} \cap \mathfrak{D}(T)$, Tu and $Tv \in \mathfrak{D}_{e^*}^{(\lambda)}$ and $P_{e^*}^{(\lambda)}u = P_{e^*}^{(\lambda)}v$ imply $P_{e^*}^{(\lambda)}Tu = P_{e^*}^{(\lambda)}Tv$; hence the operator $T_{e^*}^{(\lambda)}$ is well defined.

THEOREM 19. — *Let* T *be a closed operator in* $\mathcal{H}$ *with dense domain commuting with* E(.) *Further, let* e *and* e^* *be generating systems contained in* $\mathfrak{D}(\mathrm{T})$ *and* $\mathfrak{D}(\mathrm{T}^*)$ *respectively.*

a) $\mathrm{T}_{e^*}^{(\lambda)}$ *is a closed operator in* $\mathcal{H}_{e^*}^{(\lambda)}$ *for* $\lambda \notin \bigcup_n \Lambda_{\mathrm{T}^*e_n^*, (e^*)}$.

b) $\mathrm{T}_{e^*}^{(\lambda)}$ *has a dense domain in* $\mathcal{H}_{e^*}^{(\lambda)}$ *for*

$$\lambda \notin \left(\bigcup_n \Lambda_{\mathrm{T}^*e_n^*, (e^*)}\right) \cup \left(\bigcup_n \Lambda_{\mathrm{T}e_n, (e^*)}\right) \cup (\Lambda_{e, e^*}).$$

c) We have the following relation between the bounds (finite or infinite) of T *and* $\mathrm{T}_{e^*}^{(\lambda)}$:

$$||\mathrm{T}|| = \sup ||\mathrm{T}_{e^*}^{(\lambda)}|| \qquad \textit{for} \qquad \lambda \notin \bigcup_n \Lambda_{\mathrm{T}^*e_n^*, (e^*)}.$$

Proof. — Let $\{u_i^{(\lambda)}\} \subset \mathfrak{D}(\mathrm{T}_{e^*}^{(\lambda)})$ and suppose $u_i^{(\lambda)} \to u^{(\lambda)}$, $\mathrm{T}_{e^*}^{(\lambda)} u_i^{(\lambda)} \to w^{(\lambda)}$ in $\mathcal{H}_{e^*}^{(\lambda)}$. We prove $u^{(\lambda)} \in \mathfrak{D}(\mathrm{T}_{e^*}^{(\lambda)})$ and $\mathrm{T}_{e^*}^{(\lambda)} u^{(\lambda)} = w^{(\lambda)}$. We choose a subsequence of $\{u_i^{(\lambda)}\}$ (again denoted by the same letters) such that for $v_i^{(\lambda)} = u_i^{(\lambda)} - u_{i-1}^{(\lambda)}$ (we put $u_0^{(\lambda)} = 0$)

$$(24) \qquad \Sigma ||v_i^{(\lambda)}|| < \infty \qquad \textit{and} \qquad \Sigma ||\mathrm{T}_{e^*}^{(\lambda)} v_i^{(\lambda)}|| < \infty$$

hold. We have $u^{(\lambda)} = \Sigma v_i^{(\lambda)}$ and $w^{(\lambda)} = \Sigma \mathrm{T}_{e^*}^{(\lambda)} v_i^{(\lambda)}$ in $\mathcal{H}_{e^*}^{(\lambda)}$. For every i choose a $v_i \in \mathfrak{D}(\mathrm{T}) \cap \mathfrak{D}_{e^*}^{(\lambda)}$ such that $\mathrm{P}_{e^*}^{(\lambda)} v_i = v_i^{(\lambda)}$, $\mathrm{P}_{e^*}^{(\lambda)} \mathrm{T} v_i = \mathrm{T}_{e^*}^{(\lambda)} v_i^{(\lambda)}$. Then we choose a decreasing sequence of intervals Δ_i so small that for any Δ with $\lambda \in \Delta \subset \Delta_i$

$$(25) \quad \begin{cases} \dfrac{1}{2} ||v_i^{(\lambda)}||^2 \mu(\Delta) < ||\mathrm{E}(\Delta) v_i||^2 < 2(||v_i^{(\lambda)}|| + 2^{-i})^2 \mu(\Delta) \\ \textit{and} \\ \dfrac{1}{2} ||\mathrm{T}_{e^*}^{(\lambda)} v_i^{(\lambda)}||^2 \mu(\Delta) < ||\mathrm{E}(\Delta) \mathrm{T} v_i||^2 < 2(||\mathrm{T}_{e^*}^{(\lambda)} v_i^{(\lambda)}|| + 2^{-i})^2 \mu(\Delta), \\ \dfrac{1}{2} \theta_i^*(\lambda) \mu(\Delta) < ||\mathrm{E}(\Delta) e_i^*||^2 < 2\theta_i^*(\lambda) \mu(\Delta). \end{cases}$$

From (24) and (25) we obtain elements $u = \lim_{\mathrm{N}} \sum_1^{\mathrm{N}} \mathrm{E}(\Delta_i) v_i$ and $w = \lim_{\mathrm{N}} \sum_1^{\mathrm{N}} \mathrm{E}(\Delta_i) \mathrm{T} v_i$ in $\mathcal{H}$ and by the closedness of T then $u \in \mathfrak{D}(\mathrm{T})$ and $w = \mathrm{T}u$.

It remains to show that u, $w \in \mathfrak{D}_{e^*}^{(\lambda)}$ and $\mathrm{P}_{e^*}^{(\lambda)} u = u^{(\lambda)}$, $\mathrm{P}_{e^*}^{(\lambda)} w = w^{(\lambda)}$. We use a procedure similar to the one in the proof of Theorem 5. Let $v_i^{(\lambda)}$ and $u^{(\lambda)}$ have components $\xi_{n,i}$ and ξ_n, respectively. Then $\xi_n = \sum_i \xi_{n,i}$. We consider

$$(26) \qquad \frac{(\mathrm{E}(\Delta)u, \mathrm{E}(\Delta)e_n)}{\mu(\Delta)} = \sum_i \frac{(\mathrm{E}(\Delta)\mathrm{E}(\Delta_i)\varphi_i, \mathrm{E}(\Delta)e_n)}{\mu(\Delta)}.$$

Fix $\mathrm{N} > n$. The first N terms in (26) will approximate, for $\Delta \downarrow \lambda$, $\theta_n^*(\lambda) \sum_{i=1}^{\mathrm{N}} \xi_{n,i}$. We estimate the remainder (and consider right-hand and left hand derivatives separately, so

$$\Delta = [\lambda, \lambda'[\quad \text{or} \quad \Delta =]\lambda'', \lambda]).$$

Let $i'(\Delta)$ be the largest index for which $\Delta \subset \Delta_i^+$ (or $\Delta \subset \Delta_i^-$). Take Δ so small that $i'(\Delta) > \mathrm{N}$. Then by (25),

$$\sum_{i=\mathrm{N}+1}^{i'(\Delta)} \frac{(\mathrm{E}(\Delta)\varphi_i, \mathrm{E}(\Delta)e_n)}{\mu(\Delta)} + \sum_{i=i'(\Delta)+1}^{\infty} \frac{(\mathrm{E}(\Delta_i)\varphi_i, \mathrm{E}(\Delta)e_n)}{\sqrt{\mu(\Delta_i)}\sqrt{\mu(\Delta)}} \frac{\sqrt{\mu(\Delta_i)}}{\sqrt{\mu(\Delta)}}$$

is majorated by $2\sqrt{\theta_n^*(\lambda)} \sum_{\mathrm{N}+1}^{\infty} (\|\varphi_i^{(\lambda)}\| + 2^{-i})$. Thus we have found $d(\mathrm{E}(\lambda)u, \mathrm{E}(\lambda)e_n)/d\mu(\lambda) = \xi_n\theta_n^*(\lambda)$. To show that $d\|\mathrm{E}(\lambda)u\|^2/d\mu(\lambda)$ exists and equals $\Sigma|\xi_n|^2\theta_n(\lambda)$ we decompose

$$(27) \qquad \frac{\|\mathrm{E}(\Delta)\Sigma\mathrm{E}(\Delta_i)\varphi_i\|^2}{\mu(\Delta)} = \frac{1}{\mu(\Delta)}\left\|\sum_1^{\mathrm{N}} \cdots\right\|^2 + \frac{1}{\mu(\Delta)}\left\{\left\|\sum_{\mathrm{N}+1}^{\infty} \cdots\right\|^2 + \left(\sum_1^{\mathrm{N}}, \sum_{n+1}^{\infty}\right) + \left(\sum_{\mathrm{N}+1}^{\infty}, \sum_1^{\mathrm{N}}\right)\right\}.$$

The first term on the right side of (27) converges, for $\Delta \downarrow \lambda$, to $\left\|\sum_1^{\mathrm{N}} \varphi_i^{(\lambda)}\right\|^2$. The second term, the remainder, is evaluated for $\Delta \subset \Delta_{\mathrm{N}+1}$ by the same procedure as before:

$$\begin{aligned} \frac{1}{\mu(\Delta)}\{\cdots\} &\leqq \frac{1}{\mu(\Delta)}\left\|\sum_{\mathrm{N}+1}^{\infty} \mathrm{E}(\Delta)\mathrm{E}(\Delta_i)\varphi_i\right\|^2 \\ &\quad + 2\left[\frac{1}{\mu(\Delta)}\left\|\sum_1^{\mathrm{N}} \mathrm{E}(\Delta)\mathrm{E}(\Delta_i)\varphi_i\right\|^2\right]^{1/2}\left[\frac{1}{\mu(\Delta)}\left\|\sum_{\mathrm{N}+1}^{\infty} \mathrm{E}(\Delta)\mathrm{E}(\Delta_i)\varphi_i\right\|^2\right]^{1/2} \\ &\leqslant 2\left(\sum_{\mathrm{N}+1}^{\infty} (\|\varphi_i^{(\lambda)}\| + 2^{-i})\right)^2 \\ &\quad + 4\left(\sum_1^{\infty} (\|\varphi_i^{(\lambda)}\| + 2^{-i})\right)\left(\sum_{\mathrm{N}+1}^{\infty} (\|\varphi_i^{(\lambda)}\| + 2^{-i})\right). \end{aligned}$$

Therefore $\dfrac{d\|\mathrm{E}(\lambda)u\|^2}{d\mu(\lambda)} = \|u^{(\lambda)}\|^2$.

The same proof is repeated for the Tv_i to find $w \in \mathfrak{D}_{e^*}^{(\lambda)}$ and $P_{e^*}^{(\lambda)} w = w^{(\lambda)}$.

b) Under the present condition the identification of $\mathcal{H}_e^{(\lambda)}$ with $\mathcal{H}_{e^*}^{(\lambda)}$ is valid and $e_n(\lambda) \in \mathfrak{D}(T_{e^*}^{(\lambda)})$. Hence $\mathfrak{D}(T_{e^*}^{(\lambda)})$ is dense in $\mathcal{H}_{e^*}^{(\lambda)}$.

c) If $u \in \mathfrak{D}(T)$, $u \in \mathfrak{D}_{e^*}^{(\lambda)}$, $Tu \in \mathfrak{D}_{e^*}^{(\lambda)}$ then

$$\frac{||E(\Delta)Tu||^2}{\mu(\Delta)} \leqq ||T||^2 \frac{||E(\Delta)u||^2}{\mu(\Delta)},$$

hence, by taking $\Delta \downarrow \lambda$, we get $||T_{e^*}^{(\lambda)} u^{(\lambda)}|| \leqq ||T|| \, ||u^{(\lambda)}||$. On the other hand, for $u \in \mathfrak{D}(T)$ we have

$$||Tu||^2 = \int ||T_{e^*}^{(\lambda)} u^{(\lambda)}||^2 \, d\mu(\lambda) \leqq \sup_\lambda ||T_{e^*}^{(\lambda)}||^2 \int ||u^{(\lambda)}||^2 \, d\mu(\lambda)$$
$$= \sup_\lambda ||T_e^{(\lambda)}||^2 \, ||u^2||$$

which finishes the proof.

CHAPTER II

PROPER FUNCTIONAL HILBERT SPACES

II.1. A-Expansibility.

Let $\{\mathcal{F}, \mathcal{E}\}$ be a proper functional Hilbert space, that is, a Hilbert space $\mathcal{F}$ consisting of functions f defined everywhere on a basic set $\mathcal{E}$ such that the evaluation $f(x)$ for every fixed $x \in \mathcal{E}$ represents a bounded linear functional on $\mathcal{F}$. The reproducing kernel of $\{\mathcal{F}, \mathcal{E}\}$ is the positive matrix $K(x, y) = K_y(x)$ where every $K_y \in \mathcal{F}$ and x and y run through $\mathcal{E}$. The proper functional Hilbert space $\{\mathcal{F}, \mathcal{E}\}$ is determined by its reproducing kernel (7). The evaluation is given by the reproducing formula

$$(1) \qquad f(x) = (f, K_x) \quad \textit{for} \quad f \in \mathcal{F} \quad \textit{and} \quad x \in \mathcal{E}.$$

The elements $K_x \in \mathcal{F}$ where $x \in \mathcal{E}$ form a total subset $\mathcal{E}_{\mathcal{F}}$ in the Hilbert space $\mathcal{F}$. When there is no danger of confusion we shall sometimes write $\mathcal{F}$ for $\{\mathcal{F}, \mathcal{E}\}$ and $\mathcal{E}$ for $\mathcal{E}_{\mathcal{F}}$.

If in an abstract Hilbert space $\mathcal{H}$ a total subset $\mathfrak{U}$ is chosen then we may take $\mathfrak{U}$ as a basic set and consider the elements of $\mathcal{H}$ as functions on $\mathfrak{U}$:

$$(1') \qquad h(u) = (h, u) \quad \textit{for} \quad h \in \mathcal{H} \quad \textit{and} \quad u \in \mathfrak{U}$$

and thus obtain the proper functional Hilbert space $\{\mathcal{H}, \mathfrak{U}\}$ where $u = K_u$, $K(u, v) = (v, u)$ for all $u, v \in \mathfrak{U}$ and $\mathfrak{U}_{\mathcal{H}} = \mathfrak{U}$.

Consider a selfadjoint operator A in $\{\mathcal{F}, \mathcal{E}\}$ with resolution of identity $E(.)$ and a spectral measure μ. In analogy to sec-

(7) For the general theory of reproducing kernels and proper functional Hilbert spaces, cf. N. Aronszajn [1].

tions I.1 and I.2 we now consider the set $\mathcal{E}_{\mathcal{F}}$ instead of a generating system $\{e_n\}$.

Definition 1. — *Let* $\Lambda_{\mathcal{E}}$ *be the smallest subset of* R_1 *such that for every* $\lambda \notin \Lambda_{\mathcal{E}}$

$$(2) \qquad \frac{d(E(\lambda)K_y, E(\lambda)K_x)}{d\mu(\lambda)} \overset{\text{def}}{=\!=} K(x, y; \lambda)$$

exists and is finite for all $x, y \in \mathcal{E}$.

Theorem 2. — *For fixed* $\lambda \notin \Lambda_{\mathcal{E}}$, *the function* $K(x, y; \lambda)$ *is a positive matrix and thus defines a proper functional Hilbert space* $\mathcal{F}_{\mathcal{E}}^{(\lambda)}$ *on* $\mathcal{E}$.

Proof. — Let $x_1, \ldots, x_n \in \mathcal{E}$ and the constants $c_1, \ldots, c_n$ be arbitrary. Then

$$(3) \quad \sum_{i,j}^{1\ldots n} c_i\bar{c}_j K(x_j, x_i; \lambda)$$
$$= \lim_{\Delta \downarrow \lambda} \frac{1}{\mu(\Delta)} \sum_{i,j} c_i\bar{c}_j (E(\Delta)K_{x_i}, E(\Delta)K_{x_j}) \geqq 0,$$

since for any fixed $\Delta \ni \lambda$, the function

$$(E(\Delta)K_y, E(\Delta)K_x) = (E(\Delta)K_y)(x)$$

is the reproducing kernel for the space $E(\Delta)\mathcal{F}$ and hence a positive matrix.

Definition 3. — *For every* $f \in \mathcal{F}$ *we introduce* $\Lambda_{f,(\mathcal{E})}$ *as the smallest set containing* $\Lambda_{\mathcal{E}}$ *such that for* $\lambda \notin \Lambda_{f,(\mathcal{E})}$

$$(4) \quad \left\{ \begin{array}{l} \dfrac{d(E(\lambda)f, E(\lambda)K_x)}{d\mu(\lambda)} \overset{\text{def}}{=\!=} f(x; \lambda) \text{ exists, is finite, and equals} \\ \text{zero if } K(x, x; \lambda) = 0 \text{ for every } x \in \mathcal{E}, \end{array} \right.$$

and

$$(5) \quad f(\,.\,; \lambda) \in \mathcal{F}_{\mathcal{E}}^{(\lambda)}, \quad \frac{d\|E(\lambda)f\|^2}{d\mu(\lambda)} \text{ exists and equals } \quad \|f(\,.\,; \lambda)\|^2_{\mathcal{F}_{\mathcal{E}}^{(\lambda)}}.$$

Then we define the mapping $P_{\mathcal{E}}^{(\lambda)} : \mathcal{F} \to \mathcal{F}_{\mathcal{E}}^{(\lambda)}$ by

$$(P_{\mathcal{E}}^{(\lambda)} f)(x) = f(x; \lambda)$$

where f belongs to the domain $\mathcal{D}_{\mathcal{E}}^{(\lambda)}$ of $P_{\mathcal{E}}^{(\lambda)}$ if $\lambda \notin \Lambda_{f,(\mathcal{E})}$.

It would be of interest to study $P_{\mathcal{E}}^{(\lambda)}$, $\mathfrak{D}_{\mathcal{E}}^{(\lambda)}$ and $\mathcal{F}_{\mathcal{E}}^{(\lambda)}$ directly, but we shall not do this in the present paper. It would be of interest only in case $\mu(\Lambda_{\mathcal{E}})=0$ and $\mu(\Lambda_{f,(\mathcal{E})})=0$ for all $f \in \mathcal{F}$.

From now on we consider only separable proper functional Hilbert spaces. If the set $\mathcal{E}_{\mathcal{F}}$ is A-expansible in $\mathcal{F}$ then

$$\Lambda_{\mathcal{E}} \subset \bigcup_{x \in \mathcal{E}} \Lambda_{K_x,(e)} \stackrel{\text{def}}{=} \Lambda_{\mathcal{E},(e)}$$

for any generating system e and consequently $\mu(\Lambda_{\mathcal{E}}) = 0$.

Definition 4. — *The proper functional Hilbert space $\{\mathcal{F}, \mathcal{E}\}$ is called* A*-expansible, totally expansible, or* H.S.*-expansible, respectively, if the set $\mathcal{E}_{\mathcal{F}}$ is of that type in .$\mathcal{F}$*

Theorem 5. — *Let $\{\mathcal{F}, \mathcal{E}\}$ be* A*-expansible. Let e be a generating system obtained from some sequence $\{x_k\} \subset \mathcal{E}_{\mathcal{F}}$ which is total in $\mathcal{F}$, and let e' be another generating system. Then for every $\lambda \notin \Lambda_{(e',e)} \cup \Lambda_{\mathcal{E},(e)}$ the set $P_{e'}^{(\lambda)}\mathcal{E}_{\mathcal{F}}$ is total in $\mathcal{F}_{e'}^{(\lambda)}$, and the assignment*

$$(6) \qquad J_{\mathcal{E},e'}^{(\lambda)}(P_{e'}^{(\lambda)}K_y) = K(.,y;\lambda) \qquad \text{for} \qquad y \in \mathcal{E}$$

defines an isometric isomorphism of $\mathcal{F}_{e'}^{(\lambda)}$ onto $\mathcal{F}_{\mathcal{E}}^{(\lambda)}$. If, in particular, $e' = e$ then $\mathcal{F}_{\mathcal{E}}^{(\lambda)}$ and $\mathcal{F}_{e}^{(\lambda)}$ are isometrically isomorphic for every $\lambda \notin \Lambda_{\mathcal{E},(e)}$.

Proof. — $P_{e'}^{(\lambda)}\mathcal{E}_{\mathcal{F}}$ is total in $\mathcal{F}_{e'}^{(\lambda)}$ by Theorem 8.I. The functions $K(.;y;\lambda)$ where $y \in \mathcal{E}$ form a total subset in $\mathcal{F}_{\mathcal{E}}^{(\lambda)}$. By the assumptions on λ we have for (6):

$$\begin{aligned}(P_{e'}^{(\lambda)}K_y, P_{e'}^{(\lambda)}K_x)_{\mathcal{F}_{e'}^{(\lambda)}} &= \sum_n \xi'_n(K_y;\lambda)\bar{\xi}'_n(K_x;\lambda)\theta'_n(\lambda)\\ &= \frac{d(E(\lambda)K_y, E(\lambda)K_x)}{d\mu(x)} = K(x,y;\lambda)\\ &= (K(.,y;\lambda), K(.,x;\lambda))_{\mathcal{F}_{\mathcal{E}}^{(\lambda)}}\\ &= (J_{\mathcal{E},e'}^{(\lambda)}P_{e'}^{(\lambda)}K_y, J_{\mathcal{E},e'}^{(\lambda)}P_{e'}^{(\lambda)}K_x)_{\mathcal{F}_{\mathcal{E}}^{(\lambda)}}\end{aligned}$$

for all $x, y \in \mathcal{E}$. That is, $J_{\mathcal{E},e'}^{(\lambda)}$ maps the total subset $P_{e'}^{(\lambda)}\mathcal{E}_{\mathcal{F}}$ onto the total subset $\{K(.,y;\lambda) | y \in \mathcal{E}\}$ and leaves the scalar product invariant on them. Consequently it is an isometric isomorphism onto.

Remark. — In view of Theorem 5 and the exposition in Chapter I we can now say that the direct integral $\int \mathcal{F}_{\mathcal{E}}^{(\lambda)}\, d\mu(\lambda)$

is canonically determined and canonically ismorphic to $\mathcal{F}$. It consists of all functions $\varphi(x;\lambda)$ which for μ-almost all λ belong to $\mathcal{F}_{\mathcal{E}}^{(\lambda)}$ (this implies $\varphi(x;\lambda)=0$ for $K(x,x;\lambda)=0$), for which $\varphi(x;\lambda)$ for all $x\in\mathcal{E}$ and $\|\varphi(.\,;\lambda)\|^2_{\mathcal{F}_{\mathcal{E}}^{(\lambda)}}$ are μ-measurable, and for which $\int\|\varphi(.\,;\lambda)\|^2_{\mathcal{F}_{\mathcal{E}}^{(\lambda)}}\,d\mu(\lambda)<\infty$. The corresponding function $\varphi\in\mathcal{F}$ is given for every $x\in\mathcal{E}$ by

$$\varphi(x)=\int\varphi(x;\lambda)\,d\mu(\lambda).$$

Corollary 6. — *The elements of $\mathcal{F}_{\mathcal{E}}^{(\lambda)}$ are generalized eigenfunctions of the operator* A, *corresponding to the eigenvalue* λ. *For a function* F(A) *and every* $\lambda\notin\Lambda_{\mathcal{E},(e)}\cup\Lambda_{u,(e)}\cup\Lambda_{F(A)u,(e)}$ *one has*

$$\text{(7)}\quad \frac{d(E(\lambda)F(A)u)(x)}{d\mu(\lambda)}=(F(A)u)(x;\lambda)=F(\lambda)u(x;\lambda).$$

II.2. Hilbert-Subspaces. H.S.-Expansibility.

We recall that a subspace $\mathcal{B}_1$ of a Banach space $\mathcal{B}$ is called a Banach-subspace of $\mathcal{B}$ if $\mathcal{B}_1$ is a Banach space satisfying

$$\text{(8)}\quad \|u\|_{\mathcal{B}_1}\geq c\|u\|_{\mathcal{B}}\ \textit{for some constant } c>0 \textit{ and all } u\in\mathcal{B}_1.$$

If in a Hilbert space $\mathcal{F}$ a Banach subspace $\mathcal{G}$ with its norm $\|\ \|_{\mathcal{G}}$ is a Hilbert space then it is called a *Hilbert-subspace* of $\mathcal{F}$.

Let $\mathcal{G}$ be a Hilbert-subspace of a Hilbert space $\mathcal{F}$. Then there is a selfadjoint operator H, $0\leq H\leq cI$ with some $c>0$, mapping $\mathcal{F}$ into $\mathcal{G}$ which is defined by

$$(g,h)_{\mathcal{F}}=(g,Hh)_{\mathcal{G}}\quad \textit{for every}\quad g\in\mathcal{G}.$$

Then $H^{\frac{1}{2}}$ is an isometric isomorphism of $\mathcal{F}\ominus\mathcal{N}(H)$ onto $\mathcal{G}$: $\|h\|_{\mathcal{F}}=\left\|H^{\frac{1}{2}}h\right\|_{\mathcal{G}}$ $\left(\mathcal{N}(H)\text{ is the nullspace of H as well as of } H^{\frac{1}{2}}\right)$. If furthermore $\{\mathcal{F},\mathcal{E}\}$ is a p.f. Hilbert space with reproducing kernel $K_y(y\in\mathcal{E})$ then also $\{\mathcal{G},\mathcal{E}\}$ is a p.f. Hilbert space with reproducing kernel $L_y=HK_y$; consequently $\mathcal{E}_{\mathcal{G}}=H\mathcal{E}_{\mathcal{F}}$. Conversely, if $\{\mathcal{F},\mathcal{E}\}$ and $\{\mathcal{G},\mathcal{E}\}$ are p.f. Hilbert spaces and if $\mathcal{G}\subset\mathcal{F}$

then this embedding is continuous, and consequently $\mathcal{G}$ is a Hilbert-subspace of $\mathcal{F}$.

PROPOSITION 7. — *Let $\{\mathcal{F}, \mathcal{E}\}$ be a p.f. Hilbert space and let $\mathcal{G}$ be a Hilbert-subspace of $\mathcal{F}$. If $\{\mathcal{F}, \mathcal{E}\}$ is* H.S.*-expansible, then so is the p.f. Hilbert space $\{\mathcal{G}, \mathcal{E}\}$.*

Proof. — By hypothesis there exist $\{u_k\} \subset \mathcal{F}$ such that $\sum_k \|u_k\|_{\mathcal{F}}^2 < \infty$ and $K_y = \sum_k \zeta_k(y) u_k$ with $\{\zeta_k(y)\} \in l^2$ for all K_y, $y \in \mathcal{E}$. Consequently $L_y = \sum_k \zeta_k(y) H u_k$ for all $y \in \mathcal{E}$, and $\sum_k \|H u_k\|_{\mathcal{G}}^2 = \sum_k \|H^{\frac{1}{2}} u_k\|_{\mathcal{F}}^2 \leq c \sum_k \|u_k\|_{\mathcal{F}}^2 < \infty$. Thus also $\mathcal{G}$ is H.S.-expansible.

Let $\mathcal{E}'$ be a subset of the basic set $\mathcal{E}$ of a p.f. Hilbert space $\{\mathcal{F}, \mathcal{E}\}$. Then we denote by $\mathcal{F}|_{\mathcal{E}'}$ the space of functions on $\mathcal{E}'$ which are restrictions to $\mathcal{E}'$ of functions in $\mathcal{F}$. The restriction norm in $\mathcal{F}|_{\mathcal{E}'}$ is defined by

$$\|g\|_{\mathcal{E}'} = \min_{f \in \mathcal{F}, f|_{\mathcal{E}'} = g} \|f\|.$$

$\{\mathcal{F}|_{\mathcal{E}'}, \mathcal{E}'\}$ is a p.f. Hilbert space with this norm; its reproducing kernel is the restriction of $K(x, y)$ to $\mathcal{E}' \times \mathcal{E}'$. It is isometrically isomorphic to the subspace $\mathcal{F} \ominus \mathfrak{N}(\mathcal{E}')$ where $\mathfrak{N}(\mathcal{E}')$ consists of all functions $f \in \mathcal{F}$ which vanish identically on $\mathcal{E}'$. Furthermore $K_x \perp \mathfrak{N}(\mathcal{E}')$ for every $x \in \mathcal{E}'$.

THEOREM 8. — *In the p.f. Hilbert space $\{\mathcal{F}, \mathcal{E}\}$ let $\mathcal{E} = \bigcup_{n=1} \mathcal{E}_n$ and let every $\mathcal{F}|_{\mathcal{E}_n}$ be contained in a p.f. Hilbert space $\{\mathcal{F}_n, \mathcal{E}_n\}$. If every $\mathcal{F}_n$ is* H.S.*-expansible then $\mathcal{F}$ is* H. S.*-expansible.*

Proof. — By hypothesis every space of restrictions $\mathcal{F}|_{\mathcal{E}_n}$ is a Hilbert-subspace of the H.S.-expansible $\mathcal{F}_n$, hence by Proposition 7 it is H.S.-expansible. So there are $u_{nk} \in \mathcal{F}|_{\mathcal{E}_n}$ such that $\sum_k \|u_{nk}\|_{\mathcal{E}_n}^2 < \infty$ and $K_x|_{\mathcal{E}_n} = \sum \zeta_k^{(n)}(x) u_{nk}$, $\{\zeta_k^{(n)}(x)\} \in l^2$ for all $x \in \mathcal{E}_n$. Let $u_k^{(n)} \in \mathcal{F} \ominus \mathfrak{N}(\mathcal{E}_n)$ correspond to $u_{nk} \in \mathcal{F}|_{\mathcal{E}_n}$ under the isometric isomorphism between the two spaces. Then, since $K_x \perp \mathfrak{N}(\mathcal{E}_n)$ for all $x \in \mathcal{E}_n$, we find $K_x = \sum \zeta_k^{(n)}(x) u_k^{(n)}$, $\{\zeta_k^{(n)}(x)\} \in l^2$ in $\mathcal{F}$ for all $x \in \mathcal{E}_n$ where $\sum_k \|u_k^{(n)}\|^2 < \infty$, and hence $\left\{\mathcal{F}, \bigcup_{n=1}^{\infty} \mathcal{E}_n\right\}$ is H.S.-expansible.

CHAPTER III

EXAMPLE: SPACES OF ANALYTIC FUNCTIONS

We consider proper functional Hilbert spaces $\{\mathcal{F}, \mathcal{E}\}$ where the basic set $\mathcal{E} = \mathrm{D}$ is a domain in complex space C^n and the functions $f \in \mathcal{F}$ are analytic in D. Two particular classes of such spaces (8) are

1°) Spaces with Bergman's kernel $\mathrm{K}_{\mathrm{B}}(z, \zeta)$ as reproducing kernel; $\mathcal{F}$ is a subspace of $\mathrm{L}^2(\mathrm{D})$:

$$(1) \quad \|f\|_{\mathrm{B}}^2 = \int_{\mathrm{D}} |f(z)|^2 dx_1\, dy_1 \ldots dx_n\, dy_n < \infty \quad \textit{for} \quad f \in \mathcal{F}.$$

We shall denote these spaces by $\mathcal{F}_{\mathrm{B}}(\mathrm{D})$.

2°) Spaces with Szegö's kernel $\mathrm{K}_{\mathrm{S}}(z, \zeta)$ as reproducing kernel; $\mathcal{F}|_{\partial \mathrm{D}}$ is a subspace of $\mathrm{L}^2(\partial \mathrm{D})$:

$$(2) \qquad \|f\|_{\mathrm{S}}^2 = \int_{\partial \mathrm{D}} |f(z)|^2\, ds < \infty \quad \textit{for} \quad f \in \mathcal{F}.$$

(These spaces are usually considered only for one complex variable and for domains D with rather « nice » boundaries ∂D).

Theorem 1. — *Let* D *be the polycylinder*

$$\{|z_i| < r_i;\ i = 1, \quad \ldots, \quad n\}.$$

Then the space $\mathcal{F}_{\mathrm{B}}(\mathrm{D})$ *is* H.S.*-expansible. More precisely, a Hilbert-Schmidt operator* T *can be constructed such that*

$$\{\mathrm{K}_{\mathrm{B}}(\,.\,, \zeta) | \zeta \in \mathrm{D}\} \subset \mathrm{T}^k \mathcal{F}_{\mathrm{B}}(\mathrm{D})$$

for every k. The space $\mathrm{T}\mathcal{F}_{\mathrm{B}}(\mathrm{D})$ *(with its own norm) which guarantees the* H.S.*-expansibility of* $\mathcal{F}_{\mathrm{B}}(\mathrm{D})$ *will be denoted by* $\Phi_{\mathrm{B}}(\mathrm{D})$.

(8) Cf. N. Aronszajn [1], and for an expository treatment, M. Meschkowski [14].

Proof. — The monomials $z_1^{m_1-1} z_2^{m_2-1} \ldots z_n^{m_n-1}$,

$$m_i = 1, 2, 3, \ldots, i = 1, \ldots, n$$

are orthogonal in $\mathcal{F} = \mathcal{F}_B(D)$. We normalize them and obtain a complete orthonormal system $\{\varphi_{m_1, \ldots, m_n}\}$. Then we set

$$(3) \qquad T\varphi_{m_1, \ldots, m_n} = \frac{1}{m_1 \cdot m_2 \cdot m_n} \varphi_{m_1, \ldots, m_n} = u_{m_1, \ldots, m_n}$$

which defines a Hilbert-Schmidt operator T since

$$\sum_{\substack{m_i=1,2,\ldots \\ i=1,\ldots,n}} \|T\varphi_{m_1, \ldots, m_n}\|^2 = \sum_{\substack{m_i=1,2,\ldots \\ i=1,\ldots,n}} \frac{1}{m_1^2 m_2^2 \ldots m_n^2} < \infty.$$

For $\zeta \in D$ we have $K_\zeta = \Sigma \overline{\varphi_{m_1, \ldots, m_n}(\zeta)}\, \varphi_{m_1, \ldots, m_n}$ in $\mathcal{F}$. Now we find

$$(4) \quad \Sigma |m_1^k m_2^k \ldots m_n^k \varphi_{m_1, \ldots, m_n}(\zeta)|^2 < \infty \textit{ for every } k \textit{ and all } \zeta \in D$$

so that $D_{\mathcal{F}} \subset T^k \mathcal{F}_B(D)$. We have $\Phi_B(D) = \Phi_2(\{u_{m_1, \ldots, m_n}\})$.

Remark. — In the same way one proves H.S.-expansibility of $\mathcal{F}_B(D)$ if D is the polycylinder $D = \{|z_i - a_i| < r_i;\, i = 1, \ldots, n\}$ with center $a = \{a_i\}$.

Corollary 2. — *Let* D *be an arbitrary domain and* $\{\mathcal{F}, D\}$ *any p.f. Hilbert space of functions analytic in* D. *Then* $\{\mathcal{F}, D\}$ *is* H.S.*-expansible.*

Proof. — The domain D can be covered by countably many polycylinders D_k such that $D = \bigcup_{k=1}^{\infty} D_k$ and $\overline{D}_k \subset D$ for all k. Then the space $\mathcal{F}|_{D_k}$ of restrictions of functions in $\mathcal{F}$ to D_k consists of functions which are actually analytic in some domain containing $\overline{D}_k$, so $\mathcal{F}|_{D_k}$ is a Hilbert-subspace of $\mathcal{F}_B(D_k)$ which is H.S.-expansible by Theorem 1. Let $H_k : \mathcal{F}_B(D_k) \to \mathcal{F}|_{D_k}$ be the operator which in the general case is described in section II.2, so that $H_k L_\zeta = K_\zeta$ for $\zeta \in D_k$ (where L denotes the reproducing kernel for $\mathcal{F}_B(D_k)$ and K the r.k. for $\mathcal{F}|_{D_k}$), and let $\Phi_B(D_k) = \Phi_2(\{u^{(k)}_{m_1, \ldots, m_n}\})$ as given by Theorem 1. If we define $\Psi(D_k) = \Phi_2(\{H_k u^{(k)}_{m_1, \ldots, m_n}\})$ in $\mathcal{F}|_{D_k}$ then $\Phi_B(D)_k$ and

$\Psi(D_k)$ are isometrically isomorphic under the operator H_k, and $\Psi(D_k)$ is H.S.-expansible in $\{\mathcal{F}, D\}$ (we recall that $\mathfrak{N}(D_k) = \{0\}$ since $\mathcal{F}$ consists of analytic functions).

THEOREM 3. — *Let* A *be an arbitrary selfadjoint operator in the space* $\{\mathcal{F}, D\}$. *Let* $\Psi(D_k)$ *be the Hilbert subspaces introduced in the proof of Corollary* 2, *and let* $\Lambda_{\Psi(D_k)}$ *be the corresponding exceptional sets* (*determined as in Theorem* 11.I). *Then the generalized eigenfunctions*

$$\frac{dE(\lambda)f(z)}{d\mu(\lambda)} = f(z;\lambda) \in \mathcal{F}_D^{(\lambda)} \quad for \quad \lambda \notin \bigcup_{k=1}^{\infty} \Lambda_{\Psi(D_k)}, \quad f \in \mathfrak{D}_D^{(\lambda)}$$

are analytic in the whole domain D.

Proof. — We have to show that the functions $f(z, \lambda)$ are analytic in each variable separately. That is

$$(5) \quad \lim_{h\to 0} \frac{1}{h} [f(z + h\varepsilon_i; \lambda) - f(z; \lambda)] = \lim_{h\to 0} \left(f(\,.\,;\lambda), \frac{1}{h}[K_{z+h\varepsilon_i}^{(\lambda)} - K_z^{(\lambda)}]\right)_{\mathcal{F}_D^{(\lambda)}}$$

(where the h are complex and ε_i denotes the unit vector in the z_i-plane) exists for every $f(\,.\,;\lambda) \in \mathcal{F}^{(\lambda)}$ and every $z \in D$. ($K_\zeta^{(\lambda)}$ denotes the function $K(.,\zeta;\lambda) = P_D^{(\lambda)}K_\zeta$ in $\mathcal{F}_D^{(\lambda)}$). This is equivalent to saying that for every fixed $\zeta \in D$ the elements $\frac{1}{h}(K_{\zeta+h\varepsilon_i}^{(\lambda)} - K_\zeta^{(\lambda)}) = K_{\zeta,h}^{(\lambda)}$ converge weakly in $\mathcal{F}_D^{(\lambda)}$. We shall actually show that they converge strongly.

Since $P_D^{(\lambda)}: \Psi(D_k) \to \mathcal{F}_D^{(\lambda)}$ is continuous for $\lambda \notin \Lambda_{\Psi(D_k)}$ and $\Psi(D_k)$ and $\Phi_B(D_k)$ are isometrically isomorphic under H_k, we only have to prove that $\frac{1}{h}(L_{\zeta+h\varepsilon_i} - L_\zeta) = L_{\zeta,h}$ converge strongly in the space $\Phi_B(D_k)$. Without loss of generality we merely consider the polycylinder $P = \{|z_i| < 1; i=1, \ldots, n\}$, the space $\mathcal{B} = \mathcal{F}_B(P)$ with Bergman's kernel $\overline{L_\zeta = K_B(.,\zeta)}$, and $\Phi = \Phi_B(P)$. We have $L_\zeta = \Sigma(m_1 \ldots m_n)^2 \overline{u_{m_1,\ldots,m_n}(\zeta)}u_{m_1,\ldots,m_n}$ (where $u_{m_1,\ldots,m_n}$ is defined by (3)) and

$$L_{\zeta,h} = \Sigma(m_1 \ldots m_n)^2 \left[\frac{\overline{u_{m_1,\ldots,m_n}(\zeta + h\varepsilon_i) - u_{m_1,\ldots,m_n}(\zeta)}}{h}\right] u_{m_1,\ldots,m_n}$$

for ζ and $\zeta + h\varepsilon_i$ belonging to P. A straightforward calculation shows that

$$\left\|\frac{\partial L_\zeta}{\partial \zeta_i}\right\|_\Phi^2 = \Sigma(m_1 \ldots m_n)^4 \left|\frac{\partial u_{m_1, \ldots, m_n}(\zeta)}{\partial \zeta_i}\right|^2 < \infty$$

and that

$$\lim_{h \to 0} \left\| L_{\zeta, h} - \frac{\partial L_\zeta}{\partial \zeta_i} \right\|_\Phi^2 = \lim_{h \to 0} \Sigma(m_1 \ldots m_n)^4$$
$$\left| \frac{u_{m_1, \ldots, m_n}(\zeta + h\varepsilon_i) - u_{m_1, \ldots, m_n}(\zeta)}{h} - \frac{\partial u_{m_1, \ldots, m_n}(\zeta)}{\partial \zeta_i} \right| = 0$$

which finishes the proof.

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