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MARTIN BOUNDARY AND POSITIVE SOLUTIONS OF SOME BOUNDARY VALUE PROBLEMS

By Evgeny B. DYNKIN

In 1941 R. S. Martin [1] proposed a method of describing all positive harmonic functions in an arbitrary domain D of an n-dimensional Euclidean space. Our aim is the investigation of all positive harmonic functions satisfying some boundary conditions. It will be proved that the Martin method is valid for a wide class of such boundary value problems. We will employ it for the Neumann problem with oblique derivative.

Let D be a two-dimensional domain bounded by a smooth closed contour C and let $v(z)$ be a Hölder continuous vector field on C. We shall investigate harmonic functions in D subject to the condition that

$$(1) \qquad \frac{\partial h}{\partial v} = 0$$

on C. (Here

$$\frac{\partial h}{\partial v}(z_0) = v_1(z_0)h_{x_1}(z_0) + v_2(z_0)h_{x_2}(z_0)$$

and $h_{x_i}(z_0)$ is the limit of

$$h_{x_i}(z) = \frac{\partial}{\partial x_i}h(z) \qquad \text{as} \qquad z \to z_0.)$$

The exact statement of the question is the following. We suppose that there exist only a finite number of points at which the vector field $v(z)$ is tangent to the contour C. We call a boundary point α exceptional if the projection of $v(z)$ on the exterior normal changes its sign at α. The set of all exceptional points will be designated by Γ. Harmonic functions h in the domain D satisfying condition (1) at all points $\alpha \in C \setminus \Gamma$ will be called *solutions of the boundary value problem* (1). (No restrictions will be imposed upon the behaviour of h as $z \to \alpha \in \Gamma$.) We want to describe all positive solutions of the boundary value problem (1).

The extension of the Martin theory to a class of boundary value problems (including the problem (1)) will be given in § 1. The Martin boundary associated with the problem (1) and all positive solutions of this problem will be described in § 2. The probabilistic interpretation of the results obtained and some unsolved problems will be formulated in § 3.

1. Boundary value problems for the Laplace equation and Martin boundary.

Let D be an arbitrary domain of an n-dimensional Euclidean space. We shall investigate harmonic functions in the domain D satisfying a certain boundary condition $\mathscr{R}$. The concept of the boundary condition is defined in the following way. A set $D \setminus \mathscr{K}$ ($\mathscr{K}$ is any compact contained in D) will be called a boundary neighborhood. Let us consider all the functions defined in the boundary neighborhoods. Let a set $\mathscr{R}$ of such functions satisfy the following conditions:

A. If f_1 and f_2 coincide in a boundary neighborhood and if $f_1 \in \mathscr{R}$, then $f_2 \in \mathscr{R}$.

B. If $f_1, f_2 \in \mathscr{R}$, then $c_1 f_1 + c_2 f_2 \in \mathscr{R}$ for any real numbers c_1, c_2.

C. If a sequence of harmonic functions f_n converges to a function f in a boundary neighborhood and if $f_n \in \mathscr{R} (n = 1, 2, \ldots)$, then $f \in \mathscr{R}$.

Then $\mathscr{R}$ defines a boundary condition. The phrase « *f satisfies the boundary condition $\mathscr{R}$* » has the same meaning as « $f \in \mathscr{R}$ ». Harmonic functions satisfying the boundary condition $\mathscr{R}$ will be called solutions of the boundary value problem $\mathscr{R}$.

Let us suppose that the set $\mathscr{R}$ contains a subset $\mathscr{R}_+$ subject to the following conditions:

D. If f_1 and f_2 coincide in a boundary neighborhood and if $f_1 \in \mathscr{R}_+$, then $f_2 \in \mathscr{R}_+$.

E. If $f_1, f_2 \in \mathscr{R}_+$, then $c_1 f_1 + c_2 f_2 \in \mathscr{R}_+$ for any nonnegative numbers c_1, c_2.

F. If $f_n \to f$ uniformly in a boundary neighborhood and if

$$f_n \in \mathscr{R}_+ (n = 1, 2, \ldots),$$

$f \in \mathscr{R}$, then $f \in \mathscr{R}_+$.

G. If $f \in \mathscr{R}$ and if $f \geqslant 0$ in a boundary neighborhood, then $f \in \mathscr{R}_+$.

H. (Minimum principle) Let $\mathscr{K}$ be a compact contained in D and let $D_0 = D \setminus \mathscr{K}$. Assume that a function $f \in \mathscr{R}_+$ is continuous on the set $D_0 \cup \partial\mathscr{K}$ and is harmonic on D_0. Then either $f(z) \geqslant 0$ for any

$z \in D_0$ or there exists a point $z_0 \in \partial \mathcal{K}$ such that $f(z_0) < f(z)$ for any $z \in D_0$.

We denote the set of all functions f such that $f \in \mathcal{R}_+$ and $-f \in \mathcal{R}_+$ by $\mathcal{R}_0$. Suppose that the following two conditions are fulfilled:

I. For any $w \in \mathcal{D}$ there exists a function $h_w(z)$ harmonic in D and such that the function $g_w(z) = h_w(z) + \gamma(w - z)$ belongs to $\mathcal{R}_0$. Here $\gamma(z) = -|z|$ for $n = 1$, $\gamma(z) = -\ln |z|$ for $n = 2$ and $\gamma(z) = |z|^{2-n}$ for $n > 2$.

J. The function $g_w(z)$ has partial derivatives with respect to the coordinates of the point w, which are continuous with respect to w and z for $w \neq z$.

The minimum principle implies that the function $g_w(z)$ is uniquely defined by the condition I. We shall call it the *Green function.*

THEOREM 1. — *Let the conditions* A–J *be fulfilled. Put*

$$k_w(z) = \frac{g_w(z)}{g_w(z_0)} \qquad (w, z \in \mathcal{D}, w \neq z, w \neq z_0),$$

where z_0 is a point of the domain D. *A compactification* E *of the domain* D *can be constructed such that:*

a) *for any $z \in \mathcal{D}$, the function $k_w(z)$ of the variable w can be extended continuously to the set* $E \setminus \{z\}$*;*

b) D *is an everywhere dense subset of* E. *Every nonnegative solution $f(z)$ of the boundary problem $\mathcal{R}$ can be represented as an integral*

$$f(z) = \int_B k_w(z)\nu(dw) \qquad (z \in \mathcal{D}) \tag{2}$$

where $B = E \setminus D$ *and ν is a finite measure on Borel subsets of the set* B.

Formula (2) is a generalisation of the well-known Martin expansion. (The case of Martin is obtained if we select $\mathcal{R}$ as the set of all functions defined near the boundary of D.)

It is possible that some of functions k_w $(w \in B)$ don't belong to $\mathcal{R}$. Put $w \in \tilde{B}$ if $w \in B$ and $k_w \in \mathcal{R}$. Is the formula (2) true with the set $\tilde{B}$ instead of the set B? We prove that it is if there exist sets $\mathcal{R}_w$ $(w \in B)$ subject to conditions:

K. $$\bigcap_{w \in B} \mathcal{R}_w = \mathcal{R}$$

L. If a sequence of harmonic functions $f_n \in \mathcal{R}_w$ converges to a function f in $D \cap V$ (where V is a neighborhood of w), then $f \in \mathcal{R}_w$. (This is a « localization » of the condition C.)

The set $\tilde{B}$ will be called *the Martin boundary* associated with the boundary value problem $\mathcal{R}$.

A point $w \in \tilde{B}$ is called *minimal* if the equality $k_w(z) = f_1(z) + f_2(z)$ (f_1 and f_2 are nonnegative solutions of the boundary problem $\mathcal{R}$) implies that $f_1 = c_1 k_w, f_2 = c_2 k_w$ (c_1, c_2 are real numbers). The set of all minimal points $w \in \tilde{B}$ will be denoted by B_e. The following strengthening of the Theorem 1 can be proved with the aid of the Choquet theorem on convex cones in linear topological spaces.

THEOREM 2. — *Every nonnegative solution $f(z)$ of the boundary value problem* (*under the conditions* A–L) *can be represented as an integral*

$$f(z) = \int_{B_e} k_w(z) \nu(dw) \tag{3}$$

where ν is a finite measure on Borel subsets of the set B_e.

2. The Martin boundary associated with the Neumann problem with oblique derivative.

We will now apply the general theorems of § 1 to the boundary value problem (1).

Let α be a point of the set Γ. Let s be a canonical parameter (the arc length) on the contour C reckoned from the point α in the direction of the vector $v(\alpha)$. Let $c(s)$ be a point of C corresponding to s and let $\theta(s)$ be the angle between $v[c(s)]$ and $c'(s)$. The function $\theta(s)$ changes sign at α. Set $\alpha \in \Gamma_-$ if the sign changes from minus to plus, and set $\alpha \in \Gamma_+$ if it changes from plus to minus. The numbers of points of Γ_- and Γ_+ will be denoted by n_- and n_+. We assume that the function $\theta(s)$ has a Hölder continuous derivative $\theta'(s)$ in a neighborhood of 0 and set $\alpha \in \Gamma_+^0$ if $\alpha \in \Gamma_+$ and $\kappa = \theta'(0) = 0$.

It is proved that the boundary value problem (1) satisfies the conditions A–L (if $n_+ > 0$). Without loss of generality we may assume that the domain D is a unit circle (the general case can be reduced to the case of the unit circle with the aid of conformal mapping). The most difficult parts are the proof of the minimum principle H and the construction of the Green function. The latter may be divided into the following stages:

1. We build a pair of analytical functions S(z), T(z) which are regular in D except at the point 0 where they can have a pole; these functions are Hölder continuous near the boundary C of the circle D; they are connected by the relation $S(z)T(z) = 1$ and satisfy the

following condition: the value of $S(z)$ for $z \in C$ differs from $e^{i\theta}$ by a positive factor (here θ is the angle between $v(z)$ and the positive direction of the tangent at the point z).

2. For any $\alpha \in \Gamma_+$ we construct a bounded solution $p_\alpha(z)$ of the boundary problem (1) such that $p_\alpha(\alpha) = 1$, $p_\alpha(\gamma) = 0$ for $\gamma \in \Gamma_+, \gamma \neq \alpha$.

3. The Green function $g_w(z)$ is defined by the formula

$$g_z(w) = q(z, w) - \sum_{\alpha \in \Gamma_+} q(\alpha, w) p_\alpha(z)$$

where

$$q(z, w) = \operatorname{Re} \int_0^z T(z) z^{-1} [\overline{S(w)} L(z, \bar{w}^{-1}) - S(w) L(z, w)]\, dz.$$

Here $L(z, w) = \frac{1}{2} \frac{z + w}{z - w}$ if $n_+ \geqslant n_-$. If $n_- > n_+$, we put $m = n_- - n_+$, we select an arbitrary subset $\tilde{\Gamma}_-$ of the set Γ_- containing $2m - 1$ points, we form the functions

$$P_\gamma(w) = \gamma^{m-1} w^{1-m} \prod_{\beta \in \tilde{\Gamma}_-, \beta \neq \gamma} (w - \beta)(\gamma - \beta)^{-1} \qquad (\gamma \in \tilde{\Gamma}_-)$$

and we define the function $L(z, w)$ by the formula

$$L(z, w) = \frac{1}{2} \frac{z + w}{z - w} - \frac{1}{2} \sum_{\gamma \in \tilde{\Gamma}_-} P_\gamma(w) \frac{z + \gamma}{z - \gamma}.$$

By studying the asymptotic behavior of the Green function as $w \to \alpha \in \Gamma$ we arrive at the following result.

THEOREM 3.— *Let $n_+ > 0$. Let $\tilde{B}$ be the Martin boundary associated with the boundary value problem* (1). *Then the connected components of $\tilde{B}$ are in one-to-one correspondence with the points α of the set Γ. For $\alpha \in \Gamma_\alpha$ the component B_α consists of one minimal point $\mathscr{C}_\alpha$. For $\alpha \in \Gamma_+$ the component B_α is a closed interval. The ends $\mathscr{C}_\alpha^+$ and $\mathscr{C}_\alpha^-$ of this interval are minimal points. If $\alpha \in \Gamma_+^0$ then an interior point $\mathscr{C}_\alpha$ of the interval B_α is minimal too.*

Let us introduce special designations for the solutions $k_w(z)$ of the boundary value problem (1) corresponding to the minimal points of the Martin boundary. Let us set

$$k_{\mathscr{C}_\alpha}(z) = u_\alpha(z) \qquad (\alpha \in \Gamma_- \cup \Gamma_+^0),$$

$$k_{\mathscr{C}_\alpha^+}(z) = a_\alpha^+ p_\alpha^+(z), \qquad k_{\mathscr{C}_\alpha^-}(z) = a_\alpha^- p_\alpha^-(z) \qquad (\alpha \in \Gamma_+).$$

Here the positive constants a_α^+ and a_α^- are selected so that $p_\alpha^+(z) \to 1$ as z tends to α along the contour C from the positive side, and $p_\alpha^-(z) \to 1$ as z tends to α along C from the negative side.

We shall describe the behavior of the solutions $u_\alpha, p_\alpha^+, p_\alpha^-$ near a point $\alpha \in \Gamma$. We assume that D is a unit circle. Consider the functions

$$\varphi_\alpha(z) = \operatorname{Im}\ln\left(1 - \frac{z}{\alpha}\right) = \arg\left(1 - \frac{z}{\alpha}\right) = \arctan\frac{1-x}{y},$$

$$\psi_\alpha(z) = \operatorname{Re}\ln\left(1 - \frac{z}{\alpha}\right) = \ln\left|1 - \frac{z}{\alpha}\right| = \frac{1}{2}\ln[(1-x)^2 + y^2],$$

$$\omega_\alpha(z) = \operatorname{Re}\left(1 - \frac{z}{\alpha}\right)^{-1} = \frac{1-x}{(1-x)^2 + y^2} \qquad \left(x + iy = \frac{z}{\alpha}\right).$$

These functions are positive harmonic in D and continuous in D $\cup$ C except α. Let us agree to write $f \equiv g$ if the difference $f - g$ can be represented as the sum of a harmonic function, continuous in $\mathcal{D} \cup$ C and a linear combination of functions $\varphi_\alpha(z)(\alpha \in \Gamma_-)$.

THEOREM 4. — *The following relations*

$$u_\alpha(z) \equiv a_\alpha[\omega_\alpha(z) - \kappa\psi_\alpha(z)] \qquad (\alpha \in \Gamma_- \cup \Gamma_+^0),$$

$$p_\alpha^+(z) \equiv -c_\alpha\varphi_\alpha(z), \qquad p_\alpha^-(z) \equiv c_\alpha\varphi_\alpha(z) \qquad (\alpha \in \Gamma_+)$$

are fulfilled with positive constants a_α and c_α.

Using the Theorem 2 it is not difficult to describe the general form of nonnegative solutions of the boundary value problem (1).

THEOREM 5.— *If $n_+ = 0$ then the only nonnegative solutions of the boundary problem* (1) *are constants. If $n_+ > 0$, then every nonnegative solution f is uniquely represented in the form*

$$f(z) = \sum_{\alpha \in \Gamma_- \cup \Gamma_+^0} a_\alpha u_\alpha(z) + \sum_{\alpha \in \Gamma_+} (c_\alpha^+ p_\alpha^+(z) + c_\alpha^- p_\alpha^-(z)),$$

where a_α, c_α^+, c_α^- are nonnegative constants. A solution f is bounded if and only if $a_\alpha = 0(\alpha = \Gamma_- \cup \Gamma_+^0)$. If $\alpha \in \Gamma_+ \setminus \Gamma_+^0$, then the solutions $k_w(z)$ corresponding to the points w of the interval B_α are linear combinations of p_α^+ and p_α^-. If $\alpha \in \Gamma_+^0$, then $k_w(z)$ is a linear combination of p_α^- and u_α for $w \in [\mathscr{C}_\alpha^-, \mathscr{C}_\alpha]$, and $k_w(z)$ is a linear combination of u_α and p_α^+ for $w \in [\mathscr{C}_\alpha, \mathscr{C}_\alpha^+]$.

Let us introduce notations for points of the interval $B_\alpha(\alpha \in \Gamma_+)$. In the case $\alpha \in \Gamma_+ \setminus \Gamma_+^0$ we set $w = \mathscr{C}_\alpha^\lambda$ if

$$k_w = c[(2 + \lambda + |\lambda|)p_\alpha^+ + (2 + |\lambda| - \lambda)p_\alpha^-]$$

(c is a constant depending on λ). In the case $\alpha \in \Gamma_+^0$ we set

$$w = \mathscr{C}_\alpha^\lambda \quad (\lambda \geqslant 0)$$

if $k_w = c[u_\alpha + \lambda p_\alpha^+]$ and we set $w = \mathscr{C}_\alpha^{-\lambda}(\lambda \geqslant 0)$ if $k_w = c[u_\alpha + \lambda p_\alpha^-]$. In addition, we set $\mathscr{C}_\alpha^{+\infty} = \mathscr{C}_\alpha^+$, $\mathscr{C}_\alpha^{-\infty} = \mathscr{C}_\alpha^-$, $\mathscr{C}_\alpha^0 = \mathscr{C}_\alpha$.

Let $\alpha \in \Gamma_+$ and let s be the canonical parameter introduced at the beginning of § 2. Let $n(s)$ be the unit vector directed along the interior normal to C at the point $c(s)$ and let $w(s, t) = c(s) + tn(s)$. Considering only sufficiently small values of s and t we obtain a local coordinate system in a neighborhood of α. Put

$$\theta(s, t) = \theta(s),$$

$$\zeta = \begin{cases} 2\pi s t^{-1} |\ln(s^2 + t^2)|^{-1} & \text{for} \quad \alpha \in \Gamma_+ \setminus \Gamma_+^0, \\ -2\pi t^{-1} \theta(s, t) & \text{for} \quad \alpha \in \Gamma_+^0. \end{cases}$$

THEOREM 6. — *If $\alpha \in \Gamma_-$, then the convergence w to $\mathscr{C}_\alpha$ in the Martin topology is equivalent to the convergence w to α in the topology of the Euclidean plane. In the case $\alpha \in \Gamma_+$, w tends to $\mathscr{C}_\alpha^\lambda$ in the Martin topology if, and only if, $w \to \alpha$ and $\zeta \to \lambda$.*

3. Probabilistic interpretation. Some unsolved problems.

It is possible to give the following probabilistic interpretation to the results obtained in § 2. Consider the Wiener process in the domain D and suppose that the trajectory of the process is reflected at the boundary point $z \in c \setminus \Gamma$ in the direction of $v(z)$ (or $-v(z)$, if $v(z)$ is directed outside D). We assume that the process terminates when the trajectory hits the set Γ. It may be proved that the movement starting from the point z terminates at the point $\alpha \in \Gamma_+$ with the probability $p_\alpha(z) = p_\alpha^+(z) + p_\alpha^-(z)$. The probability of hitting the set Γ_- is equal to 0. The probability of nontangential approach to $\alpha \in \Gamma_+$ is equal to 0 too. The quantities $p_\alpha^+(z)$ and $p_\alpha^-(z)$ are equal to the probabilities of tangential approach to $\alpha \in \Gamma_+$ from the positive and negative sides, respectively. These probabilistic conclusions were obtained earlier in a different way by Malyutov [2]. Using the Theorem 6 it is not difficult to obtain more precise information concerning the behavior of the trajectory near the terminal point α.

It would be very interesting to study the behavior of the trajectory of a many-dimensional Wiener process with reflection near the boundary points where the direction of reflection is tangential. As in the two-dimensional case, this problem is closely connected with the structure of the set of all positive solutions of the boundary value problem (1). The many-dimensional boundary value problem (1) in the classical statement (where solutions continuous in a closed domain are sought) meets considerable difficulties. In our statement the problem is actually local, therefore it is possibly simpler.

We conclude with an open problem in the general theory of Martin boundaries. Under very broad assumptions the measure ν in formula (3) is uniquely determined by the function f. Hence, to every point β of the Martin boundary $\tilde{\mathbf{B}}$ there corresponds a measure ν_β on the set $\mathbf{B}_e$, which is defined by the formula

$$k_\beta(z) = \int_{\mathbf{B}_e} k_w(z)\nu_\beta(w).$$

Let us agree to say that the point β_1 is subordinate to the point β_2 if the measure ν_{β_1} is differentiable with respect to the measure ν_{β_2}. The points β_1 and β_2 are called equivalent if the measures ν_{β_1} and ν_{β_2} are differentiable with respect to each other. The Martin boundary falls into the classes of equivalent points, and there exists a definite relation of subordination between these classes. Each minimal point is a separate class. In the case investigated in § 2 the remaining classes are open intervals. It would be very interesting to study the set of classes in the general case, to learn when this set is a finite complex, etc.

The following question is closely connected with the last one: what continuous functions on the Martin boundary $\tilde{\mathbf{B}}$ are limits of the harmonic functions under consideration? Apparently these boundary functions must be in some sense harmonic on each class of equivalent points. The exact definition of this harmonicity is now clear only for some special cases.

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Added in proof. The proofs of the Theorems 1–6 are published in:

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