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A REMARK ON A LOWER ENVELOPE PRINCIPLE

by Masanori KISHI

Introduction.

Let Ω be a locally compact Hausdorff space, every compact subset of which is separable, and let $G(x, y)$ be a positive continuous (in the extended sense) function defined on $\Omega \times \Omega$, which is finite at any point $(x, y) \in \Omega \times \Omega$ with $x \neq y$. This function G is called a positive continuous kernel on Ω. The kernel $\check{G}$ defined by $\check{G}(x, y) = G(y, x)$ is called the adjoint kernel of G. For a given positive measure μ, the potential $G\mu(x)$ and the adjoint potential $\check{G}\mu(x)$ are defined by

$$G\mu(x) = \int G(x, y)\, d\mu(y) \qquad \text{and} \qquad \check{G}\mu(x) = \int \check{G}(x, y)\, d\mu(y)$$

respectively. The G-energy of μ is defined by $\int G\mu(x)\, d\mu(x)$. Evidently this is equal to $\int \check{G}\mu(x)\, d\mu(x)$.

We shall say that G satisfies the *compact lower envelope principle* when for any compact subset K of Ω and for any $\mu \in \mathcal{E}_0$ and $\nu \in \mathfrak{M}_0$ (1), the lower enveloppe $G\mu \wedge G\nu$ (2) coincides G-p.p.p. on K with a potential $G\lambda$ of a positive measure λ supported by K(3). It is seen by an existence theorem obtained in [4] that if the adjoint kernel $\check{G}$ satisfies the continuity

(1) $\mathfrak{M}_0$ is the totality of positive measures with compact support and $\mathcal{E}_0$ is the totality of positive measures in $\mathfrak{M}_0$ with finite G-energy.

(2) $(G\mu \wedge G\nu)(x) = \inf \{ G\mu(x), G\nu(x) \}$.

(3) We say that a property holds G-p.p.p. on K when it holds on K almost everywhere with respect to any μ in $\mathcal{E}_0$.

principle ([4]) and G satisfies the ordinary domination principle ([5]), then G satisfies the compact lower envelope principle (cf. [6]). In this paper we examine what we can say about the converse.

We consider a positive continuous kernel G satisfying the continuity principle and we assume that any open subset of Ω is of positive G-capacity ([6]). We shall show that such a kernel satisfies the ordinary domination principle if it is not a finite-valued kernel on a discrete space, provided that G or $\check{G}$ is non-degenerate ([7]) and G satisfies the compact lower envelope principle. The exceptional kernel G satisfies the invere domination principle ([8]).

1. Elementary weak balayage principle.

1. We say that G satisfies the elementary weak balayage principle, if for any compact set K and any point $x_0 \notin K$, there exists $\mu \in \mathfrak{M}_0$, supported by K, such that

$$G\mu = G\varepsilon_{x_0} \quad \text{G-p.p.p. on } K,$$

where ε_{x_0} is the unit measure at x_0.

First we show that the compact lower envelope principle is stronger than the elementary weak balayage principle.

Lemma. — *If a positive continuous kernel* G *satisfies the compact lower envelope principle, then it satisfies the elementary weak balayage principle.*

Proof. — Without loss of generality, we may suppose that $\mathcal{E}_0$ is not empty. Let K be a compact set and x_0 be a point not on K. Since $G\varepsilon_{x_0}$ is bounded on K and $\mathcal{E}_0 \neq \emptyset$, there exists a positive measure λ in $\mathcal{E}_0$ such that $G\lambda \geqslant G\varepsilon_{x_0}$ on K.

([4]) This means that if $\check{G}\mu$ is finite continuous as a function on the support $S\mu$ of μ, then $\check{G}\mu$ is finite continuous in Ω.

([5]) Namely the following implication is true for G: $G\mu \leqslant G\nu$ on $S\mu$ with $\mu \in \mathcal{E}_0$ and $\nu \in \mathfrak{M}_0 \Longrightarrow G\mu \leqslant G\nu$ in Ω.

([6]) This means that for any non-empty open subset ω of Ω there exists $\lambda \neq 0$ in $\mathcal{E}_0$ such that $S\lambda \subset \omega$.

([7]) We say that G is non-degenerate when for any two different points x_1 and x_2, $G\varepsilon_{x_1}/G\varepsilon_{x_2} \not\equiv$ any constant in Ω, where ε_{x_i} is the unit measure at x_i, $(i = 1, 2)$.

([8]) Namely the following implication is true for G: $G\mu \leqslant G\nu$ on $S\nu$ with $\mu \in \mathcal{E}_0$ and $\nu \in \mathfrak{M}_0 \Longrightarrow G\mu \leqslant G\nu$ in Ω.

Then, by the compact lower envelope principle, there exists a positive measure μ, supported by K, such that

$$G\mu = G\lambda \wedge G\varepsilon_{x_0} \quad \text{G-p.p.p. on K.}$$

Hence $G\mu = G\varepsilon_{x_0}$ G-p.p.p. on K and G satisfies the elementary weak balayage principle.

2. In [5] we obtained the following results concerning the elementary weak balayage principle.

PROPOSITION 1. — *Let* G *be a positive continuous kernel on* Ω *such that* G *or* $\check{G}$ *is non-degenerate and* G *satisfies the continuity principle. Assume that every open subset of* Ω *is of positive* G-*capacity. If* G *satisfies the elementary weak balayage principle, then it satisfies the ordinary domination principle or the inverse domination principle.*

PROPOSITION 2. — *Under the same assumption as above,* G *satisfies the ordinary domination principle, if it satisfies the elementary weak balayage principle and there exists a point* x_0 *in* Ω *such that* $G(x_0, x_0) = +\infty$.

By these propositions and Lemma 1 we have

THEOREM 1. — *Assume that a positive continuous kernel* G *on* Ω *satisfies the continuity principle and that every open subset of* Ω *is of positive* G-*capacity. If* G *satisfies the compact lower envelope principle and* G *or* $\check{G}$ *is non-degenerate, then it satisfies the ordinary domination principle or the inverse domination principle.*

THEOREM 2. — *Assume the same as aobve. If* G *satisfies the compact lower envelope principle,* G *or* $\check{G}$ *is non-degenerate and there exists a point* x_0 *in* Ω *such that* $G(x_0, x_0) = +\infty$, *then* G *satisfies the ordinary domination principle.*

From these theorems follows

COROLLARY. — *Assume the same as above. If* G *satisfies the compact envelope principle and dose not satisfy the ordinary domination principle, then it is a finite continuous kernel* [9] *satisfying the inverse domination principle.*

[9] Namely it is a finite-valued and continuous kernel.

2. Finite continuous kernels.

3. Throughout this section we consider a finite continuous kernel G on Ω. We shall prove several lemmas on G.

Lemma 2. — *Let* G *satisfy the inverse domination principle. Then it is non-degenerate if and only if*

$$\Gamma(x_1, x_2) = G(x_1, x_1)G(x_2, x_2) - G(x_1, x_2)G(x_2, x_1) < 0$$

for any two different points x_1 *and* x_2 *in* Ω.

Proof. — Since G satisfies the inverse domination principle, $\Gamma(x_1, x_2) \leqslant 0$ for any two different points x_1 and x_2 in Ω. In fact, the identity $G\varepsilon_{x_1}(x_1) = aG\varepsilon_{x_2}(x_1)$ with

$$a = G(x_1, x_1)/G(x_1, x_2)$$

and the inverse domination principle yield

$$G\varepsilon_{x_1}(x) \geqslant aG\varepsilon_{x_2}(x) \tag{1}$$

for any x in Ω. Therefore $G\varepsilon_{x_1}(x_2) \geqslant aG\varepsilon_{x_2}(x_2)$ and hence $\Gamma(x_1, x_2) \leqslant 0$.

Now suppose that $\Gamma(x_1, x_2) = 0$. Then

$$G\varepsilon_{x_1}(x_2) = aG\varepsilon_{x_2}(x_2).$$

Hence by the inverse domination principle

$$G\varepsilon_{x_1}(x) \leqslant aG\varepsilon_{x_2}(x)$$

for any x in Ω. This together with (1) shows that G is degenerate. Consequently $\Gamma(x_1, x_2) < 0$ if G is non-degenerate. The converse is evidently true.

Corollary. — *Under the same assumption as above* G *is non-degenerate if and only if its adjoint kernel* $\check{G}$ *is non degenerate.*

Proof. — This is an immediate consequence of Lemma 2, since G satisfies the inverse domination principle when and only when $\check{G}$ satisfies the principle (see Theorem 2′ in [5]).

LEMMA 3. — *If* G *satisfies the inverse domination principle, then* G *satisfies the compact upper envelope principle, i.e., for any* μ, $\nu \in \mathfrak{M}_0$ *and any compact subset* K *of* Ω, *there exists* $\tau \in \mathfrak{M}_0$, *supported by* K, *such that*

$$G\tau = G\mu \vee G\nu \quad \text{on } K(^{10}).$$

Proof. — Put $u = G\mu \vee G\nu$. Then by the inverse existence theorem (cf. Theorem 4′ in [5]) there exists a positive measure τ, supported by K, such that

$$G\tau \leqslant u \quad \text{on } K,$$
$$G\tau = u \quad \text{on } S\tau.$$

By these inequalities and the inverse domination principle we obtain

$$G\tau = u \quad \text{on } K.$$

COROLLARY. — *If* G *satisfies the inverse domination principle, then its adjoint* $\check{G}$ *satisfies the compact upper envelope principle.*

LEMMA 4. — *If* G *is non-degenerate and satisfies the inverse domination principle, then it satisfies the unicity principle* (11).

Proof (12). — Let K be a compact subset of Ω and $\mathcal{C}$ be the space of all finite continuous functions on K with the uniform convergence topology. We put

$$\mathfrak{D} = \{ f \in \mathcal{C} ; f = \check{G}\mu_1 - \check{G}\mu_2 \text{ on K with } \mu_i \in \mathfrak{M}_0 \}.$$

First we show that $\mathfrak{D}$ is dense in $\mathcal{C}$. By the corollary of Lemma 3 we easily see that $\mathfrak{D}$ is closed with respect to the operations $\vee$ and $\wedge$, *i.e.*, if $f_i \in \mathfrak{D} (i = 1, 2)$, then $f_1 \vee f_2$ and $f_1 \wedge f_2$ belong to $\mathfrak{D}$. Let x_1 and x_2 be different points on K. Since G is non-degenerate, $\Gamma(x_1, x_2) \neq 0$ by Lemma 2. Hence for any given real numbers a_1 and a_2, there exists f in $\mathfrak{D}$ such that

$$f = t_1 \check{G}\varepsilon_{x_1} + t_2 \check{G}\varepsilon_{x_2} \quad (t_i, \text{ real})$$
$$f(x_i) = a_i \quad (i = 1, 2).$$

(10) $(G\mu \vee G\nu)(x) = \max\{G\mu(x), G\nu(x)\}$.
(11) Namely the equality $G\mu = G\nu$ in Ω with $\mu, \nu \in \mathfrak{M}_0$ implies $\mu = \nu$.
(12) Cf. [3] and [6].

Thus we can apply the theorem of Weierstrass and Stone (cf. [1], p. 53) and we obtain that $\mathfrak{D}$ is dense in $\mathcal{C}$.

Now let $G\mu_1 = G\mu_2$ in Ω with $\mu_i \in \mathfrak{M}_0$ and take a compact set K which contains $S\mu_1 \cup S\mu_2$. We shall show that

$$\int f\,d\mu_1 = \int f\,d\mu_2$$

for any f in $\mathcal{C}$. By the above remark there exists, for any positive number ε, a function g in $\mathfrak{D}$ such that $|f(x) - g(x)| < \varepsilon$ on K. Then

$$\left|\int f\,d\mu_i - \int g\,d\mu_i\right| < \varepsilon \int d\mu_i \quad (i = 1, 2).$$

Since $\int g\,d\mu_1 = \int g\,d\mu_2$,

$$\left|\int f\,d\mu_1 - \int f\,d\mu_2\right| < 2\varepsilon \max\left(\int d\mu_1, \int d\mu_2\right)\cdot$$

Consequently $\int f\,d\mu_1 = \int f\,d\mu_2$. This completes the proof.

Lemma 5. — *Assume that* G *is non-degenerate and satisfies the compact lower envelope principle and the inverse domination principle, Let* λ_0 *be a positive measure such that*

$$G\lambda_0 = G\mu \wedge G\nu \quad \textit{on } S\mu \cup S\nu,$$
$$S\lambda_0 \subset S\mu \cup S\nu.$$

Then for any x *in* Ω

$$G\lambda_0(x) = (G\mu \wedge G\nu(x).$$

Proof. — Let K be a compact set containing $S\mu \cup S\nu$ and λ be a positive measure supported by K such that

$$G\lambda = G\mu \wedge G\nu \quad \text{on K}.$$

By Lemma 3, there exists a positive measure τ, supported by K, such that

$$G\tau = G\mu \vee G\nu \quad \text{on K}.$$

Then

$$G\lambda + G\tau = G\mu \wedge G\nu + G\mu \vee G\nu = G\mu + G\nu$$

on K. Since $\lambda + \tau$ and $\mu + \nu$ are supported by K, we obtain by the inverse domination principle that

$$G(\lambda + \tau) = G(\mu + \nu) \quad \text{in } \Omega.$$

Hence by Lemma 4, $\lambda + \tau = \mu + \nu$ and λ is supported by $S\mu \cup S\nu$. Consequently again by the inverse domination principle, we have $G\lambda = G\lambda_0$ and hence $\lambda = \lambda_0$. This shows that

$$G\lambda_0 = G\mu \wedge G\nu \quad \text{in } \Omega.$$

LEMMA 6. — *Assume that* G *is non-degenerate and satisfies the compact lower envelope principle and the inverse domination principle. Then for any points* x_1, x_2 *and* x *in* Ω *either*

$$\frac{G(x, x_1)}{G(x, x_2)} = \frac{G(x_1, x_1)}{G(x_1, x_2)}$$

or

$$\frac{G(x, x_1)}{G(x, x_2)} = \frac{G(x_2, x_1)}{G(x_2, x_2)}.$$

Proof. — Without loss of generality we may assume that $G(x, x) = 1$ for any x in Ω, since $G'(x, y) = G(x, y)/G(x, x)$ is a non-degenerate finite continuous kernel which satisfies the compact lower envelope principle and the inverse domination principle. We take three different points x_1, x_2 and x_3 in Ω and put

$$g_{ij} = G(x_i, x_j).$$

By Lemma 2

$$(2) \qquad g_{12}g_{21} > 1.$$

Hence we can take positive measures $\mu = a_1\varepsilon_1 + a_2\varepsilon_2$, $\nu = b_1\varepsilon_1 + b_2\varepsilon_2$ such that

$$(3) \qquad G\mu(x_1) < G\nu(x_1) \qquad \text{and} \qquad G\mu(x_2) > G\nu(x_2),$$

where ε_i is the unit measure at x_i. Then by our assumption there exists a positive measure $\lambda = c_1\varepsilon_1 + c_2\varepsilon_2$ such that

$$G\lambda(x_i) = (G\mu \wedge G\nu)(x_i) \quad i = 1, 2.$$

By Lemma 5 this equality holds at x_3. Suppose that

$$G\lambda(x_3) = G\mu(x_3).$$

Then

$$c_1 + c_2 g_{12} = a_1 + a_2 g_{12},$$
$$c_1 g_{21} + c_2 = b_1 g_{21} + b_2,$$
$$c_1 g_{31} + c_2 g_{32} = a_1 g_{31} + a_2 g_{32}.$$

Therefore the following determinant vanishes;

$$\begin{vmatrix} 1 & g_{12} & a_1 + a_2 g_{12} \\ g_{21} & 1 & b_1 g_{21} + b_2 \\ g_{31} & g_{32} & a_1 g_{31} + a_2 g_{32} \end{vmatrix} = 0.$$

Hence

$$(g_{32} - g_{12}g_{31})\{(a_1 g_{21} + a_2) - (b_1 g_{21} + b_2)\} = 0,$$

namely $(g_{32} - g_{12}g_{31})(\mathrm{G}\mu(x_2) - \mathrm{G}\nu(x_2)) = 0$. Hence by (3), $g_{32} = g_{12}g_{31}$, that is,

$$\mathrm{G}(x_1, x_1)\mathrm{G}(x_3, x_2) = \mathrm{G}(x_1, x_2)\mathrm{G}(x_3, x_1).$$

Similarly we obtain

$$\mathrm{G}(x_2, x_2)\mathrm{G}(x_3, x_1) = \mathrm{G}(x_2, x_1)\mathrm{G}(x_3, x_2)$$

if $\mathrm{G}\lambda(x_3) = \mathrm{G}\nu(x_3)$. This completes the proof.

4. We are still making preparations.

LEMMA 7. — *Let* K *be a compact subset of* Ω, x_0 *a point on* K *and put*

$$h(z) = \inf \{\mathrm{G}\mu(z);\ \mu \in \mathfrak{M}_0,\ \mathrm{S}\mu \subset \mathrm{K},\ \mathrm{G}\mu(x_0) \geqslant 1\}$$

for any $z \in \Omega$. *If* G *satisfies the compact lower envelope principle, there exists a positive measure* μ, *supported by* K, *such that*

$$h = \mathrm{G}\mu \text{ on } \mathrm{K}.$$

Proof (13). — Put

$$\Phi = \{\mathrm{G}\mu;\ \mu \in \mathfrak{M}_0,\ \mathrm{S}\mu \subset \mathrm{K},\ \mathrm{G}\mu(x_0) \geqslant 1\}.$$

We first show that for any n given points $x_1, \ldots, x_n$ on K, there exists a potential $\mathrm{G}\mu \in \Phi$ such that

$$\mathrm{G}\mu(x_i) = h(x_i) \ (1 \leqslant i \leqslant n).$$

By the definition of $h(z)$, to each x_i corresponds a sequence $\{\mathrm{G}\mu_k^{(i)}\}$ of potentials in Φ in such a way that $\mathrm{G}\mu_k^{(i)}(x_i) \to h(x_i)$ as $k \to \infty$. We may assume that $\mathrm{G}\mu_k^{(i)}(x_0) = 1$ and hence the total masses of $\mu_k^{(i)}$ are bounded. Therefore a subsequence $\{\mu_{k_p}^{(i)}\}$ converges vaguely to $\mu^{(i)}$. Then $\mathrm{G}\mu^{(i)} \in \Phi$ and

(13) We assume the separability of K in the proof. However this assumption is not essential. We can verify our lemma without the separability (cf. Lemma 3 in [7]).

$G\mu^{(i)}(x_i) = h(x_i)$. By the compact lower envelope principle, $G\mu^{(1)} \wedge G\mu^{(2)} \wedge \ldots \wedge G\mu^{(n)}$ coincides with a potential $G\mu$ on K. This potential fulfills our requirements.

Now let $\{x_i\}(i = 1, 2, \ldots)$ be a dense subset of K. By the above remark there exists a positive measure μ_n, for each n, such that

$$\begin{aligned} &G\mu_n \in \Phi \\ &G\mu_n(x_0) = 1 \\ &G\mu_n(x_i) = h(x_i) \quad i = 1, 2, \ldots, n. \end{aligned}$$

Then a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ converges vaguely to a positive measure μ, supported by K. Evidently $G\mu$ belongs to Φ and $G\mu(x_i) = h(x_i)(i = 1, 2, \ldots)$. By the upper semicontinuity of h, $G\mu(z) \leqslant h(z)$ for any $z \in K$. Therefore $G\mu = h$ on K.

LEMMA 8. — *Let* G *be a non-degenerate kernel on* Ω *which satisfies the compact lower envelope principle and the inverse domination principle, and let* Ω_0 *be a compact subset of* Ω. *Then there exists a mapping* φ *from* Ω_0 *into* Ω_0 *such that*

$$(4) \qquad \varphi(x) \neq x \quad \textit{for any } x \textit{ in } \Omega_0,$$
$$(5) \qquad G(y, \varphi(x))G(\varphi(x), x) = G(y, x)G(\varphi(x), \varphi(x))$$

for any

$$x \neq y \textit{ in } \Omega_0.$$

Proof. — Without loss of generality we may assume that $G(x, x) = 1$ for any x in Ω. We take an arbitrary fixed point x in Ω_0, and we put

$$(6) \quad h_x(z) = \inf \{G\mu(z);\ \mu \in \mathfrak{M}_0,\ S\mu \subset \Omega_0,\ G\mu(x) \geqslant 1\}$$

for any z in Ω. Then by Lemma 7 there exists $\mu \in \mathfrak{M}_0$, supported by Ω_0, such that

$$h_x(z) = G\mu(z) \quad \text{for any } z \text{ in } \Omega_0.$$

By Lemma 4, μ is uniquely determined by a given point x. We shall show that there exists a unique point x' in Ω_0 such that $\mu = a\varepsilon_{x'}$ with $a^{-1} = G(x, x')$. If the assertion is false, $S\mu$ contains different points x' and x''; take a compact neighborhood K of x' such that $K \not\ni x''$. We put $\mu = \mu_K + \mu_K'$, where μ_K is the restriction of μ to K and $\mu'_K = \mu - \mu_K$. Then we can put

$$(7) \qquad G\mu_K(x) = \theta \quad \text{and} \quad G\mu'_K(x) = 1 - \theta$$

with $0 < \theta < 1$. By (6) and (7)

$$G\mu_K(z) \geqslant \theta\, h_x(z) \quad \text{for any } z \in \Omega$$
$$G\mu'_K(z) \geqslant (1 - \theta) h_x(z) \quad \text{for any } z \in \Omega.$$

Since $G\mu(z) = G\mu_K(z) + G\mu'_K(z) = h_x(z)$, it follows from the above inequalities that

$$G\mu_K = \theta h_x \quad \text{and} \quad G\mu'_K = (1 - \theta) h_x$$

in Ω. Hence $\theta^{-1} G\mu_K = (1 - \theta)^{-1} G\mu'_K$ in Ω, which contradicts the unicity principle. Therefore there exists a unique point x' in Ω_0 such that

$$(8) \qquad h_x(z) = a G\varepsilon_{x'}(z) \quad \text{for any } z \text{ on } \Omega_0,$$

with $a^{-1} = G(x, x')$. Thus we define a mapping $\varphi : \Omega_0 \to \Omega_0$ by $\varphi(x) = x'$ [14].

Now we shall show the validity of (4). Contrary suppose that $\varphi(x) = x$, and take a point $x'' \neq x$ in Ω_0. Then by (6)

$$G\varepsilon_x \leqslant G(x, x'')^{-1} G\varepsilon_{x''} \text{ on } \Omega_0.$$

On the other hand by the inverse domination principle

$$G\varepsilon_x \geqslant G(x, x'')^{-1} G\varepsilon_{x''} \text{ in } \Omega.$$

Therefore G is degenerate. This is a contradiction.

Next we shall show the equality (5). Take different points x and y in Ω_0. Then by (6)

$$G(x, \varphi(x))^{-1} G\varepsilon_{\varphi(x)}(y) \leqslant G(x, y)^{-1} G\varepsilon_y(y),$$

that is

$$G(y, \varphi(x)) G(x, y) \leqslant G(x, \varphi(x)).$$

Hence by (2)

$$\frac{G(y, \varphi(x))}{G(y, x)} < G(x, \varphi(x)).$$

(14) This mapping was first defined by Choquet-Deny [2].

Therefore Lemma 3 yields

$$\frac{G(y, \varphi(x))}{G(y, x)} = \frac{1}{G(\varphi(x), x)}.$$

This completes the proof.

Remark. — Just as Choquet and Deny did in [2], we can show that $\varphi^{-1}(x)$ is uniquely determined.

3. Main theorem.

5. We now prove the following main theorem.

THEOREM 3. — *Let* G *satisfy the continuity principle and the compact lower envelope principle. Assume that* Ω *is not discrete that any open subset of* Ω *is of positive* G*-capacity and that* G *or* $\check{G}$ *is non-degenerate. Then* G *satisfies the ordinary domination principle.*

Proof. — By the corollary of Theorems 1 and 2 it is sufficient to show that if G is a non-degenerate finite continuous kernel which satisfies the compact lower envelope principle and the inverse domination principle, then Ω is discrete. We take an arbitrary fixed point x_0 and its compact neighborhood Ω_0. Then by Lemma 8 we have a mapping $\varphi : \Omega_0 \to \Omega_0$ such that

$$\varphi(x) \neq x$$
$$G(y, \varphi(x))G(\varphi(x), x) = G(y, x)G(\varphi(x), \varphi(x))$$

for any $x \neq y$ in Ω_0. Then x_0 is an isolated point of Ω_0. In fact, if $\{y_n\}$ converges to x_0, then

$$\begin{aligned} G(x_0, \varphi(x_0))G(\varphi(x_0), x_0) &= \lim G(y_n, \varphi(x_0))G(\varphi(x_0), x_0) \\ &= \lim G(y_n, x_0)G(\varphi(x_0), \varphi(x_0)) = G(x_0, x_0)G(\varphi(x_0), \varphi(x_0)). \end{aligned}$$

This contradicts the non-degeneracy of G. Therefore Ω is discrete.

6. *Remark* 1. — When G is a non-degenerate finite continuous kernel satisfying the compact lower envelope principle and the inverse domination principle, so that Ω is discrete,

the mapping φ in Lemma 8 maps Ω_0 onto Ω_0 and the kernel G^{φ} on Ω_0 defined by

$$G^{\varphi}(x, y) = G(x, \varphi(y))$$

satisfies the ordinary domination principle. This corresponds to Choquet-Deny's theorem on « Modeles finis » (cf. Theoreme 3 in [2]).

Remark 2. — Let Ω be discrete. Then there always exists a non-degenerate finite continuous kernel G on Ω which satisfies the compact lower envelope principle and the inverse domination principle. For example, G defined by

$$G(x, y) = \begin{cases} 1 & \text{for } x = y \\ 2 & \text{for } x \neq y \end{cases}$$

fulfills all the requirements.

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