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On infinitesimal transformations preserving the curvature tensor field and its covariant differentials


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We shall say that a transformation \( \varphi \) of a Riemannian manifold \( M \) is \textit{strongly curvature-preserving} if it preserves the curvature tensor field \( R \) and all its successive covariant differentials \( \nabla^m R \). Similarly, an infinitesimal transformation \( X \) on \( M \) is strongly curvature-preserving if
\[
L_X (\nabla^m R) = 0, \quad m = 0, 1, 2, \ldots,
\]
where \( L_X \) denotes Lie differentiation with respect to \( X \) and \( \nabla^0 R = R \).

Of course, an affine transformation or an infinitesimal affine transformation is strongly curvature-preserving. In the present note, we shall prove the converse in the following form. Recall that an infinitesimal transformation \( X \) is conformal, homothetic, or Killing according as \( L_X g = f g \) (\( f \): function), \( L_X g = c g \) (\( c \): constant), or \( L_X g = 0 \), respectively, where \( g \) denotes the metric tensor.

**Theorem 1** (1). — Let \( M \) be an irreducible analytic Riemannian manifold of dimension \( \geq 2 \). Then a strongly curvature-preserving infinitesimal transformation is necessarily homothetic. If \( M \) is furthermore complete, then \( X \) is Killing.

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(2) We have since extended theorem 1 to the case of a global transformation; this result will appear elsewhere.
Note that the additional assertion is a consequence of a result of Kobayashi [2]. The proof of Theorem 1 will depend on the following results.

**Theorem 2.** — Let $M$ be an irreducible Riemannian manifold of dimension $> 2$. An infinitesimal conformal transformation $X$ is homothetic if $L_X R = 0$.

**Theorem 3.** — Let $M$ be an irreducible analytic Riemannian manifold of dimension 2. An infinitesimal transformation $X$ is homothetic if $L_X R = 0$ and $L_X (\nabla R) = 0$.

The proof of Theorem 2 makes use of a result of Guillemin and Sternberg [1] on the prolongations of the conformal algebra.

Finally, we shall prove the following generalization of Theorem 1.

**Theorem 4.** — Let $M$ be a connected, complete and analytic Riemannian manifold which has no Euclidean part (i.e., the restricted homogeneous holonomy group $\mathfrak{h}_0$ has no non-zero fixed vector). Then any strongly curvature-preserving infinitesimal transformation $X$ is a Killing vector field.

1. Preliminaries.

For an arbitrary infinitesimal transformation $X$ on $M$, we shall define a tensor field $K$ of type $(1, 2)$ which measures the deviation of $X$ from being affine; $X$ is affine if and only if $K = 0$. For any vector field $Y$, consider the derivation

$$K(Y) = [L_X, \nabla_Y] - \nabla_{[X,Y]}$$

of the algebra of tensor fields. It is easy to verify that $K(Y)$ is actually a tensor field of type $(1, 1)$ and that $K(fY) = fK(Y)$ for any differentiable function $f$. This means that $K$ is a tensor field of type $(1, 2)$ which associates to a vector field $Y$ the tensor field $K(Y)$ of type $(1, 1)$.

Using the formula $L_X = A_Y + \nabla_Y$, where $A_X$ is the tensor field of type $(1, 1)$ defined by $A_X Y = - \nabla_Y X$ (cf. [3], p. 235), we may express $K(Y)$ as follows:

$$K(Y) = R(X, Y) - \nabla_Y (A_X).$$
In fact, we have
\[ K(Y) = [A_x + \nabla_x, \nabla_Y] - \nabla_{[x, y]} \]
\[ = [A_x, \nabla_Y] + [\nabla_x, \nabla_Y] - \nabla_{[x, y]} \]
\[ = - \nabla_Y(A_x) + R(X, Y). \]

We now prove

**Lemma 1.** — The tensor field \( K \) corresponding to a vector field \( X \) has the following properties:

1) \( K(Y)Z = K(Z)Y \) for any vector fields \( Y \) and \( Z \);
2) \( (\nabla_u K)(Y)Z = (\nabla_u K)(Z)Y \) for any vector fields \( Y, Z, \) and \( U \);
3) If \( L_X R = 0 \), then \( (\nabla_1 K)(Z) = (\nabla_2 K)(Y) \) for any vector fields \( Y \) and \( Z \);
4) If \( X \) is conformal: \( L_X g = \alpha g \), then
\[ (\nabla_u K)(Y)g = - (\nabla_u \alpha)(Y)g \]
for any vector field \( Y \), where \( \alpha = df \).

5) If \( X \) is conformal, then, for the form \( \alpha \) in 4), we have
\[ (\nabla_u K)(Y)g = - (\nabla_u \alpha)(Y)g \]
for any vector fields \( Y \) and \( U \).

**Proof.** — 1) By using (2), we have
\[ K(Y)Z = R(X, Y)Z - [\nabla_Y(A_x)]Z \]
\[ = R(X, Y)Z - \nabla_Y(A_x Z) + A_x(\nabla_Y Z) \]
and hence
\[ K(Y)Z = R(X, Y)Z + \nabla_Y \nabla_Z X - \nabla_{YZ} X \]
by definition of \( A_x \). Thus alternating with respect to \( Y \) and \( Z \), we have
\[ K(Y)Z = K(Z)Y \]
\[ = R(X, Y)Z - R(X, Z)Y + ([\nabla_Y, \nabla_Z] - \nabla_{[Y, Z]} )X = 0 \]
by virtue of Bianchi’s identity:
\[ R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0, \]
and the definition of the curvature tensor:
\[ [\nabla_Y, \nabla_Z] - \nabla_{[Y, Z]} = R(Y, Z). \]
2) We take $\nabla_v$ of 1) and obtain
\[
(\nabla_v K)(Y)Z + K(\nabla_v Y)Z + K(Y)\nabla_v Z = (\nabla_v K)(Z)Y + K(\nabla_v Z)Y + K(Z)\nabla_v Y,
\]
from which, using 1) again, we find
\[
(\nabla_v K)(Y)Z = (\nabla_v K)(Z)Y.
\]

3) By using (2), we have
\[
(\nabla_y K)(Z) = \nabla_y (K(Z)) - K(\nabla_y Z) = (\nabla_y R)(X, Z) + R(\nabla_y X, Z) + R(X, \nabla_y Z) - \nabla_y \nabla_x (A_x) - R(X, \nabla_y Z) - R(x, z)(A_x)
\]
or
\[
(\nabla_y K)(Z) = (\nabla_y R)(X, Z) - R(A_x Y, Z) - (\nabla_y \nabla_x - \nabla_{[x, y]})(A_x).
\]
Alternating with respect to $Y$ and $Z$, we find
\[
(\nabla_y K)(Z) - (\nabla_z K)(Y) = (\nabla_y R)(X, Z) - (\nabla_z R)(X, Y) - R(A_x Y, Z) + R(A_x Z, Y) - (\nabla_x + A_x)R(Y, Z) = 0,
\]
by virtue of Bianchi's identity:
\[
(\nabla_x R)(Y, Z) + (\nabla_y R)(Z, X) + (\nabla_z R)(X, Y) = 0
\]
and the assumption $L_x R = 0$.

4) By definition of $K(Y)$, we have
\[
K(Y) = L_x \nabla_y - \nabla_y L_x - \nabla_{[x, y]}.
\]
Applying this derivation to $g$, we find
\[
K(Y)g = -\nabla_y L_x g.
\]
Thus if $L_x = fg$, then we have
\[
K(Y)g = -\alpha(Y)g,
\]
where $\alpha = df$.

5) Taking $\nabla_v$ of the equation in 4), we have
\[
(\nabla_v K)(Y)g + K(\nabla_v Y)g = - (\nabla_v \alpha)(Y)g - \alpha(\nabla_v Y)g,
\]
which implies

\[ (\nabla_{\nu}K)(Y)g = - (\nabla_{\nu}\alpha)(Y)g, \]

since \( K(\nabla_{\nu}Y)g = - \alpha(\nabla_{\nu}Y)g \) by (4).

We shall now interpret Lemma 1 above in terms of the prolongations of the conformal algebra [1]. By the conformal algebra over an \( n \)-dimensional real vector space \( V \) with inner product, we mean the following. Let \( \text{co}(V) \) be the set of all linear endomorphisms \( A \) of \( V \) such that

\[ (AX, Y) + (X, AY) = c(X, Y) \]

for all \( X, Y \) in \( V \), where \( c \) is a constant which depends on \( A \). With respect to the usual bracket \([A, B] = AB - BA\), \( \text{co}(V) \) forms a Lie algebra.

Suppose \( X \) is conformal. Property 4) means that for any \( Y \) in the tangent space \( T_x(M) \) at a point \( x \in M \), the endomorphism \( K(Y) \) is in the conformal algebra \( \text{co}(x) \) over \( T_x(M) \), of course, with respect to the metric \( g_x \). Property 1) means that the linear mapping \( K : Y \in T_x(M) \rightarrow K(Y) \in \text{co}(x) \) is an element of the first prolongation \( \text{co}(x)^{(1)} \). Property 5) means that for any \( U \in T_x(M) \), the endomorphism \( (\nabla_{\nu}K)(Y) \) belongs to \( \text{co}(x) \) for any \( Y \in T_x(M) \). Property 2) means that the linear mapping \( \nabla_{\nu}K : Y \in T_x(M) \rightarrow (\nabla_{\nu}K)(Y) \in \text{co}(x) \) is an element of \( \text{co}(x)^{(2)} \).

Now assume that \( L_xR = 0 \). Property 3) means that the linear mapping \( \nabla_{\nu}K : U \in T_x(M) \rightarrow \nabla_{\nu}K(\in \text{co}(x)^{(2)}) \) is actually an element of the second prolongation \( \text{co}(x)^{(2)} \). It is known [1], however, that \( \text{co}(x)^{(2)} = 0 \) when \( \dim M > 2 \). Thus we arrive at the following consequence of the lemma above:

\text{If } X \text{ is conformal and } L_xR = 0, \text{ then the corresponding tensor field } K \text{ satisfies } \nabla K = 0.\]

2. Proof of Theorem 2.

From the preceding interpretation of the Lemma, we see that \( \nabla K = 0 \). Let \( \gamma \) be the 1-form defined by \( \gamma(Y) = \text{trace of } K(Y) \). We have then \( \nabla \gamma = 0 \). Since \( M \) is irreducible, we have \( \gamma = 0 \), that is, trace \( K(Y) = 0 \) for any \( Y \). Since \( K(Y) \) is in \( \text{co}(x) \), it follows that \( K(Y) \) is skew-symmetric. In equation (3), we have \( K(Y)g = - \alpha(Y)g = 0 \) for any \( Y \), which means that \( \alpha = 0 \). Since \( \alpha = df \) in the proof of equation (3), we see that \( f \) is a constant, that is \( X \) is homothetic.
3. Proof of Theorem 3.

In a two-dimensional irreducible Riemannian manifold, the Ricci tensor $S$ has the form

$$S = \lambda g,$$

where $\lambda$ is a function which is not identically zero. From this we have

$$\nabla_Y S = (Y\lambda)g$$

for any vector $Y$.

If the infinitesimal transformation $X$ satisfies $L_X R = 0$ and $L_X (\nabla R) = 0$, then it satisfies $L_X S = 0$ and $L_X (\nabla S) = 0$. From $S = \lambda g$ and $L_X S = 0$, we obtain

$$(X\lambda)g + \lambda(L_X g) = 0.$$  \hfill (4)

From $\nabla_Y S = (Y\lambda)g$ and $L_X (\nabla S) = 0$, we obtain

$$0 = L_X \nabla_Y S - \nabla_{[X,Y]} S = (XY\lambda)g + (Y\lambda)L_X g - ([X,X]\lambda)g = (XY\lambda)g + (Y\lambda)L_X g,$$

that is,

$$(YX\lambda)g + (Y\lambda)(L_X g) = 0.$$  \hfill (5)

Taking $\nabla_Y$ of (4) and taking (5) into account, we get

$$\lambda \nabla_Y (L_X g) = 0.$$

Since our manifold is real analytic, the set of zero points of $\lambda$ is nowhere dense. Hence we have

$$\nabla L_X g = 0.$$  \hfill (6)

Since the manifold is irreducible, we get

$$L_X g = cg,$$

where $c$ is a constant.

4. Proof of Theorem 1.

Since $M$ is an analytic Riemannian manifold, the holonomy algebra $h_x$ (Lie algebra of the restricted holonomy group at $x$) is generated by all endomorphisms of the form

$$R(Y, Z), (\nabla_{vR})(Y, Z), \ldots, (\nabla^{mR})(Y, Z; U_1; \ldots; U_m), \ldots,$$

where $Y, Z, U_1, \ldots, U_m$ are arbitrary vectors at $x$. 
(cf. [3, p. 152]). From the assumption $L_x(\nabla^m R) = 0$, it follows that $A_x(\nabla^m R) = -\nabla_x(\nabla^m R)$. It is easy to see that

$$[A_x, (\nabla^m R)(Y, Z; U_1; \ldots; U_m)] \in h_x$$

and hence

$$[A_x, h_x] \subset h_x.$$

The tensor $L_x g = A_x g$ at $x$ is then invariant by $h_x$. In fact, for any $B \in h_x$, we have

$$B(A_x g) = A_x(Bg) + [A_x, B]g = 0,$$

since $B$ and $[A_x, B]$ are skew-symmetric as elements in $h_x$. Since $h_x$ is irreducible, $A_x g$ at $x$ is a scalar multiple of the tensor $g_x$. This being the case at every point $x$ of $M$, we have $A_x g = f g$, that is, $L_x g = f g$, where $f$ is a function. This means that $X$ is conformal.

Thus, if the dimension of $M > 2$, then Theorem 2 implies that $X$ is homothetic.

If the dimension of $M$ is 2, then Theorem 1 is a special case of Theorem 3.

5. Proof of Theorem 4.

We may assume that $M$ is simply connected. Let $M = M_1 \times \cdots \times M_k$ be the de Rham decomposition, where $M_1, \ldots, M_k$ are irreducible, complete and analytic Riemannian manifolds. We shall show that the vector field $X$ decomposes naturally, that is, there exists a strongly curvature-preserving infinitesimal transformation $X_i$ on $M_i$, $1 \leq i \leq k$, such that

$$X_{(x_1, \ldots, x_k)} = (X_1)_{x_1} + \cdots + (X_k)_{x_k}$$

for any point $x = (x_1, \ldots, x_k) \in M_1 \times \cdots \times M_k$. Once this is shown, we see that $X_i$ is Killing on $M_i$ by Theorem 1 and hence $X$ is Killing on $M$.

In order to prove a natural decomposition of $X$, we proceed as follows. Let $(T_1), \ldots, (T_k)$ be the parallel distributions corresponding to the de Rham decomposition $M_1 \times \cdots \times M_k$.

**Lemma 2.** — $L_x(T_i) \subset (T_i)$ for each $i$, in the sense that if $Y$ is a vector field belonging to the distribution $(T_i)$, then

$$L_x(Y) = [X, Y]$$

belongs to $(T_i)$. 

Proof. — Since $L_x = \nabla_x + A_x$ and since $\nabla_x(T_i) \subseteq (T_i)$ because $(T_i)$ is parallel, it is sufficient to show that $A_x(T_i) \subseteq (T_i)$. Let $x$ be an arbitrary point. In the proof of Theorem 1, we have seen that $(A_x)_x$ lies in the normalizer of the holonomy algebra $h_x$. Thus the 1-parameter group of linear transformations $\exp tA_x$ of $T_x(M)$ lies in the normalizer of the holonomy group $\mathcal{H}_x$. It follows that, for each $t$, $(\exp tA_x)(T_i)_x$ coincides with some $(T_j)_x$ by virtue of the uniqueness of the de Rham decomposition

$$T_x(M) = (T_1)_x + \cdots + (T_k)_x$$

(cf. Theorem 5.4, (4), p. 185, and Lemma, p. 186, in [3]). By continuity, we see that $(\exp tA_x)(T_i)_x = (T_i)_x$ for every $t$. This implies $A_x(T_i)_x \subseteq (T_i)_x$.

Lemma 3. — Let $\Delta$ be a differentiable distribution on a differentiable manifold $M$. If a vector field $X$ on $M$ satisfies $L_x(\Delta) \subseteq \Delta$, then a local 1-parameter group $\varphi_t$ of local transformations generated by $X$ preserves the distribution.

Proof. — Let $Y_1, \ldots, Y_r$ be a local basis for $\Delta$ in a neighborhood of $x$. It is sufficient to show that $(\varphi_t(\langle X, Y \rangle)_x$ belongs to $\Delta_x$ for every $t$. We recall the formula

$$\frac{d}{dt}\langle \varphi_t(\langle X, Y_i \rangle)_x = -(\varphi_t(\langle X, Y_i \rangle)_x$$

(Corollary 1.10, p. 16, [3]).

Since $[X, Y_i]$ belongs to $\Delta$, we have

$$[X, Y_i] = \sum_{j=1}^{r} f_{ij} Y_j,$$

where $f_{ij}$ are certain functions. Therefore

$$\frac{d}{dt}\langle \varphi_t Y_i\rangle_x = -(\varphi_t(\sum_{j=1}^{r} f_{ij} Y_j))_x$$

$$= -\sum_{j=1}^{r} (f_{ij} \circ \varphi_t^{-1}) \langle \varphi_t Y_j \rangle_x.$$
values in $T_x(M)$ satisfy a system of differential equations

$$\frac{dY_i(t)}{dt} = \sum_{j=1}^{r} g_{ij}(t) Y_j(t),$$

where $g_{ij}(t) = -f_{ij}(\varphi_t^{-1}(x))$. The initial conditions are $Y_i(0) = (Y_i)_x$. It follows that $Y_i(t)$ has to be a linear combination

$$Y_i(t) = \sum_{j=1}^{r} F_{ij}(t)(Y_j)_x$$

of the vectors $(Y_1)_x, \ldots, (Y_r)_x$, that is, $Y_i(t) \in \Delta_x$. $(F(t) = [F_{ij}(t)]$ is the matrix function which is a unique solution of

$$\frac{dF}{dt} = G(t)F(t)$$

with initial condition $F(0) = [\hat{\varepsilon}_{ij}]$. The existence of such a solution is a special case of Lemma, p. 69, [3].) This proves Lemma 3.

Now we can prove that $X$ decomposes naturally. Let $\varphi_t$ be a local 1-parameter group of local transformations generated by $X$ in a neighborhood of a point $x$. By Lemma 2,

$$L_X(T_t) \in (T_t).$$

By Lemma 3, $\varphi_t$ preserves each distribution $(T_t)$ and hence its maximal integral manifold. It follows, by an argument similar to the proof of Theorem 3.5, p. 240, in [3], that there exists, for each $t$ a local transformation $\varphi_t^{(i)}$ of $M_t$ such that

$$\varphi_t(x_1, \ldots, x_k) = (\varphi_t^{(1)}(x_1), \ldots, \varphi_t^{(k)}(x_k)).$$

Each $\varphi_t^{(i)}$ is a local 1-parameter group and defines a vector field $X_i$ on $M_t$. It is clear that $X = X_1 + \ldots + X_k$. Since the curvature tensor $R$ and its successive covariant differentials $\nabla^m R$ decompose naturally, it is obvious that each $X_i$ is strongly curvature-preserving on $M_t$. 

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