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GLOBAL PHASE-PORTRAIT OF A PLANE AUTONOMOUS SYSTEM

by **Czeslaw OLECH** (Kraków).

Introduction.

Consider an autonomous system on the plane R^2

$$(S) \qquad x' = f(x) \quad (' = d/dt),$$

where $x = (x_1, x_2)$, $f(x) = (f_1(x), f_2(x))$.

The purpose of the present note is to study the global structure of (S) under the main assumption that the divergence of the right-hand side of (S) in nonpositive everywhere on R^2; that is we assume the following condition

H_1. $f(x)$, in (S), is of class C^1 and

$$\operatorname{div} f(x) = \partial f_1/\partial x_1 + \partial f_2/\partial x_2 \leqslant 0 \qquad \text{on} \qquad R^2.$$

Some other restrictions of $f(x)$ will be also introduced below, one concerning the behaviour of $f(x)$ in a neighbourhood of infinity and the other concerning the number of singular points of (S).

This note is a continuation of the author's paper [1]. In particular Theorems 5 and 6 of [1] are special cases of the situation we are going to discuss here.

1. Further introductory remarks and propositions.

Let $x(t, P)$ denote the unique solution of (S) satisfying the initial condition $x(O, P) = P$. Let D be a region of R^2 with regular boundary C. An important and well known consequence of H_1 is that the area of $D(t) = x(t, D)$ is non-increasing function of t. This may suggest that solutions of (S), if H_1 is assumed, should exhibit some stability property as t increases as well as instability for negative t. However for establishing such properties of solutions of (S), the fact that the area of $D(t)$ is decreasing seems to be rather useless since the shape of a particular $D(t)$ may change considerably with t. Thus we shall make use of H_1 rather through the Geren's formula

$$\int\int_{D} \operatorname{div} f(x)\, dx_1\, dx_2 = \int_{C} f_1\, dx_2 - f_2\, dx_1. \tag{1.1}$$

Notice that the right-hand side of (1.1) is zero along those parts of C which are segments of solution curves of (S). Consider now the system

$$x' = f^*(x), \tag{S*}$$

where $f^*(x) = (f_2(x), -f_1(x))$. (S*) is a system for orthogonal trajectories to solution curves of (S). Let $x^*(t)$ be a solution of (S*) and suppose $P = x^*(t_P)$ and $Q = x^*(t_Q)$, $t_P < t_Q$. By $I^*(P, Q)$ we denote the set $\{x : x = x^*(t),\ t_P < t < t_Q\}$ and will call it a segment of solution curve of (S*). Suppose now that $I^*(P, Q) \subset C$, where C is as in (1.1). Then it is easy to see that

$$\int_{I^*(P,Q)} f_1\, dx_2 - f_2\, dx_1 = \pm \int_{t_P}^{t_Q} |f(x^*(t))|^2\, dt, \tag{1.2}$$

where the sign « — » should be choosen if solution curves of (S) crossing $I^*(P, Q)$ enter D as t increases and « + » in the opposite case. (Notice that the way of crossing $I^*(P, Q)$ by solutions of (S) is the same at each point of the segment). The absolute value of the integral in the right-hand side of (1.2) we denote shortly by $L(P, Q)$. Therefore $L(P, Q)$ is a function of two points of R^2 defined for such pairs P, Q which can be joined by a solution curve of (S*) and unless the solution curve of (S*) passing through P and Q is a closed orbit, $L(P, Q)$ is well defined. However we shall use $L(P, Q)$ also in the case $I^*(P, Q)$ is a segment of a closed orbit and the context will exclude any misunderstanding.

The following two propositions can be obtained from (1.1) and (1.2) if H_1 holds.

PROPOSITION 1.1. — *Suppose the boundary* C *of* D *in* (1.1) *can be decomposed into disjoined parts*

$$C = I^*(P, Q) \cup C_1 \cup C_2$$

where C_1 *is composed of segments of solution curves of* (S) *and*

$$\left|\int_{C_2} f_1\, dx_2 - f_2\, dx_1\right| < L(P, Q).$$

Then, if H_1 holds on D, solution curves of (S) crossing $I^*(P, Q)$ enter D as t increases.

The other proposition involves the following construction. Suppose we have given a segment $I^*(P, Q)$; that is there is a solution $x^*(t)$ of (S^*) such that, say, $x^*(0) = P$ and $x^*(t_0) = Q$ for some $t_0 > 0$. Consider solutions $x(t, P)$ and $x(t, Q)$ of (S) defined by initial conditions $x(0, P) = P$ and $x(0, Q) = Q$, respectively. Put $P(t) = x(t, P)$ for $t \geqslant 0$, hence $P(0) = P$. There is $\varepsilon > 0$ such that for any $0 \leqslant t < \varepsilon$, the solution $x^*(\tau, P(t))$ $(x^*(0, P(t)) = (P(t))$ of (S^*) crosses $x(t, Q)$; more precisely, there is $\tau(t)$ and $s(t)$ for any $\theta \leqslant t < \varepsilon$ such that $x^*(\tau(t), P(t)) = x(s(t), Q)$ and if we require $s(t)$ and $\tau(t)$ to be continuous functions of t satisfying conditions $s(0) = 0$ and $\tau(0) = t_0$, then $s(t)$ and $\tau(t)$ are uniquely defined and $s(t)$ is increasing function of t. Since $s(t)$ can be defined locally it can be continued to the maximal interval $[0, \omega)$ open to the right. Put $Q(t) = x(s(t), Q)$ for $0 \leqslant t < \omega$, of course $Q(0) = Q$, and we can consider segments

$$I^*(P(t), Q(t)) = \{x : x = x^*(\tau, P(t)), \quad 0 < \tau < \tau(t)\}.$$

We have now the following.

PROPOSITION 1.2. — *If* H_1 *is assumed then the function*

$$(1.3) \quad \varphi(t) = L(P(t), Q(t)) = \int_0^{\tau(t)} |f(x^*(\tau, P(t))|^2\, d\tau$$

is continuous and non-increasing for $t \in [0, \omega)$.

2. Negative limiting sets and singular points.

In the sequel we shall use the following notations. $x(t, P)$, as above, will stand for the solution of (S) satisfying the initial condition $x(0, P) = P$. The maximal interval on which

$x(t, P)$ exists we denote by $(a(P), b(P))$. By $I(P)$ we denote the set of R^2 defined by $\{x : x = x(t, P), a(P) < t < b(P)\}$. $I(P)$ will be called solution curve or trajectory of (S) passing through P. Correspondingly, $I^+(P)$ and $I^-(P)$ will stand for the positive and negative half trajectory of (S) issuing from P. We shall write I, I^+ and I^- for trajectory, positive half trajectory and negative half trajectory respectively, if we do not wish to indicate any particular point they initiate from. By $\alpha(I)$ and $\omega(I)$ we denote, as usual, the negative and the positive limiting set of I respectively.

We wish to discuss in this section some results concerning the negative limiting sets and isolated singular points of (S). We start with the following theorem which can be obtained using Proposition 1.1.

THEOREM 2.1. — *Assume* H_1 *holds.* Let $\alpha(I(P)) \neq \emptyset$ ($\emptyset$ stands *for the empty set*) *for some non-singular* $P(f(P) \neq 0)$. *Then for all* Q, $Q \in I^*(P)$, $Q \neq P$, *with one possible exception we have*

$$\alpha(I(P)) \cap \alpha(I(Q)) = \emptyset.$$

In other words, it follows from H_1 that non-singular trajectories of (S) with non-empty negative limiting set are isolated in the sense that its neighbouring trajectories have negative limiting set either empty or disjoined with that of $I(P)$. In particular, such trajectories are orbitally unstable for $t < 0$.

As a consequence of Theorem 2.1 we have

THEOREM 2.2. — *If* H_1 *holds and if for a given non-singular trajectory* I *of* (S) *the negative limiting set* $\alpha(I)$ *is not empty then either* $\alpha(I) = I$, *hence* I *is a closed orbit, or* $\alpha(I)$ *is composed of singular point of* (S) *only.*

To see this theorem from Theorem 2.1, notice that if $\alpha(I)$ contains a regular point P which does not belong to I then $I^*(P)$ has to be crossed by I at least in three different points, but this contradicts Theorems 2.1.

In the case singular points of (S) are isolated we have

THEOREM 2.3. — *Assume* H_1 *and suppose the singular points of* (S) *are isolated. Then for any trajectory* I *of* (S) *we* *have*: either $\alpha(I)$ is empty, or $\alpha(I) = I =$ closed orbit or $\alpha(I) = \{P\}$, *where* P *is a singular point of* (S).

Suppose now x_0 is an isolated singular point of (S). x_0 is said to be stable if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any solution $x(t)$ of (S), $|x(0) - x_0| < \delta$ implies that

$$|x(t) - x_0| < \varepsilon \qquad \text{for} \qquad t > 0.$$

Let us notice that x_0 is not stable if and only if there exists a trajectory I of (S) such that $\alpha(I) = \{x_0\}$. Suppose the last is the case for x_0, then by Theorem 2.1 there is a finite number, at the most, of trajectories of (S) having x_0 as negative limiting set, since any of them must be isolated. Thus also eliptic sectors in a neighbourhood of x_0 are excluded (cf. for terminology [2]). Between two trajectories of (S) approaching x_0 as $t \to -\infty$ there must exist a trajectory I for which

$$\omega(I) = \{x_0\}$$

or an attractive fan (cf. [2], p. 208). Therefore in a neighbourhood of an isolated and unstable singular point of (S) we have the following picture. There is a neighbourhood N of x_0 which can be decomposed into disjoined parts

$$N_1 \cup N_2 \cup N_3 \cup \{x_0\} = N$$

in such a way that 1) if $P \in N_1$ then $I^-(P) \subset N_1$ and $\alpha(I(P)) = x_0$, and N_1 is contained in a union of finite number of trajectories of (S), 2) N_2 is composed of a finite number of half trajectories or attractive fans and if $P \in N_2$ then $I^+(P) \subset N_2$ and

$$\omega(I(P)) = x_0,$$

3) N_3 is composed of finite number of hyperbolic sectors (cf. [2]) and if $P \in N_3$ then neither $I^+(P) \subset N$ nor $I^-(P) \subset N$, 4) N_1 and N_2 are closed in $N - \{x_0\}$ while N_3 is open, but $x_0 \in \overline{N_3}$ — the closure of N_3. Such a singular point we shall call in the sequel a generalised saddle point. Manifestly a saddle point in the usual sense is a generalized saddle point, too.

Consider now the case when x_0 is stable isolated singular point of (S). Then either x_0 is asymptotically stable or any neighbourhood of it contains a closed orbit of (S). Suppose the second possibility is the case and let I be a closed orbit of (S) containing inside x_0 as only one singular point of (S). Then by H_1 it follows that $\operatorname{div} f(s) \equiv 0$ inside I. On the other hand it follows from Theorem 2.2 that any solution curve inside I has to be a closed orbit. Therefore x_0 is a center in that case. We have the following result.

THEOREM 2.4. — *Assume* H_1. *Then the singular points of* (S) *if they are isolated, fall under the following categories: generalized saddle points, points of attraction and centers.*

A stable singular point of (S) is a center if and only if there is a neighbourhood of it on which $\operatorname{div} f(x) \equiv 0$.

3. Positive limiting sets and behaviour at infinity.

In order to avoid some undesired behaviour of trajectories near infinity we adopt the following hypothesis in which follows. H_2. There is a finite number of singular points of (S), if any. H_3. For any fixed P and any $Q_n \in I^*(P)$ $(n = 1, 2, \ldots)$ the boundness of $L(P, Q_n)$ implies boundness of $|P - Q_n|$, where $|\ \ |$ indicates any norm on R^2.

Let us remark that H_3 will be satisfied if

$$\int^{\infty} \min_{|x|=r} |f(x)|\, dr = \infty. \tag{3.1}$$

The following result is a crucial point of our argument.

THEOREM 3.1. — *Suppose* H_1, H_2 *and* H_3 *hold true. Let* I(P) *be a non-singular trajectory of* (S) *and assume* $\omega(I(P))$ *does not reduce to a singular point, hence is empty or contains regular points.*

Then

i) there is an open segment $I^*(Q_1, Q_2)$ of the orthogonal trajectory $I^*(P)$, containing P and such that for any

$$Q \in I^*(Q_1, Q_2), \qquad \omega(I(Q))$$

does not reduce to a singular point,

ii) for each $Q \in I^*(Q_1, Q_2)$ the function $s(t)$ of section 1 (cf. p. 3) can be defined for $t \in [0, b(P))$ and $s(t) \to b(Q)$ as $t \to b(P)$, and the function $\varphi(t)$ in (1.3) in non-increasing.

Theorem 3.1. can be derived easily from Proposition 1.2. Let us remark that assumption H_3 is needed only in the case I(P) is unbounded for $t > 0$. For such trajectories the above result has been proved in slightly less general form in [1] (cf. Lemma 2). For a generalization of it to n-dimentional case cf. [3].

In the case I(P) is a closed orbit, hence $x(t, P)$ is periodic, it follows from Theorem 3.1 that for any neighbouring solution $x(t)$ there is a function $s(t)$ such that either

$$|x(s(t)) - x(t, P)|$$

is periodic in t or $|x(s(t)) - x(t, P)|$ tends to zero as $t \to +\infty$. The first case is the only possible for $x(t)$ from inside I(P).

The second case takes place if and only if $\operatorname{div} f(x)$ is not identically zero in any neighbourhood of I(P). If this is the case then I(P) is a one side limit cycle.

Thus we have for any closed orbit I of (S) : either I is a one side limit cycle for positive t or there is a neighbourhood of I filled up by closed orbits.

Notice that inside of each limit cycle there must be at least one singular point of (S) and there cannot be any other limit cycle. Hence if H_2 is assumed, then there is a finite number of limit cycles if any.

As for a neighbourhood of infinity we have the following concequence of Theorem 3.1. Following V. V. Niemytskii (cf. [4], p. 434) we say that (S) has a saddle point at infinity or an improper saddle point if there is a sequence of points P_n and two sequences t_n and τ_n of reals, $0 < \tau_n < t_n$ such that $P_n \to P$, $x(t_n, P_n) \to Q$ while $x(\tau_n, P_n)$ does not contain any convergent subsequence, hence $|x(\tau_n, P_n)| \to \infty$.

We have the following result.

Theorem 3.2. — *If* $H_1 - H_3$ *hold then there is no saddle point at infinity.*

Theorem 3.1. shows that the generalized saddle points and the trajectories of (S) which tend to a saddle point as $t \to +\infty$ are exceptional in the sense that the neighbouring trajectories may behave differently as $t \to +\infty$. This is the very property of system (S) which leads to the global phase-portrait described in the next section.

4. The global structure of (S).

Le us assume $H_1 — H_3$. Denote by Z the set composed by the union of all generalized saddle points of (S), of all trajectories of (S) tending to a saddle point as $t \to +\infty$ and of all limit cycles of (S). Since there is a finite number of saddle points and limit cycles, if any, therefore Z is closed in R^2. Let W be the complement of Z. Therefore W is open. One can prove that W is a union of a finite number of regions, connected components, say, $W_1, \ldots, W_m$. Of course each

$$W_i (i = 1, \ldots, m)$$

is invariant for (S), that is if $P \in W_i$ then $I(P) \subset W_i$.

Let us fixe W_i, for simplicity we shall write simly W without any subscript. Let us distinguish three cases

A. W. is bounded.

B. W is unbounded and contains a singular point of (S).

C. W is unbounded and does not contain any singular point of (S).

In the case A we have two subcases.

A_1. W contains a singular point. Then it contains exactly one singular point and it is a center. In this case W can be mapped topologically onto a two-cell $|x| < r_0$ in such a way that the singular point goes into $x = 0$ and any other trajectory contained in W is mapped onto a circumference $|x| = r < r_0$.

A_2. W does not contain any singular point of (S), then W is filled up with closed orbits and topologically it is an annulus $0 < r_1 < |x| < r_2$ with trajectories beeing the circumferences $|x| = r$, $r_1 < r < r_2$.

In the case B we have also two subcases.

B_1. The singular point contained in W is a point of attraction, then W is the set of attraction to this point, that is the union of all trajectories tending to the point of attraction as $t \to +\infty$.

B_2. W contains a center, then $\operatorname{div} f(x) = 0$ on W and any trajectory of (S) within W is a closed orbit. Actually, W must be identical with R^2 in this case and there is a homeomorphism of R^2 onto R^2 mapping the center into 0 and any trajectory of (S) onto a circumference $|x| = r$.

At last, let us consider the case C. The picture now may be as follows.

C_1. W contains a solution curve I of (S) such that $\omega(I) = \emptyset$, then the same holds for any trajectory contained in W and W can be mapped topologically onto R^2 such that each trajectory of (S) contained in W has as the image a straight line parallel to a fixed direction. Hence the family of trajectories of (S) composing W is, as we say, parallelizable.

C_2. W contains a trajectory I of (S) for which $\omega(I) \neq \emptyset$. Then $\omega(I)$ is bounded and we have two subcases :

C_{21}. $\omega(I) \neq I$. Hence $\omega(I)$ is either a limit cycle or a closed trajectory-polygon and W is identical with the union of all trajectories of (S) which spirals towards $\omega(I)$ as $t \to +\infty$ (W is the set of attraction of $\omega(I)$, as one may say).

C_{22}. $\omega(I) = I$, then each trajectory of (S) within W is a closed orbit and the boundary of W reduces to a bounded trajectory-polygon. Hence the complement of W is bounded. Again as in A_2, topologically W is an annulus $0 < r_0 < |x|$ with trajectories beeing circumferences $|x| = r$, $r > r_0$.

Among seven posibilities we have listed for the phase-portrait of a connected component W_i of the set W, B_2 excludes any other and C_{22} excludes all but A_1 and A_2. Notice also that A_1, A_2, B_2, C_{21} and C_{22} (either one of them) implies the existence of a center for (S). Therefore if there is no center then only B_1 and C_1 are possible cases for the picture within any W_i.

If either one of A_1, B_1, B_2 or C_1 is the case for a particular W_i, then W_i must be simply connected. In the remaining cases W_i is always homeomorphic to an annulus. It follows from the above discussion also that each W_i can contain one singular point of (S), at the most and it does contain in the cases A_1, B_1 and B_2. Therefore the number of stable singular points of (S) can be evaluated if we are able to estimate the number m of connected components of W.

We may state now the following theorem.

THEOREM 4.1. — Assume H_1, H_2 and H_3 and consider three cases: i) $\operatorname{div} f(x) \equiv 0$ on R^2, ii) $\operatorname{div} f(x) \not\equiv 0$ on R^2, iii)

$$\operatorname{div} f(x) \not\equiv 0$$

on any neighbourhood of each singular point of (S). Let Z and W be as defined above. Then Z is closed. W is open and composed of a finite number of components W_i each of them is homeomorphic either with $\{x: |x| < 1\}$ or with

$$\{x: 0 < |x| < 1\}.$$

Moreover in the case i) we have: there is no limit cycle of (S), if there is a closed orbit inside which are all singular points of (S), then Z is bounded and if Z is empty then B_2 holds for $W = R^2$, if Z is not empty, then there is exactly one component of W, say W_1, which is unbounded and for W_1, C_{22} is the case while for any W_i $(i = 2, \ldots, m)$ either A_1 or A_2 holds. If there is a solution curve I of (S) such that $\omega(I) = \emptyset$, then on any unbounded W_i the picture is as in C_1 and of course for each bounded W_i A_1 or A_2 holds.

In the case (ii) the phase portrait within any W_i is one of A_1, A_2, B_1, C_1 and C_{21}.

Finally, in the case (iii) any stable singular point has to be asymptotically stable, thus there is no center of (S), each W_1 is unbounded and the phase-portrait within W_i is either that of B_1 or that of C_1.

We close up the discussion with a special case in which we are able to evaluate the number m of components of W provided we know the number of saddle points. We have the following.

THEOREM 4.2. — *Suppose* $H_1 — H_3$ *hold and let* $x^{(1)}, \ldots, x^{(N)}$ *are the singular points of* (S). *Assume additionally that*

$$(4.1) \quad \operatorname{div} f(x) < 0, \quad \det J(x) \neq 0 \text{ for } x = x^{(i)}, i = 1, \ldots, N,$$

where $J(x)$ *is the Jacobian matrix for* $f(x)$.

If the number of saddle points of (S) is $n (n \leqslant N)$, then the number m of components W_i of W is equal $m = n + 1$, and each W_i is a set of attraction to a singular point of attraction or the family of trajectories contained in W_i is parallellizable. We have also the inequality

$$N \leqslant 2n + 1,$$

therefore if $N = 2n + 1$ then each W_i contains a singular point and each W_i must be a set of attraction. In the last case the system (S) has the following property : for any solution curve I of (S) the positive limiting set $\omega(I) = \{x_0\}$, where x_0 is a singular point of (S).

BIBLIOGRAPHY

[1] C. OLECH, On the global stability of an autonomous system on the plane, *Contributions to Differential Equations*, vol. I (1963), 389-400.

[2] S. LEFSCHETZ, *Differential equations: geometric Theory*, Interscience Publishers, New York, London, (1957).

[3] P. HARTMAN and C. OLECH, On global asymptotic stability of solutions of differential equations, *Trans. Amer. Math. Soc.*, 104 (1962), 154-178.

[4] V. V. NIEMYTSKII and V. V. STIEPANOV, *Qualitative theory of Differential equations* (Second edit., Russian), Moskow, (1949).