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CONNEXIONS ASSOCIATED WITH FOLIATED STRUCTURES

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1. — It has been suggested by A. G. Walker that the problem of calculating concomitants associated with any given structure on a differentiable manifold is closely related to that of determining global connexions associated with the structure. Usually it is not possible to associate a unique connexion with the structure, but it is possible to construct a class of connexions. From these connexions may be constructed tensor fields and operators which are independent of the particular connexion from the class and which are therefore invariants of the structure.

Walker's method has been applied with success to almost-product structures, including complex and almost-complex structures as particular cases. The method is also applicable to manifolds which admit a foliated or multi-foliated structure. For example, Ehresmann's foliation group appears as the holonomy group of the special class of connexions associated with the foliated structure.

The detailed calculations of Walker's work in this direction are given in a series of papers [2], [3], [4], [5] and [6]. I shall summarize here sufficient of the results to enable me to raise a number of questions which appear to be of interest. For reasons of simplicity of exposition, I shall consider only manifolds which admit a foliated structure.

Let the C^∞-manifold M admit a system of distributions T′, T″ such that (i) at any point the planes of T′, T″ are disjoint (that is, they have only the zero vector in common) and (ii) $T' + T'' = T$ where T is the distribution of tangent planes over M. We shall follow the notation used in [6]. In particular, if h, k are vector 1-forms, then hk denotes the vector 1-form given by $hk(u) = h(ku)$ where u is a tangent vector field.

Moreover, if P is a tensor field of type (1, 2) and h is a vector 1-form, three other tensor fields can be constructed from P and h as follows:

$$hP(u, v) = h(P(u, v)), \qquad Ph(u, v) = P(hu, v),$$
$$P.h(u, v) = P(u, hv),$$

where u and v are vector fields.

The system $\{T', T''\}$ determines projectors a', a'' which are vector 1-forms satisfying

$$(1) \qquad a'^2 = a', \qquad a''^2 = a'', \qquad a'a'' = a''a' = 0,$$
$$a' + a'' = \underset{\sim}{1}.$$

Such a structure admits a torsion tensor H, a skew-symmetric tensor field of type (1, 2), defined by

$$(2) \qquad H = \frac{1}{2}\, a'[a', a'] + \frac{1}{2}\, a''[a'', a'']$$

where $[h, k]$ is a skew-symmetric tensor field, the Nijenhuis product of the vector 1-forms h and k given in [1]. An alternative expression for the torsion is

$$(3) \qquad a'Ha''.a'' + a''Ha'.a'.$$

The condition for the distribution T′ to be integrable is

$$(4) \qquad a''Ha'.a' = 0.$$

A foliated structure consists essentiably of a manifold M which admits an integrable distribution T′ of plane fields. Let $\underset{\sim}{g}$ be a positive definite Riemannian metric of class ∞ defined over M (such metrics are known to always exist) and define T″ to be the complementary distribution to T′ with respect to $\underset{\sim}{g}$. Then the given foliated structure gives rise to an almost-product structure $\{T', T''\}$ whose torsion tensor H satisfies equation [4].

2. Special connexions.

Associated with the almost-product structure $\{T', T''\}$ are certain affine connexions on M called *special connexions*, defined as follows:

A global affine connexion Γ is special if it is torsion-free and such that T', T'' are parallel with respect to the connexion $L = \Gamma + H$.

Let P be a tensor field of type (r, s) and let L be any affine connexion over M. The covariant derivative of P with respect to L is denoted by $\dot{P}(L)$ and is a tensor field of type $(r, s+1)$. We define a tensor field $A(L)$ of type $(1, 2)$ as follows:

$$A(L) = a'\dot{a}'(L) + a''\dot{a}''(L). \tag{5}$$

Then it can be proved that *if* Γ *is a global torsion-free connexion, then* Γ *is special if and only if* $A(\Gamma)$ *is skew-symmetric, and that when this condition is satisfied then* $A(\Gamma) = H$.

Let Γ' be any torsion-free connexion defined globally over M; for example, Γ' could be taken to be the Christoffel connexion corresponding to the metric tensor $\underset{\sim}{g}$. Then Walker has proved that $\Gamma = \Gamma' + X$ is a special connexion where

$$X = B + a'(Ba' + B.a') + a''(Ba'' + B.a'') + a'Ca' + a''Ca'', \tag{6}$$

B is the symmetric part of $A(\Gamma')$ and C is any symmetric tensor field of type $(1, 2)$ defined on M. Moreover, every special connexion is given by (6) for some symmetric tensor field C.

Special connexions have the following geometrical properties. The distributions T', T'' are both *path-parallel* with respect to a special connexion Γ i.e. every path of Γ which is tangent to T' at some point is an integral curve of T', and similarly for T''. Moreover T' is parallel relative to T'' with respect to any special connexion Γ, and T'' is parallel relative to T'.

In particular when M admits an integrable distribution T', then the leaves form geodesic submanifolds with respect to any special connexion. Moreover, the parallelism of T'' relative to T' is strict and is independent of the particular special connexion chosen. Thus, in a sufficiently small neighbourhood of M, there is a unique mapping of T''_x onto T''_y where x, y are any two points on the same leaf. In particular, parallel transport relative to a small closed arc through x, lying in the leaf F_x through x, will leave T''_x vectorwise invariant. Also, if y is any point on F_x, the mapping $T''_x \to T''_y$ given by parallel transport along any arc xy in the leaf F_x depends only upon the homotopy class of the arc xy in the intrinsic topolopy of F_x. In particular we obtain a group of automorphisms of T''_x given by parallel transport round closed arcs in the leaf F_x. This group is a factor group of the fundamental group of F_x and can be identified with Ehresmann's foliation group. Thus the foliation group of a leaf is realised by the holonomy of special connexions associated with the foliated structure.

3. Completeness relative to special connexions.

Suppose now that the leaf F_x is complete with respect to all special connexions. Then this property, being independent of the particular connexion chosen, must be an invariant property of the structure.

We say that the leaf F_x is *specially complete* if it is complete with respect to all special connexions associated with the foliated structure. The problem arises of characterising foliated structures whose leaves are specially complete.

Consider the real cartesian plane R^2 and the foliation given by the leaves $x =$ constant. At each point (x, y), T' is parallel to Oy and T'' parallel to Ox.

Consider now R^2 metrized by the riemannian metrics

A : $$ds^2 = dx^2 + dy^2.$$

B : $$ds^2 = dx^2 + dy^2/(1 + y^2)^2.$$

It is easy to see that the riemannian connexions corresponding to both metrics A and B are special connexions of the structure. Moreover, the leaf $x = x_0$ is complete with respect to A but incomplete with respect to B. The latter assertion follows because if l denotes the length along the leaf $x = x_0$ from the point $(x_0, 0)$ we have

$$l = \int_0^y \frac{dy}{(1 + y^2)} = \tan^{-1} y,$$

and as $y \to \infty$ so $l \to \frac{1}{2}\pi$.

In other words, it is impossible to prolong the geodesic $x = x_0$ in the direction of increasing y more than a distance $\frac{1}{2}\pi$. Thus we have a very simple example of a foliation none of whose leaves is specially complete.

It seems reasonable *to conjecture that if a leaf is specially complete, then it is necessarily compact.* This, however, remains an open question.

4. Metric special connexions.

Another problem concerns conditions when a special connexion can be the Christoffel connexion of a positive definite riemannian metric. In this case the almost-product structure must have zero torsion, as both T′ and T″ would necessarily be integrable. Moreover, if dim $T' = r$, the manifold M would admit a harmonic r-form. Thus for a compact orientable manifold M, the r^{th} Betti number must be necessarily non-zero. A set of sufficient conditions is unknown.

Since the Euler characteristic of the 3-sphere S^3 is zero, we know that S^3 admits a non-singular vector field T′. However S^3 cannot admit a metric connexion specially related to any structure $\{T', T''\}$, since the first Betti number of S^3 is zero.

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