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SOME PROPERTIES OF FOLIATIONS (*)

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1. Introduction.

Reeb [4] pointed out that if a manifold has a foliated structure of co-dimension one it is reasonable to classify its leaves as proper, locally dense, or exceptional, and he raised the question whether all leaves must be proper or locally dense if the manifold is compact and the foliation is of class C^2. At that time, an affirmative answer to the question was known for the case of oriented foliations of the torus, Denjoy [2], [7], [8]. Reeb showed that the arguments used in the torus case can also be used, in certain situations, in higher dimensions.

Recently, A. J. Schwartz has generalized the Denjoy theorem. His theorem can be stated as follows:

THEOREM (A. J. Schwartz [6]). — *Let* M *be a compact two-dimensional manifold. Suppose that a vector field* X *of class* C^2 *is defined on* M *and that* $x = x(t)$ *is a solution curve of* X *which contains no singularities in in its closure. Then, either* $x = x(t)$ *is dense in* M *and* M *is the torus, or the* α *and* ω *limit sets of* $x = x(t)$ *are closed orbits.*

Schwartz's Theorem can easily be extended to the case where the foliation of M is not oriented and M is not compact, provided it is assumed that closure of the orbit $x = x(t)$ is compact.

Here we shall investigate the possibility of applying some of Schwartz's techniques to questions which are related to

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the one Reeb has raised. The main difficulties which arise in dimensions greater than two are: 1. It is no longer true that a leaf is either compact or has a trivial holonomy group. 2. The pseudo-group which acts on a transversal arc is no longer generated by a single element.

2. General Remarks.

This section is devoted to some general remarks. Let M be a manifold of dimension $n (n \geq 2)$ with a foliated structure of co-dimension $k (0 < k < n)$. If L is any leaf of the foliation, one can obtain in the usual way a homomorphism from the fundamental group $\pi_1(L)$ of L to the linear holomony group of the leaf. In a given coordinate system, an element of the linear holonomy group is represented by a $k \times k$ matrix. Let $M(\gamma)$ be the matrix corresponding to the element $\gamma \in \pi_1(L)$. Then the map $\gamma \to \operatorname{Log} |\det M(\gamma)|$ defines a homomorphism from $\pi_1(L)$ to the additive group of the real numbers; that is, it defines a real cocycle $\theta_L : H_1(L, R) \to R$. The cocycle θ_L is clearly independent of the coordinate system.

There is another way to obtain a cocycle on each leaf of a foliation. Suppose that the 1-forms $\omega^1, \ldots, \omega^k$ are orthonormal with respect to some metric and define the foliated structure locally. The integrability conditions imply that there are 1-forms θ^i_j such that $d\omega^i = \sum_{j=1}^{k} \theta^i_j \wedge \omega^j$ holds for $i = l, \ldots, k$. Let ω be the k-form defined by $\omega = \omega^1 \wedge \ldots \wedge \omega^k$. Simple calculations show that $d\omega = \theta \wedge \omega$, where $\theta = \sum_{i=1}^{k} \theta^i_i$, and the restriction of θ to the leaf L is independent of the choice of $\omega^1, \ldots, \omega^k$. This implies that, given a metric, the restriction of θ to L, say $\theta | L$ is well defined globally on L; however, $\theta | L$ can be an odd form if L is not orientable. Since

$$d^2\omega = 0 = d\theta \wedge \omega,$$

$\theta | L$ is a de Rham cocycle on L, and another simple computation shows that the cohomology class of $\theta | L$ is independent of the metric on M. In fact, one can prove the following, cf. [3] p. 115.

THEOREM 1. — θ_L *and* $\theta|L$ *are in the same cohomology class.*

The application made of $\theta|L$ in the proof of Theorem 2 requires only the case $k = 1$; however, the cocycle $\theta|L$ arises in other connections too. For example, Chern's condition [1] that the quotient space of a Lie group by a closed subgroup have an invariant measure is that the form $\theta|L$ should be identically zero if $\omega^1, \ldots, \omega^k$ are chosen to be invariant under the action of the group.

3. The Main Theorems.

M will denote a manifold of dimension $n (n \geq 2)$ with a C^2 foliated structure of dimension $n - 1$. If $x \in M$, L_x will denote the leaf containing x, and C_x the closure of L_x in M. A compact subset $C \subset M$ will be called *minimal* if $x \in C$ implies that $C_x = C$.

THEOREM 2. — *Let* $x \in M$ *be such that*

1) C_x *is minimal and* 2) *for every* $y \in C_x$, *the first real Betti number of* L_y *is zero. Then either* $C_x = M$ *or* C_x *is a compact leaf.*

Since every compact set C such that $x \in C$ implies $L_x \subset C$ contains a minimal set, one can weaken the hypothesis 1) to « C_x is compact » if the conclusion is weakened to « either $C_x = M$ or C_x contains a compact leaf ».

Theorem 3 is somewhat more in the spirit of Reeb's Theorems [4].

THEOREM 3. — *Let* M *be a* C^2 *fibre space with fiber* S^1 *and base space* B, *where* B *is an* $(n - 1)$*-manifold. Suppose that the foliation of* M *is transversal to the fibers and the fundamental group of* B *is Abelian. Then if* $x \in M$ *is such that* C_x *is minimal, either* $C_x = M$ *or* C_x *is a compact leaf.*

4. Lemmas.

Two techniques can be used to prove Theorem 2. One is based on an analysis of the action of a certain pseudogroup on a small arc transversal to the leaves of the foliation near a point. This method gives a proof which is more analagous to Schwartz's proof of his theorem. The other technique, which seems simpler, is based on the following lemma.

LEMMA 1. — *Let* $x = x(\xi, t)$ *be the solution of the differential equation* $x' = F$ *satisfying* $x(\xi, 0) = \xi$, *where* F *is a continuous function defined on the set* $R = \{(x, t): |x| < a, 0 \leqq t < T\}$. *Suppose that* $\partial F/\partial x$ *exists and satisfies*

$$|\partial F/\partial x(x_1, t) - \partial F/\partial x(x_2, t)| \leqq A|x_1 - x_2|$$

for all

$$(x_1, t), (x_2, t) \in R,$$

where A *is a constant. Moreover, suppose that* $F(0, t) \equiv 0$,

$$B = \int_0^T \exp\left\{\int_0^t \partial F/\partial x(0, s)ds\right\} dt < \infty,$$

and

$$C = \sup\left\{\exp\int_0^t \partial F/\partial x(0, s)ds : 0 \leqq t < T\right\} < \infty.$$

Let $\lambda > 1$ *be a constant. Then* $x(\xi, t)$ *exists for all* t, $0 \leqq t < T$ *and* $\partial x/\partial\xi(\xi, t) \leqq \lambda\partial x/\partial\xi(0, t)$ *if* $|\xi| < |\xi| < \lambda^{-1}\operatorname{Min}(a/C, \operatorname{Log}\lambda/AB)$, $\operatorname{Min}(a, a/C, \operatorname{Log}\lambda/AB)$.

In the application of the lemma to the proof of Theorem 2 the rectangle R is immersed in M in such a way that the x-direction is always orthogonal to the leaves of the foliation and the image of the set where $x = 0$ is a geodesic on a leaf.

The proof of Theorem 3 is somewhat easier than the proof of Theorem 2 because one is able to deal with a group of diffeomorphisms of S^1, rather than with a pseudo-group. In fact, Theorem 3 is implied by Lemma 2.

LEMMA 2. — *Let* G *be a finitely generated Abelian group of* C^2 *diffeomorphisms of* S^1. *Then any minimal set under* G *is either finite or is all of* S^1.

5. An Extension of Theorem 2.

The purpose of the assumption 2) in Theorem 2 is to assure that the cocyles $\theta|L_\gamma$ are zero, where θ is the form defined in section 2. One could, of course, postulate this condition directly. More generally, it can be shown that 2) cen be replaced by the following weaker hypothesis in Theorem 2.

2) *For every* $y \in C_x$, *either* $\theta | L_y$ *is cohomologous to* 0, *or there are cycles* h_1 *and* h_2 *on* L_y *such that* $\int_{h_1} \theta | L_y$ and $\int_{h_2} \theta | L_y$ *are rationally independent.*

6. A Counterexample.

(added September, 1963).

Theorems 2 and 3 give cases in which there cannot exist exceptional leaves of a foliation of co-dimension one. However, it is possible to construct an example of a compact 3-manifold M with a C^∞ foliation of co-dimension one, which has an exceptional leaf. Thus the answer to Reeb's question is, in general, no. In the example, $M = S^1 \times B$, where B is a 2-sphere with two handles. The foliation is transversal to S^1 in M. Therefore, all of the hypotheses of Theorem 3 are satisfied except that the fundamental group of B is, of course, not Abelian. The details will be given in [5].

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