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ON SETS FILLED BY ASYMPTOTIC SOLUTIONS
OF DIFFERENTIAL EQUATIONS

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Consider an ordinary differential equation

\[ x' = f(t, x) \]

\[ x = (x_1, \ldots, x_n), \quad f = (f_1, \ldots, f_n). \]

Assumption I. Suppose that the domain D of \( f(t, x) \) is open, \( f(t, x) \) is continuous on D and through each point \( A : t = a_0, \quad x = a = (a_1, a_2, \ldots, a_n) \) of D passes only one integral \( x = x(t, A) \) of (1).

Denote by \((\alpha(A), \beta(A))\) the maximal interval on which there exists the integral passing through \( A \). We shall denote

\[ X(t, A) = (t, x(t, A)) \quad \text{for} \quad t \in (\alpha(A), \beta(A)). \]

Let \( E \) be an open subset of D. In the following we shall deal with the set \( Z(E) \) of such points \( A \), that \( X(t, A) \in E \) for \( a_0 \leq t < \infty \). Obviously set \( Z(E) \) depends on both set \( E \) and system (1). It is evident that \( E \subset F \) implies \( Z(E) \subset Z(F) \). Let \( \varphi \) be a family of subsets \( F \) of D. We shall consider the following properties of equation (1).

Property I (of equation (1) in respect to \( E \) and \( \varphi \)). — For every \( F \in \varphi \) \( Z(E) \cap F \) is empty or consists of one point.

Property II. — For every \( F \in \varphi \) \( Z(E) \cap F \) is not empty.

Let \( I^+(A) \) denote the set of all points \( B = X(t, A) \) for \( t \geq a_0 \).
We say that the point \( A \in P(G) \cap D \), where \( P(G) \) denotes the boundary of an open set \( G \), is the point of egress from \( G \) (with respect to equation (1) and set \( D ) \) if there exists such an integral \( x(t) \) of (1) and a positive number \( \varepsilon > 0 \) that
\[
x(a_0) = a \quad \text{and} \quad (t, x(t)) \in G
\]
for \( a_0 - \varepsilon < t < a_0 \) (under Assumption I, \( X(t, A) \in G \) for \( a_0 - \varepsilon < t < a_0 \)). If no point of \( P(G) \) is a point of egress from \( G \) then \( A \in G \) implies \( I^+(A) \subset G \). If Property I is satisfied and \( B \in Z(E) \cap F \) then \( (F - B) \cap Z(E) = \emptyset \), where \( F - B \) denotes the set of all points of the set \( F \) except the point \( B \).

It follows that for every \( A \in F \), \( A \neq B \) either \( I^+(A) \sim e \in E \) or \( \beta(A) < \infty \). Let \( G \) be such a set that \( G \cap E \) has no common point with a plane \( t = c > a_0 \), where \( \overline{G} \) denotes the closure of \( G \), then \( I^+(A) \subset \overline{G} \) implies \( A \sim e \in Z(E) \).

**Lemma.** — Suppose Assumption I and the following conditions. For each set \( G_i (i = 1, \ldots) \cap E \) is contained in a halfspace \( t < c_i \). No point of \( P(G_i) \) is a point of egress. Set \( F \) satisfies inclusion \( F - 0 \subset \bigcup_{i=1}^{\infty} G_i \).

Then \( (F - O) \cap Z(E) = \emptyset \).

**Theorem 1.** — Suppose Assumption I and the following conditions. The intersection \( E(s) \) of a given set \( E \) and the plane \( t = s \) satisfies the inequality \( \text{diam}(E(t)) < p(t) \), where \( p(t) \) is a positive function continuous on \( (-\infty, \infty) \). No point of \( P(G_i) \) is a point of egress in respect to the equation
\[
x' = f(t + a_0, x + a(t)) - f(t + a_0, a(t)),
\]
where \( a_0 \) is a real number and \( x = a(t) \) is such a Lipchitzian function that the right side of the equation is defined. Set \( F \) satisfies inclusion \( F - O \subset \bigcup_{i=1}^{\infty} G_i \). For any \( i \) and \( s \) there exists a constant \( c(i, s) \) that dist \( (G_i(t), 0) \geq p(t + s) \) for \( t \geq c(i, s) \), where \( G_i(s) \) is the intersection of \( G_i \) and the plane \( t = s \).

Under these assumptions if \( A \in Z(E) \), then
\[
(F(A) - A) \cap Z(E) = \emptyset,
\]
where \( F(A) \) denotes set obtained from \( A \) by translation of \( \mathbb{R}^{n+1} \) transforming \( O \) on \( A \).
Theorem 2. — If assumptions of Theorem 1 are satisfied and $F$ is a plane then equation (1) possesses property I in respect to $E$ and the family of planes parallel to $F$ (and of the same dimension).

Suppose now that set $F$ is a plane and in the coordinate system $t, x = (u, \nu), u = (u_1, \ldots, u_k), \nu = (\nu_1, \ldots, \nu_{n-k})$ it has the equation $t = 0, u = 0$. Now Property I (for the family of planes $t = c_0, u = (c_1, \ldots, c_k), c_i$ arbitrary) is necessary and sufficient for set $Z(E)$ to be the graph of a single-valued function $\nu = q(t, u)$. Putting $g = (f_1, \ldots, f_k), h = (f_{k+1}, \ldots, f_n)$ system (1) takes the form

$$u' = g(t, u, \nu), \quad \nu' = h(t, u, \nu).$$

The following result formulated in terms of inequalities can be obtain from Theorem 1 formulated in terms of sets (1)

Theorem 3. — Suppose that system (2) satisfies Assumption I and that the functions $g(t, u, \nu), h(t, u, \nu)$ for

$$(t, u, \nu) \in D, \quad (t, \bar{u}, \bar{\nu}) \in D$$

satisfy inequalities

$$g(t, u, \nu) - g(t, \bar{u}, \bar{\nu}) \leq \gamma(t) (u - \bar{u})^2$$

for $|\nu - \bar{\nu}| = |u - \bar{u}|$, where $|z|$ denotes Euclidean distance of point $z$ from 0,

$$(h(t, u, \nu) - h(t, \bar{u}, \bar{\nu})) (\nu - \bar{\nu}) \geq \gamma(t) (\nu - \bar{\nu})^2,$$

for

$$|u - \bar{u}| \leq |\nu - \bar{\nu}|,$$

where $\gamma(t)$ is a function summable in every finite interval, and such that

$$\int_0^\infty \gamma(s) \, ds = \infty,$$

then set $Z$ of points $A$ lying on the integrals of (2) (remaining in $D$) bounded for $a_0 \leq t < \infty$ is a graph of a single-valued function $\nu = q(t, u)$ defined in a certain set $S(S \subset \mathbb{R}^{k+1})$ satisfying the Lipschitz condition with respect to all the variables.

(1) Such kind of formulation was suggested by T. Wazewski.
and in particular the condition
\[ |q(t, u) - q(t, \bar{u})| \leq |u - \bar{u}| \]
in the set S or the set Z is an empty set.

Theorem 3 is a particular case of theorem 2 in [1].

Now for illustration of Property II we present a variant of an example from [2].

Let system (2) satisfy Assumption I on a neighbourhood D of the set H: \(|u| \leq 1, |\varphi| \leq 1, -\infty < t < \infty\). Moreover suppose that \(g(t, u, \varphi)u < 0\) for \(|u| = 1, |\varphi| \leq 1\) and arbitrary \(t, h(t, u, \varphi) > 0\) for \(|\varphi| = 1, |u| \leq 1\) and arbitrary \(t\).

Under these assumptions for every \(\bar{u}, |\bar{u}| < 1\) and arbitrary \(\bar{t}\), there exists \(\bar{\varphi}\), that \(I^+(\bar{t}, \bar{u}, \bar{\varphi}) \in H\).

BIBLIOGRAPHY
