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EXTREME HARMONIC FUNCTIONS AND BOUNDARY VALUE PROBLEMS

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INTRODUCTION

The extreme elements of any base of the cone of positive harmonic functions on any open domain of R^n play an important role in the theory of harmonic functions. R. S. Martin [19], while generalising the Poisson-Stieltjes formula, gives an integral representation for non-negative harmonic functions with measures on the set of extreme elements of a base of the set of non-negative harmonic functions. Martin proves that all the extreme functions (and in fact some other functions too) form boundary elements of D in a suitable metric topology (which induces on D the topology of the euclidean space); this boundary is known after Martin as the Martin boundary (1) of D. M. Brelot [2, 3, 4, 5] continues the study of the (Martin) boundary and further extends the results to the case of Green spaces. He considers the Dirichlet problem (the first boundary value problem) on any Green space for the Martin boundary and moreover the relativized problem with the limits at the boundary of quotients of functions by a fixed positive harmonic function h. He demonstrates that for continuous functions on the boundary the solution by the Perron method exists, even in the case of the latter problem and for every h.

(1) In fact Martin arrived at this boundary by considering the possible limits of $G(x, y)/G(x, y_0)$ where G is the Green's function of the domain. He introduced the extreme elements (not in the modern sense) but as minimal functions.

L. Naïm [21] introduces, with the help of the Green potentials a new definition of thinness at elements of the Martin boundary Δ of a Green space Ω, and proves that the whole space is not thin only at the extreme elements of Δ. Suppose Δ_1 is the set of extreme elements in Δ. She gives a criterion for the thinness of any set E in Ω at an element h in Δ_1, viz., the reduced function of h on E (which is the infimum of all non-negative superharmonic functions on Ω majorising h on E) is not identically equal to h. The sets E in Ω such that $\complement$ E is thin at h form a filter $\mathfrak{T}_h$, called the fine filter corresponding to h. With these filters she proves the existence of a topology on $\Omega \cup \Delta$ such that (i) Ω has for the induced topology its fine topology and (ii) the trace on Ω of the filter of neighbourhoods of h in Δ_1 is the fine filter $\mathfrak{T}_h$. The limits at Δ_1 of functions on Ω in this topology, that is the limits following the filters $\mathfrak{T}_h$ are called the fine limits and they play a fundamental role in the discussions. Another form of Dirichlet problem with the aid of the fine limits is posed which enables a further study of the problem. In the study of all these questions Naïm makes heavy use of the symmetry of Green's function as well as the existence of the Lebesgue measure (at least locally).

In the consideration of Brownian motion on a Green space J. L. Doob [11] studies, by probabilistic methods the behaviour of positive superharmonic functions (and also the quotient of such functions by a fixed harmonic function >0) along the Brownian paths, and proves the existence of limits along « almost » all paths. Expressed differently, the result is that any quotient of the form v/h as considered above has a fine limit « almost everywhere » in a suitable sense at Δ_1. In [12] he gives, by using heavily the results of Naïm, a non-probabilistic proofs of these results which are similar to the classical theorems of Fatou type on the existence of angular limits [14, 13bis] and actually these have been recently proved to be a real generalisation by M. Brelot and J. L. Doob [8].

Our interest is to consider the situation in the case of axiomatic theories of harmonic functions on a locally compact Hausdorff space (which include the study of solutions of certain types of elliptic and parabolic equations). We shall extend the *earlier results of the classical case to this set up* (1 bis). Many proofs carry over; but some of the fundamental points need, entirely different approach.

Our starting point consists in taking for definition the criterion of thinness (mentioned above) in terms of the reduced functions. More generally we consider first (in Ch. I) on any set X two families of non-negative valued functions P and U (corresponding to potentials and non-negative harmonic functions on a domain) satisfying a few conditions (abstracted from the properties of superharmonic functions $\geqslant 0$). Then we define the thinness of a set E relative to any minimal element h in U (2), by h not identically equal to the reduced function of h on E (suitably defined as in the classical case). The sets E contained in X such that the complement of E is thin relative to h form a filter $\mathfrak{T}_h$ on X and is called the fine filter corresponding to h; some simple limit theorems extend without any difficulty. The limits following the filters $\mathfrak{T}_h$ are called the fine limits at the minimal elements. By choosing suitably U and P we deduce the corresponding results in the case of the axiomatic theories of Doob [13], Brelot [6] and Bauer [1]. Moreover, in the last two cases, where the harmonic functions $\geqslant 0$ and the potentials play the role of U and P respectively, we prove the existence of a topology on $X \cup \Delta_1$ (where Δ_1 is the set of minimal functions on any base of the cone of positive harmonic functions), such that the induced topology on X is its fine topology and the trace on X of the filter of neighbourhoods of h in Δ_1 is exactly $\mathfrak{T}_h$.

In order to study the question further, we have to suppose more. We restrict our attention to the axiomatic theory of Brelot with the important developments of Mrs R. M. Hervé. We shall suppose suitably strong conditions. The theory

(1 bis) The limits considered above along brownian paths are in fact particular cases of a more general study of the Markov processes by Doob and Hunt. The relation to these processes of the extensions of Fatou-Naïm-Doob results that we shall consider is becoming clearer thanks to a paper of Meyer in this volume: this author succeds to include the axiomatic study of Brelot [6] in the general study of Hunt.

(2) An element in U is minimal if it is proportional to every element of U which it majorises. The minimal elements are exactly the elements which are not the sum of two non-proportional elements in case U has the property that u, w in U and $u \leqslant w$ implies w-u is in U.

starts with the assignment to every open set of a locally compact, connected and locally connected Hausdorff space Ω with a countable base, of a real vector space of finite continuous functions, called harmonic functions, satisfying three axioms. These are (1) the sheaf axiom for harmonic functions; (2) the existence of a base for the open sets of Ω consisting of open sets for which the Dirichlet problem has a unique (and increasing) solution; and (3) an axiom of convergence (axiom 3′) for harmonic functions. We shall need (and we shall recall later) the integral representation of the superharmonic functions $\geqslant 0$ with measures on the set of extreme elements of a base of the cone of these functions provided with a suitable topology which has been established without other restrictions by Mrs. Hervé [17]. We suppose moreover a fourth axiom (axiom D) which ensures a convergence theorem for the superharmonic functions which we need in the sequel.

In Chapter II, after recalling certain results of the theory, we extend to the axiomatic case a result of Naïm and one of Doob; these are quite important for our study. The theorem of Naïm that the reduced function on E of a positive harmonic function u is a potential if and only if E is thin « u-almost everywhere » on Δ_1 extends completely. The proof of Naïm cannot be extended to our case. Our proof is based essentially on some results of the balayage theory developed by Mrs. Hervé and this replaces the use of Lebesgue measure. This is a key result for us to prove a minimum principle with the fine filters which in turn enables us to study a relativized first boundary value problem. The other result on the fine cluster values is more general than the corresponding result of Doob in [12] and this is useful to us at many places. We then make a preliminary study of the Dirichlet problem. But we do not know anything about the resolutivity of the continuous functions (in the topology of compact convergence on Δ_1), which is necessary for the study of the problem.

In Chapter III we introduce a new axiom called the *axiom of resolutivity*, which requires that the class of resolutive functions contains a certain family of continuous functions. This enables us to give in a familiar form an integral representation for the solutions, connecting it with the integral representation of Mrs. Hervé. Then by using the reasonings of Naïm and Doob [12] *we prove that the solution for any function on the boundary has a fine limit « almost everywhere » on the boundary and from that we get the generalisation of the Fatou-Naïm-Doob results.* Finally we observe that certain equivalent forms of resolutivity given by BRELOT [5] carry over to our case, but with modified proofs.

In chapter IV we are interested in showing that the axiom of resolutivity is valid in a particular but important case. It is the so called case of « unicité » (Mrs Hervé) or proportionality in which it is supposed that for every point of Ω the non-zero potentials that are harmonic outside this point are proportional. In this case Ω can be identified with such potentials belonging to a base (compact in the topology of Mrs. Hervé) of the cone of non-negative superharmonic functions. Hence the boundary Δ of the set of these potentials on the base can be considered as the boundary of Ω (and this is exactly the Martin boundary in the classical case). This boundary and the corresponding limit conditions given by the trace of filter of neighbourhoods of the elements in the boundary allow a second Dirichlet problem. With the techniques introduced by Brelot in [5] we prove that the solution for this Dirichlet problem has an integral respresentation (with Borel measures) on the boundary and that the solution by the Perron method of the continuous functions exist. It is true that the boundary of Ω contains the set of minimal harmonic functions of the base. Now we prove that the trace of the neighbourhoods of every minimal element in Δ is less fine than the fine filter we had earlier introduced. This results in a relation between the upper solutions of the two problems. We deduce finally the validity of the resolutivity axiom and the equivalence of resolutivity for both the problems.

Most of the results contained here had been published in two announcements in Comptes Rendus [15, 16].

It is a great pleasure for me to record my indebtness to Professor Brelot, who suggested to me the problem and who offered me his valuable guidance prior to and during the preparation of this paper. I thank the Centre National de la Recherche Scientifique (France) for offering me a grant enabling me to complete my work in Paris.

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I. — ABSTRACT THINNESS AND FINE LIMITS

1. Fundamental Classes of Functions.

Let Ω be a non-void set and U and P two non-empty families of real valued functions on Ω. Let the elements of U be finite valued (functions) $\geqslant 0$ on Ω and such that with every $\alpha \geqslant 0$ and h in U, αh belongs to U. The functions in P possibly taking the value $+\infty$ are assumed to be $\geqslant 0$, and further for every $\alpha_1 \geqslant 0$, $\alpha_2 \geqslant 0$ and p_1, p_2 in P let $\alpha_1 p_1 + \alpha_2 p_2$ belong to P (here, by definition $0.\infty$ will be 0). Further il we call Σ, the class of functions on Ω which can be expressed in the form $u + p$ where u belongs to U and p to P then let Σ satisfy (i) v_1, v_2 in Σ implies the function inf. (v_1, v_2) belongs to Σ and (ii)

$$v_1 = u_1 + p_1 \leqslant v_2 = u_2 + p_2$$

implies $u_1 \leqslant u_2$. Therefore every function in Σ is written in the form $u + p$ in an unique way and further u in U and $u \leqslant p$ in P gives that u is identically zero.

Minimal functions.

A function h in U is *minimal* if u in U and $u \leqslant h$ gives $u = \alpha h$ where $0 \leqslant \alpha \leqslant 1$.

If moreover we assume the property « A » that for every u, w in U and $u \leqslant w$ implies that $w - u$ also belongs to U, then h in U is minimal if and only if h cannot be expressed as the sum of two non-proportional elements in U.

Reduced function.

If f is a real valued function on $E \subset \Omega$, $f \geqslant 0$ and majorised by some element in Σ, then the corresponding reduced function is defined as,

$$R_f^E = \inf_{\substack{v \in \Sigma \\ v \geqslant f \text{ on } E}} v.$$

THEOREM I.1. — *Let $h \not\equiv 0$ be a minimal element of* U *and* $E \subset \Omega$. *Then either* R_h^E *is identically h or* R_h^E *does not majorise any element of U other than* 0.

Proof. — In view of the property (i) of Σ, the function R_h^E is also equal to the inf. v for $v \leqslant h$ on Ω ($v \in \Sigma$ and $v \geqslant h$ on E). Now if v belongs to Σ and minorises h, supposing $v = u + p$ (where $u \in U$ and $p \in P$), we have $u \leqslant h$ and since h is a minimal element u equals αh where $0 \leqslant \alpha \leqslant 1$. Hence any v in Σ minorising h is of the form $p + \alpha h$ where p is in P and $0 \leqslant \alpha \leqslant 1$. Now assume that R_h^E is not identically h, in which case there is at least one v in Σ with $v \leqslant h$, and $v \geqslant h$ on E and v different from h. Obviously this v equals $p + \alpha h$ where p is in P and $\alpha \neq 1$. Hence $p \geqslant (1 - \alpha)h$ on E and this gives $R_h^E \leqslant [1/(1 - \alpha)]p$. Now the assertion of the theorem follows since any U-minorant of R_h^E minorises also $[1/(1 - \alpha)]p$ which is an element of P, and is hence zero identically.

DEFINITION I.1. — E *contained in* Ω *is thin relative to h, a minimal element (not identically zero) of U, if there exists an element v in Σ such that $v \geqslant h$ on* E *but not on the whole space, or equivalently if* $R_h^E \not\equiv h$.

Criterion of thinness. — E is thin relative to h (not identically zero) if and only if there exists an element p in P such that $p \geqslant h$ on E.

During the course of the proof of the Theorem I.1 we showed that if $R_h^E \neq h$, then there is a p in P such that $p \geqslant h$ on E, and the converse is obvious.

Remark. — The points where h ($\not\equiv 0$) a minimal function of U equals zero form a set thin relative to h.

THEOREM I.2. — *Let $h \not\equiv 0$ be a minimal function of U. Then the union of two sets thin relative to h is again thin relative to h; i.e., the family $\mathfrak{T}_h$ of the sets E contained in Ω such that $\complement$ E is thin relative to h is a filter on Ω.*

Proof. — Let E and F be two sets thin relative to h. Then there are two elements p and q in P (none of them identically zero) such that p majorises h on E and q majorises h on F.

Hence $p + q$ majorises h on EUF and the thinness of EUF relative to h is a consequence. Now $\mathfrak{T}_h$ is a non void collection of non empty subsets of Ω. The above property shows that it is a filter.

DEFINITION I.2. — *We call the sets of $\mathfrak{T}_h$ the fine approach neighbourhoods of h and any limit (lim. sup. or inf.) following $\mathfrak{T}_h$, a fine limit (fine lim. sup. or inf.).*

THEOREM I.3. — *Let h $(\not\equiv 0)$ be a minimal element of U and $v \in \Sigma$. Then on the set Λ where v/h has a sense (everywhere for instance if $h > 0$), v/h tends to a limit following $\mathfrak{T}_h$ and this limit equals infimum of $[v(x)/h(x)]$ for x in Λ. This limit is zero for $v \in$ P.*

Proof. — The set $\complement \Lambda$ is contained in the set of zeros of h and hence $\complement \Lambda$ is thin relative to h. Now if, $\alpha = \inf_{x \in \Lambda} v(x)/h(x)$, then on Λ $\liminf_{\mathfrak{T}_h} v/h \geqslant \alpha$. Suppose now α to be finite and for $\varepsilon > 0$, let

$$E_\varepsilon = \{x \in \Lambda : [v(x)/h(x)] < \alpha + \varepsilon\}.$$

This set is non-void and $v/(\alpha + \varepsilon) \geqslant h$ on $\Lambda - E_\varepsilon$; but $v/(\alpha + \varepsilon)$ does not majorise h on the whole space. Hence $\Lambda - E_\varepsilon$ is thin relative to h; the same is true of

$$(\Lambda - E_\varepsilon) \cup (\complement \Lambda).$$

It follows $E_\varepsilon \in \mathfrak{T}_h$. This gives on Λ $\limsup_{\mathfrak{T}_h} v/h \leqslant \alpha$. This completes the proof.

COROLLARY 1. — *If h and h′ (none of them identically zero) are two non-proportional minimal elements of U then on the set where h'/h has sense $\lim_{\mathfrak{T}_h} h'/h = 0$.*

This is true since the assumption that the limit is different from zero will lead to the contradictory result that h and h' are proportional.

COROLLARY 2. — *Let h and h′ be as in the corollary 1. Then there are sets in $\mathfrak{T}_h$ and $\mathfrak{T}_{h'}$ which do not intersect each other. Hence there is no filter on Ω finer than both $\mathfrak{T}_h$ and $\mathfrak{T}_{h'}$.*

It suffices to take the sets $\{h/h' < 1,\ h,\ h' \neq 0\}$ and

$$\{h'/h < 1, \quad h, h' \neq 0\}.$$

2. Application to the Discrete Potential Theory (J.L. Doob).

First we recall briefly an axiomatic study of Doob [13]. We shall then see that the results of the previous article hold good in this axiomatic set up.

Let Ω be a measurable space, i.e., a set with a Borel field $\mathcal{B}$ of subsets. Let p be a stochastic transition function with state space Ω; that is p is a real valued function on $\Omega \times \mathcal{B}$ such that (i) $p(., A)$ is a measurable function for every A and (ii) $p(x,.)$ is a substochastic measure (measure with $p(x, \Omega) \leqslant 1$) on $\mathcal{B}$.

Define $p^n(x, A)$ recursively by

$$\begin{aligned} p^0(x, A) &= 1 \text{ if } x \text{ is in A} \\ &= 0 \text{ if } x \text{ is not in A} \\ p^n(x, A) &= \int p^{n-1}(y, A)\, p(x, dy) \quad \text{for } n > 1. \end{aligned}$$

p^n are all substochastic transition functions.

A measurable function u on Ω is called superregular if (i) $-\infty < u \leqslant \infty$ and (ii) for every integer $n > 0$, $\int u(y)\, p^n(x, dy)$ is finite and $u(x) \geqslant \int u(y)\, p(x, dy)$.

A function is called regular if both u and $-u$ are superregular.

Suppose φ is a non-negative superregular function on Ω then $\int \varphi(y)\, p^n(x, dy)$ decreases with n and tends to a regular function when n tends to infinity. This limit $p^{(\infty)}\varphi$ is the greatest regular minorant of φ. Further $\varphi - p^{(\infty)}\varphi$ is the potential of a non-negative measurable function; the potential of a non-negative measurable function f is defined to be $\int f(y)\, g(x, dy)$ where

$$g(x, A) = \sum_{n=0}^{\infty} p^n(x, A).$$

Further, the potential w of a non-negative measurable function is a superregular function with the waiving that

$$\int p^n(x, dy)\, w(y)$$

could be infinite for some $n's$ (called loosely superregular);

and has zero for the greatest regular minorant. Finally the decomposition of v into the sum of a regular function and a potential is unique.

Now let us take for U the non-negative regular functions and for P the potentials of non-negative measurable functions. Then Σ is precisely the set of non-negative loosely superregular functions. And from what we have recalled it is clear that all the hypothesis regarding U and P (I.1) are satisfied. We note moreover that U satisfies the property « A ».

3. Application to the axiomatic harmonic functions of M. Brelot.

We first recall the axioms and some of the developments briefly. The details are to be found in [6].

Let Ω be a locally compact Hausdorff space connected and locally connected. To each open set ω of Ω is assigned a vector space of finite valued continuous functions called harmonic functions on Ω satisfying the following axioms.

Axiom 1. — A function harmonic in ω is also harmonic in every open subset of ω. A finite valued function defined and continuous on ω and harmonic in an open neighbourhood of each point of ω is itself harmonic on ω.

DEFINITION I.2. — *Any relatively compact open set ω with the property that for every finite continuous function f on $\partial\omega$ (the boundary of ω in Ω, which is necessarily non-empty) there exists a unique harmonic function H_f^ω on ω such that H_f^ω continued by f on $\bar{\omega}$ is continuous and $H_f^\omega \geqslant 0$ if $f \geqslant 0$, is called a regular open set.*

For any regular open set ω, $f \to H_f^\omega(x)$, for any fixed x in ω defines a positive Radon measure which we denote by ρ_x^ω. Note that the connected components of a regular open set ω are also regular and conversely.

Axiom 2. — Ω has a base for open sets formed by regular open sets (and hence connected regular open sets).

Axiom 3. — On every domain $\omega \subset \Omega$, the limit of any increasing directed family of harmonic functions is $+\infty$ identically or harmonic on ω.

Consequences.

A real valued function v on an open set ω is called hyperharmonic if (i) $v > -\infty$, and lower semi-continuous and (ii) for every regular open set $\delta \subset \bar{\delta} \subset \omega$, $v(x) \geqslant \int v(y)\, d\rho_x^\delta(y)$.

In a domain ω, a hyperharmonic function v is either $+\infty$ identically or finite on an everywhere dense subset; if $v \geqslant 0$ then $v > 0$ everywhere or $v \equiv 0$ on ω.

A hyperharmonic function on an open set ω is called superharmonic if it is not identically $+\infty$ on any connected component of ω.

If a superharmonic function has a harmonic minorant it has a greatest harmonic minorant. A potential is a superharmonic function $p \geqslant 0$ for which the greatest harmonic minorant is 0.

If there exists no potential > 0 on Ω then all the superharmonic functions $\geqslant 0$ on Ω are harmonic and proportional.

Let $w \geqslant 0$ be a superharmonic function on Ω. If we denote by w also the restriction to $E \subset \Omega$ of w on Ω, we have, $\widehat{R_w^E}$ (the lower semi-continuous regularisation at each point of Ω of R_w^E) is a superharmonic function and satisfies

$$0 \leqslant \hat{R}_w^E \leqslant w.$$

There exists a topology on Ω called the « fine topology » which is the coarsest among the topologies finer than the initial topology of Ω such that all the superharmonic functions are continuous.

Let us now assume that the harmonic functions satisfy the axioms 1, 2, 3 and that there exists a potential > 0 on Ω.

Take for U and P (in I.1) the set H^+ of non-negative harmonic functions and the set of potentials on Ω respectively. Σ is nothing but the set of all non-negative superharmonic functions on Ω. It is clear that all the hypotheses of I.1 are satisfied. Hence we have the filters $\mathfrak{T}_h$ for every h, a positive (> 0) minimal harmonic function on Ω. Moreover in this case we shall define a topology on $\Omega \cup Y$, where Y is the following set of classes of positive minimal functions. The relation $h \sim h'$ if and only if $h = \alpha h'$ with a $\alpha > 0$, is an equivalence relation in the set of all positive minimal harmonic functions. We denote by $\bar{h}$ the equivalence class containing the function h. Let Y be the set of the equivalence classes. The filter $\mathfrak{T}_h$ is the same for all the functions in the same class and can be unambiguously denoted by $\mathfrak{T}_{\bar{h}}$ whatever be the particular member chosen from the class $\bar{h}$, and depends only on the class.

We note first that for every $\bar{h} \in Y$, the filter $\mathfrak{T}_{\bar{h}}$ on Ω has a base formed by fine open sets in Ω. For if E is in $\mathfrak{T}_h$, then $R_h^{\complement E} \neq h$ and for F the fine closure of $\complement E$ in Ω, $R_h^F \neq h$, now the fine open set $\complement F$ belongs to $\mathfrak{T}_h$ and $\complement F$ is contained in E.

Now using the definition and study of boundaries by Myskis [20] (3) we can define a topology on $\Omega \cup (\mathfrak{T}_h)_{\bar{h} \in Y}$ or equivalently $\Omega \cup Y$, since Y can be identified with $(\mathfrak{T}_h)_{\bar{h} \in Y}$. More precisely:

Theorem I.4. — *On $\Omega \cup Y$ there are topologies such that the induced topology on Ω is the fine topology and the trace on Ω of the filter of neigbrahoods of any point $\bar{h}$ in Y is $\mathfrak{T}_h$. There is a coarsest one*; *this is a Hausdorff topology for which a fundamental system of neigbourhoods of $\bar{h} \in Y$ is formed by sets of the form $E \cup E'$ where $E \in \mathfrak{T}_h$ and E' depending on E is defined by*

$$E' = \{\bar{h}' \in Y : \textit{there is } F \in \mathfrak{T}_{h'}, \textit{ with } F \subset E\} \quad (^{3\ \text{bis}}).$$

(3) Theorem (Myskis). Let X be a Hausdorff space. Let M be a class of families of open sets forming bases of filters, that is any element $\mathfrak{F}$ of M is a non-void collection of non-void open sets of X, such that the intersection of any finite number of elements of $\mathfrak{F}$ contains an element of $\mathfrak{F}$. Then a necessary and sufficient condition that $X \cup M$ is a Hausdorff space such that the induced topology on X is the original topology and a base of neighbourhoods of $\mathfrak{F}$ in M consists of the sets of the form $F \cup N_F$ where F is in $\mathfrak{F}$ and $N_F = \{ \mathfrak{G} \in M : \text{there is a } G \in \mathfrak{G} \text{ with } G \subset F \}$ is the simultaneous fulfillment of the following two conditions.

1. Let $\mathfrak{F}$, $\mathfrak{F}'$ belong to M and $\mathfrak{F} \neq \mathfrak{F}'$ then there are sets F in $\mathfrak{F}$ and F' in $\mathfrak{F}'$ with $F \cap F' = \emptyset$.

2. For any $\mathfrak{F}$ in M and x in X there are sets, a neighbourhood V of x in X and an element F in $\mathfrak{F}$ such that $F \cap V$ is void.

According to M. Brelot, this topology ou $X \cup M$, *even without the two conditions and for any topological space* X, is the coarsest topology such that.

(i) the induced topology on X is the original one.

(ii) the neigbourhoods of a point $\mathfrak{F} \in M$ intersect X to form a filter which is generated by $\mathfrak{F}$.

(3 bis) M. Brelot indicates that in the classical case of Green space, this coarsest topology on $\Omega \cup \Delta_1$ is exactly the fine topology of Naïm [21] defined on the Martin space with the help of Θ-functions.

Proof. — The proof is an immediate application of the considerations of Myskis and Brelot Since the first condition has already been proved (cor. 2, Theorem I.3) the verification of the second condition completes the proof. For that we note that for any point x in Ω if V is a relatively compact neighbourhood of x in Ω, then $\complement \overline{V}$ is in $\mathfrak{T}_h$ for every $h > 0$ and minimal.

4. Application to other axiomatic theories of H. Bauer and Doob.

The results of I.1 hold good in the case of the latest axiomatic theory of Bauer [1], which includes the study of the solutions of certain elliptic as well as parabolic equations. The theory is built up on four axioms. Ω is a locally compact Hausdorff space. To each open set of Ω is assigned a real vector space of finite valued continuous functions.

Axiom 1. — Exactly the same as the Axiom 1 in I.3.

An open set is called regular if it has a non-void boundary and it is regular according to the definition in I.3.

Axiom 2. — There is a base for open sets of Ω consisting of regular open sets.

Axiom 3. — If the limit of an increasing directed family of harmonic functions is finite on an everywhere dense set of ω, then the limit is a harmonic function on ω. (Doob's axiom for sequences).

Axiom T. — There is a harmonic function $h > 0$ on Ω and the hyper-h-harmonic functions separate the points of Ω. (The definition of hyper-h-harmonic functions is similar to the definition of the hyperharmonic functions but starting with the functions which are quotients by h of the harmonic functions on each open set).

With this set of axioms Bauer has proved [1] important results, for instance that for every superharmonic function (i.e., hyperharmonic function finite on a dense set) with a harmonic minorant there is a greatest harmonic minorant; hence the definition of potential follows. Now, if we take for U the set of non-negative harmonic functions on Ω and for P the potentials then we note that the hypotheses of I.1 are satisfied.

We observe that in this case also the property « A » is satisfied. Moreover we have a topology as defined in theorem I.4.

We also remark that the same application hold good for asimilar axiomatic which was given by Doob ([3 ter]) before the previous ones [same axioms 1, 2, 3; moreover metrisable space and constants harmonic and instead of T a more complicated condition].

We note that this axiomatixon of Bauer contains the previous one (I.3) as particular case when in the latter one there is a positive harmonic function on Ω. But in order to go further we need the restrictions of the theory of Brelot and in fact even some more assumptions.

II. — DIRICHLET PROBLEM WITH THE FINE FILTERS

In what follows we shall be concerned only with the axiomatic set up of M. Brelot. Let Ω be a locally compact, locally connected and connected Hausdorff space provided with the system of harmonic functions on each of its open sets. We shall indicate at suitable places the axioms required.

1. Some Known results of the axiomatic theory of M. Brelot.

We recall here certain basic notions, results and two supplementary axioms that are required in the sequel.

a) *The generalised Dirichlet Problem* [6]

axioms 1, 2 et 3;
existence of a potential > 0 on Ω.

A class Σ of hyperharmonic functions is said to be *saturated* if (i) the infimum of any two elements of Σ belongs to Σ (ii) for any φ in Σ and a regular domain δ (contained with its closure in ω the domain of definition of the functions) the function φ' defined by $\varphi' = \varphi$ on $\omega - \delta$, and $\varphi' = \int \varphi \, d\rho_x^\delta$ in δ belongs to Σ. Further Σ is said to be *completely saturated*

([3 ter]) J. L. Doob, *Proc. third Berkeley Symposium on Prob. and Statistics*, 2, 1954-55, 49-80.

if, moreover, any linear combination with positive coefficients of the elements of Σ and every hyperharmonic majorant of an element of Σ belong to Σ.

Let $\mathfrak{L}$ be a class of filters on Ω none of them having any adherence in Ω. Let Σ be a saturated class of superharmonic functions on Ω. Then Σ and $\mathfrak{L}$ are said to be associated to each other if for any v in Σ, $\liminf_{\mathfrak{T}} v \geqslant 0$ for every $\mathfrak{T}$ in $\mathfrak{L}$ implies $v \geqslant 0$.

Then corresponding to any real valued function f (possibly taking the values infinity) on $\mathfrak{L}$, if we define $\overline{\mathfrak{L}}(f)$ the lower envelope of v in Σ such that $\liminf_{\mathfrak{T}} v \geqslant f(\mathfrak{T})$ for every $\mathfrak{T}$ in $\mathfrak{L}$; then $\overline{\mathfrak{L}}(f)$ is $+\infty$ or $-\infty$ identically or else harmonic in Ω. A similar result is true for the function $\underline{\mathfrak{L}}(f)$ defined by $\underline{\mathfrak{L}}(f) = -\overline{\mathfrak{L}}(-f)$. If moreover we assume that Σ is completely saturated then $\overline{\mathfrak{L}}(f) \geqslant \underline{\mathfrak{L}}(f)$ on Ω.

DEFINITIONS. — $\overline{\mathfrak{L}}(f)$ *and* $\underline{\mathfrak{L}}(f)$ *are called the upper and lower solutions (respectively) for the problem corresponding to any function f on* $\mathfrak{L}$. *A function is said to be* $\mathfrak{L}$*-resolutive if* $\overline{\mathfrak{L}}(f) = \underline{\mathfrak{L}}(f)$ *and finite on* Ω.

Let us now assume that every $\mathfrak{L}$*-resolutive function is « absolutely resolutive » that is to say is the difference of two non-negative* $\mathfrak{L}$*-resolutive functions.* Then the functional $f \rightarrow \overline{\mathfrak{L}}(f)(x)$, that is the value of the envelope at the point x in Ω, for fixed point x in Ω, on the set of all $\mathfrak{L}$-resolutive functions on $\mathfrak{L}$ which is a vector space, defines a Daniell measure $\mathfrak{L}_x$ and this is such that the $\mathfrak{L}$-resolutivity and $\mathfrak{L}_x$-summability are equivalent. Further the $\mathfrak{L}_x$-summability of a function for some x in Ω already ensures the $\mathfrak{L}_y$-summability for every y in Ω.

b) *The axiom* 3′ *and the integral representation.*

For most of our considerations we shall need a stronger axiom in the place of the axiom 3. This new one implies the axiom 3 when we suppose the axioms 1 and 2.

Axiom 3′. — On every domain δ, any harmonic function $u \geqslant 0$ is either identically 0 or everywhere > 0; moreover the set of positive harmonic functions on δ taking the value 1 at any fixed point of δ is equicontinuous at that point.

Notations.

H^+ the set of non-negative harmonic functions on Ω.

P the set of potentials on Ω.

and

S^+ the set of non-negative superharmonic functions on Ω.

Let us now suppose that the harmonic functions on Ω satisfy the axioms $1,2,3'$ and further that there is a potential > 0 on Ω. Moreover let there be a *countable base* for the open sets of Ω. Under these conditions R.M. Hervé [17] gives an integral representation for the elements of S^+, with measures on the set of extreme elements of a compact base of S^+: the generalised Riesz-Martin representation ([4]). The method consists in applying the results of G. Choquet [9] on the existence and uniqueness of measures on extreme elements, to the vector space of differences of functions in S^+. The essential point in setting the ground to appeal to Choquet's theorem consists in providing this vector space with a proper order and a suitable topology.

Now, in $S^+ \times S^+$ define an equivalence relation

$$(v, v') \sim (w, w')$$

by setting $v + w' = v' + w$. The set S of the quotient classes is a vector space and contains S^+ with the identification $v \leftrightarrow (v, 0)$. The order « Specific Order » in S that

$$(v, v') > (w, w') \quad \text{if} \quad (v, v') \quad \text{equals} \quad (w, w') + u$$

where u is in S^+, makes S a Riesz space (that is the order is consistent with the vector space structure and further it is a lattice for this order) and the positive cone for this order is exactly S^+.

To meet the other requirements, a topology « T-topology » ([5])

([4]) A much simpler proof of the integral representation but with an additional assumption, that there is a base for the open sets consisting of « completely determinating » domains, was first given by Brelot [7]. Mrs Hervé al so givesa simpler proof in the particular case of « uniqueness ».

([5]) *T-topology on S.* — For any open set $\omega \subset \Omega$, Mrs. Hervé decomposes any function V in S^+ into the sum of two functions V_ω and V'_ω in S^+, where V'_ω is harmonic in ω and the greatest one satisfying these conditions. Note that in the classical case V_ω is the Greenian potential of the restriction to ω of the Riesz measure associated to V. In the general case, let us take the Alexandroff compactification $\overline{\Omega}$ of Ω; let $\mathcal{A}$ be the point at infinity. Now corresponding to any open set ω, is defined on S which makes S a locally convex and Hausdorff topological vector space. Further it is proved that the set S is metrisable for this topology and that it has a compact base. Hence by the Theorem of Choquet every element in S^+ is the centre of gravity of a uniquely determined measure on the compact base and this measure charges only the set of the extreme elements of the base.

An extreme superharmonic function of any base of S^+ is either an extreme harmonic function (equivalently a minimal one) or one of the potentials whose support [6] is a point.

Moreover in the integral representation for any element in P (respectively in H^+) the corresponding measure charges only the set of extreme potentials (resp. the minimal harmonic functions).

Notations. — We suppose that Λ is a compact base of S^+. Let Δ_1 be the set of minimal harmonic functions on this base. For any element u in H^+ and $u > 0$ we shall denote the measure on Λ corresponding to u by μ_u. μ_u charges only the set Δ_1 and further $u = \int h \, d\mu_u(h)$.

c) *The axiom of domination and swept out measure.*

DEFINITION. — *A set E in Ω is a polar set if it is contained in the set of points where a function v in S^+ takes the value $+\infty$.*

Axiom D. — If v is in S^+ and p a locally bounded potential on Ω then $v \geqslant p$ on the support S_p of p implies that v majorises p everywhere on Ω; or equivalently if $R_p^{S_p} = p$.

$\mathcal{A} \in \omega \subset \overline{\Omega}$ we define V_ω as equal to the sum of the greatest harmonic minorant of V and $V_{\Omega \cap \omega}$. $V_\omega(x)$ for fixed x in Ω is a function of $\omega \subset \Omega - \{x\}$; and this may be continued as a measure $\mu_V'^X$ on $\overline{\Omega} - \{X\}$, and is such that for any element (v, v') in S, $\mu_V'^X - \mu_{V'}'^X$ is the same for all the elements in the same equivalent class. On the space S the T-*topology* is defined to be the coarsest topology for which the mappings $(v, v') \to \mu_V'^X - \mu_{V'}'^X$ of S into the space of measures on $\overline{\Omega} - \{X\}$ are continuous when the latter is provided with vague topology; and for all X in Ω.

On H (the subspace of difference of functions in H^+) this topology induces the one of local uniform convergence.

On the subset E^+ of all the potentials with point supports in Ω and of all the harmonic functions $\geqslant 0$, considered with the support $\mathcal{A}$, T is the least fine topology such that the support $\varphi(p)$ of p is continuous and $p(x)$ is continuous in p when $\varphi(p) \neq x$.

[6] The support of a potential is by definition the smallest closed set in the complement of which it is harmonic.

Note that if the axiom D is true for Ω then the axiom D is true for every domain contained in Ω.

If we suppose that Ω *has a countable base* for its open sets then the axiom D gives a convergence theorem for superharmonic functions. This convergence theorem in particular provides, for any set E and $v \in S^+$, $\hat{R}_v^E = R_v^E$ except on a polar set. [6].

Let us assume in addition that axiom 3′ is satisfied. We recall the following result (Theorem 10.1, [17]).

Suppose that μ is a positive (> 0) Radon measure on Ω supported by a compact subset of Ω and let E be any subset of Ω. Then there exists a Radon measure μ^E and only one called the swept out measure corresponding to E and μ (« mesure balayée du μ ») such that for every function $v \in S^+$,

$$\int \hat{R}_v^E \, d\mu = \int v \, d\mu^E.$$

2. Some new Properties of the reduced function of an element in H^+.

Axioms 1, 2, 3′ and D.
Existence of a potential > 0.
Countable base for open sets of Ω.

LEMMA II.1. — $\hat{R}_h^E(x)$ *for any fixed* x *in* Ω *and any set* $E \subset \Omega$ *is a lower semi-continuous function of* $h \in H^+$ *provided with the topology of uniform convergence on compact subsets of* Ω *(which is also the topology induced on* H^+ *by the T-topology on* S^+*).*

Proof. — Let $h_n \in H^+$ and h_n tend to h in H^+. Let $v_n = \hat{R}_{h_n}^E$. Let $v = \liminf_n v_n$ (pointwise). Then by the convergence theorem [6] we have $\hat{v} = v$ quasi everywhere (i.e., except on a polar set) on Ω, (where $\hat{v}$ is the lower semi-continuous regularisation of v). But because of axiom D, $v_n = h_n$ quasi everywhere on E and hence $\hat{v} = h$ quasi everywhere on E; and this in turn implies that $\hat{v} \geqslant \hat{R}_h^E$. Hence we have, $\liminf \hat{R}_{h_n}^E(x) = v(x) \geqslant \hat{v}(x) \geqslant \hat{R}_h^E(x)$. This clearly gives the required lower semi-continuity since H^+ being metrisable it is sufficient to consider the convergence of sequences.

THEOREM II.1. — *The set of all points $\mathcal{E}_E$ of Δ_1 where* $E(\subset \Omega)$ *is thin is the intersection with Δ_1 of a K_σ set in* H^+.

Proof. — Suppose δ is a regular domain of Ω. If h is in H^+ $\left(\text{since} \int \hat{R}_h^E(y)\, d\rho_x^\delta(y) \leqslant h(x) \text{ for every } x \text{ in } \delta\right)$,

$$h(x) - \int R_h^E \, d\rho_x^\delta$$

is a non-négative harmonic function in δ and is hence zero identically or never zero in δ. Hence if we pose,

$$F'_\delta = \left\{ u \in H^+ \cap \Lambda : \int \hat{R}_h^E(y)\, d\rho_x^\delta(y) < u(x) \right\}$$

then F'_δ is well defined (and is independent of the reference point x in δ).

Let now $\delta_1, \ldots, \delta_n, \ldots$ be a sequence of regular domains in Ω forming a covering of Ω. We assert that $\bigcup_{n=1}^{\infty} F_{\delta_n} = \mathcal{E}_E$, where $F_{\delta_n} = \Delta_1 \cap F'_{\delta_n}$.

For, if h is in $\mathcal{E}_E$, then $R_h^E \not\equiv h$, and hence there is at least one x in Ω and that is x_0 in δ_m (say) such that $\hat{R}_h^E(x_0) < h(x_0)$. Hence, $\int \hat{R}_h^E(y)\, d\rho_{x_0}^{\delta_m}(y) \leqslant \hat{R}_h^E(x_0) < h(x_0)$. Hence $\mathcal{E}_E \subset \bigcup_n F_{\delta_n}$. On the other hand if h is not in $\mathcal{E}_E$ (but h in Δ_1) then by the definition of thinness $h \equiv R_h^E$. It follows that

$$h(x) = \int \hat{R}_h^E(y)\, d\rho_x^{\delta_n}(y)$$

for all x in δ_n and the same is true for all n. This implies that h is not in F_{δ_n} for any n, i.e., $h \notin \bigcup_n F_{\delta_n}$ and that establishes $\mathcal{E}_E = \bigcup_n F_{\delta_n}$.

The function $\varphi : H^+ \to R$ defined by $\varphi(u)$ equal to

$$\int \hat{R}_u^E(y)\, d\rho_x^\delta(y) - u(x)$$

is a lower semi-continuous function for fixed x, and where δ is a regular domain in Ω. It is enough to see that

$$u \to \int \hat{R}_u^E(y)\, d\rho_x^\delta(y)$$

is lower semi-continuous. To verify the same let u_n converge (in the compact convergence topology of H^+) to an element u in H^+. Then,

$$\text{lim.inf.}\int \hat{R}^E_{u_n}(y)\, d\rho^\delta_x(y) \geqslant \int \text{lim.inf.}\ \hat{R}^E_{u_n}(y)\, d\rho^\delta_x(y) \qquad \text{(Fatou's Lemma)}$$

$$\geqslant \int \hat{R}^E_u(y)\, d\rho^\delta_x(y) \quad \text{(Lemma II.1)}.$$

This clearly establishes the lower semi-continuity of φ and hence we have that F'_δ is the countable union of closed sets of $H^+ \cap \Lambda$. Since $H^+ \cap \Lambda$ is compact it follows that F'_δ is a K_σ set and this is true for all regular domains of δ. Now the proof is easily completed.

THEOREM II.2. — *Let u be a positive harmonic function ($u > 0$). Let $E \subset \Omega$ and $\mathcal{E}_E$ the set of points of Δ_1 where E is thin. Then the necessary and sufficient condition that $R^E_u = u$ is that $\mu_u(\mathcal{E}_E)$ is zero (where μ_u is the measure on Δ_1 corresponding to u).*

Proof. — First assume that $\mu_u(\mathcal{E}_E) = 0$. Then for every h in Δ_1 except for a set of μ_u measure zero, $R^E_h = h$. Then the equation

$$\hat{R}^E_u(x) = \int \hat{R}^E_h(x)\, d\mu_u(h) \quad \text{(Th. 22.3, [17])}$$

gives that $\hat{R}^E_u(x) = \int h(x)\, d\mu_u(h) = u(x)$; and this is true for all the elements x in Ω.

Conversely, let us suppose that $R^E_u = u$ and we shall prove that the μ_u measure of $\mathcal{E}_E$ is zero.

Let $\delta_1, \ldots, \delta_n, \ldots$ be a covering of Ω by regular domains. Consider the sets $F_{\delta_n} = \left\{h \in \Delta_1 : \int \hat{R}^E_h(y)\, d\rho^\delta_x(y) < h(x)\right\}$ as defined in Theorem II.1.

Now, let us take a fixed point x in δ_k. Let ν be the swept out measure corresponding to E and the measure $\rho^{\delta_k}_x$ which is on the compact set $\partial\delta_k$ in Ω. The measure ν is such that for every v in S^+

$$\int \hat{R}^E_v(y)\, d\rho^{\delta_k}_x = \int v(y)\, d\nu(y).$$

Then we have,

$$\begin{aligned}\int \hat{R}_u^E(y)\, d\rho_x^{\delta_k}(y) &= \int u(y)\, d\nu(y)\\ &= \int d\nu(y) \int h(y)\, d\mu_u(h)\\ &= \int d\mu_u(h) \int h(y)\, d\nu(y)\\ &\qquad\qquad \text{(Lebesgue-Fubini Theorem)}\\ &= \int d\mu_u(h) \int \hat{R}_h^E(y)\, d\rho_x^{\delta_k}(y).\end{aligned}$$

Now,

$$\begin{aligned}\int h(x)\, d\mu_u(h) &= u(x)\\ &= \hat{R}_u^E(x) \quad \text{(hypothesis)}\\ &= \int \hat{R}_u^E(y)\, d\rho_x^{\delta_k}(y)\\ &= \int d\mu_u(h) \int \hat{R}_h^E(y) \quad d\rho_x^{\delta_k}(y) \text{ (from above)}\end{aligned}$$

i.e.,

$$\int \Big[h(x) - \int \hat{R}_h^E(y)\, d\rho_x^{\delta_k}(y)\Big]\, d\mu_u(h) = 0.$$

The integrand in the last equation is always $\geqslant 0$ and hence it follows that

$$h(x) = \int \hat{R}_h^E(y)\, d\rho_x^{\delta_k}(y) \ \mu_u\text{-almost everywhere on } \Delta_1.$$

But then the set of h on Δ_1 for which $h(x) = \int \hat{R}_h^E\, d\rho_x^{\delta_k}$ is exactly $(\Delta_1 - F_{\delta_k})$. Hence we have, $\mu_u(F_{\delta_k}) = 0$, and the same property is surely true for the countable union, viz., $\mu_u\Big(\bigcup_k F_{\delta_k}\Big) = 0$. But we have proved in Theorem II.1 that this countable union is precisely $\mathcal{E}_E$ the set of points of Δ_1 where the set E is thin. Hence $\mu_u(\mathcal{E}_E) = 0$ completing the proof of the theorem.

Corollary. — *If u is in H^+, then the greatest harmonic minorant of $\hat{R}_u^E$ is $\int h(y)\, d\mu_u^E(h)$ where μ_u^E is the restriction of μ_u to the set of points where E is not thin (that is $\Delta_1 - \mathcal{E}_E$). Hence $\hat{R}_u^E$ is a potential if and only if $\mu_u(\Delta_1 - \mathcal{E}_E) = 0$.*

Proof. — We have always $\hat{R}_u^E(x) = \int \hat{R}_h^E(x)\ d\mu_u(h)$. (Th. 28.2 [17]). Hence,

$$\hat{R}_u^E \geqslant \int \hat{R}_h^E\ d\mu_u^E(h) = \int h\, d\mu_u^E(h).$$

On the other hand, if u' is the greatest harmonic minorant of $\hat{R}_u^E$ we have $u' = R_{u'}^E$, and hence by the above theorem $\mu_{u'}(\mathcal{E}_E) = 0$. But $u' \leqslant u$ and hence $\mu_{u'} \leqslant \mu_u$ and in turn $\mu_{u'} = \mu_{u'}^E \leqslant \mu_u^E$ and now the opposite inequality follows completing the proof of the first part of the corollary. $\hat{R}_u^E$ is a potential if and only if $\mu_u\ (\Delta_1 - \mathcal{E}_E) = 0$ follows immediately from the first part.

Consequence : *Behaviour of a u-potential at* Δ_1.

Theorem II.3. — *Let* v *be a potential and* u *in* H^+ $(u > 0)$ *then* v/u *tends to zero following the fine filters except for a set on* Δ_1 *of* μ_u *measure zero (that is* v/u *has a fine limit zero at* Δ_1 μ_u *almost everywhere).*

Proof. — Consider the set $E_\varepsilon = \{(v/u) > \varepsilon\}$ for $\varepsilon > 0$. We have immediately $\hat{R}_u^{E_\varepsilon} \leqslant v/\varepsilon$ and hence it is a potential. By the corollary to Theorem II.2 it follows that

$$\mu_u(\Delta_1 - \mathcal{E}_{E_\varepsilon}) = 0,$$

in other words E_ε is thin μ_u almost everywhere on Δ_1. Now the proof is completed by considering for ε a sequence of rational numbers (positive) and tending to zero.

3. A Minimum Principle with the fine filters.

The following theorem concerning the fine adherent values of a function is a slightly stronger version of an analogous result of J. L. Doob [12] in the case of the Green space; the method adopted here is essentially the same.

Theorem II.4. — *Let* $E \subset \Omega$ *be not thin at* h *in* Δ_1. *Let* f *be a real valued function (possibly taking the values infinity) on* E. *Let* $a =$ *fine* $\limsup_{x\in E,\, x\to h} f(x)$ *Then there is a set* F *contained in* E *and not thin at* h *such that* $a =$ fine $\lim_{x\in F,\, x\to h} f(x)$. *a is also equal to the limit of* $f(x)$, x *following the base of the filter formed by* $F \cap \complement K$ *for all compact sets* $K \subset \Omega$.

Proof. — Let K_n be an increasing sequence of compact sets of Ω with the union of $K_n = \Omega$. Let m, n and p be positive integers. Define,

$$M_{m,n,p} = \{x \in E : |f(x) - a| < (1/m) \quad \text{and} \quad x \in K_p \cap \complement K_n\}.$$

The sets $M_{m,n,p}$ form a sequence increasing with p and their union (with m and n fixed) for p running through the set of positive integers is not thin at h. (This is true because a is the fine lim. sup. of $f(x)$ when x tends to h following the trace of the fine filter $\mathfrak{T}_h$ on E). Hence $R_h^{\bigcup_p M_{m,n,p}} = h$. Since the axiom D is satisfied, $\hat{R}_h^{M_{m,n,p}}$ increases to h (for every m and n).

Let x_0 be a fixed point of Ω. Let $p_1 = 1$ and p_2 be a positive integer such that $\hat{R}_h^{M_{1,1,p_2}}(x_0) > h(x_0)/3$. Having chosen $p_2, \ldots, p_n$ successively, let p_{n+1} be a positive integer $\geqslant$ max. (n, p_n) such that $\hat{R}_h^{M_{n,p_n,p_{n+1}}}(x_0) > h(x_0)/3$. Let $F = \bigcup_{n=1}^{\infty} M_{n,p_n,p_{n+1}}$. We assert that F (obviously contained in E) satisfies the required conditions.

Firstly suppose F is thin at h. Then since $\hat{R}_h^F$ is different from zero (by the choice of the set F, $\hat{R}_h^F(x_0) > h(x_0)/3$), $\hat{R}_h^F$ is a potential. Moreover since

$$F \cap \complement K_{p_n} \supset M_{n,p_n,p_{n+1}}$$

we have $R_h^{F \cap \complement K_{p_n}}(x_0) > h(x_0)/3$ by the choice of the latter set. But $R_h^{F \cap \complement K_{p_n}}$ is a decreasing sequence of potentials and tends to zero (the limit being a harmonic minorant of any one of the potentials) when n tends to infinity. This is clearly a contradiction and hence the assumption that F is thin at h is impossible. It is clear from the nature of definition of the set F that $f(x)$ tends to the limit a following the trace of the filter $\mathfrak{T}_h$ on F.

THEOREM II.5. *(Minimum Principle).* — *Let $u > 0$ be a harmonic function on Ω. Let v be a lower bounded super-u-harmonic function on Ω such that for every h in $\Delta_1 - B$ $(B \subset \Delta_1)$,* *there is a set* $E_h \subset \Omega$ *not thin at* h *with fine lim.sup.*$v(x) \geqslant 0$; *where the set* B *is of inner* μ_u *measure zero. Then* v *is* $\geqslant 0$.

Proof. — We may assume (as an immediate application of Theorem II.4) that the function v has actually a fine limit $\geqslant 0$ at h following the trace of $\mathfrak{T}_h$ on the set E_h; and this being true for all h in $\Delta_1 - B$. Now, for a $\varepsilon > 0$ let,

$$X_\varepsilon = \{x \in \Omega : v(x) > -\varepsilon\}.$$

Then X_ε is a non-empty set and further it is not thin at any point of $\Delta_1 - B$. Hence the set $\mathcal{E}_\varepsilon$ of points where X_ε is thin is contained in B and has hence inner μ_u measure zero. But the set of points on Δ_1 where X_ε is thin being measurable, the μ_u measure of the set $\mathcal{E}_\varepsilon$ is zero.

Let $v' = \inf. (v, 0)$. Since v is lower bounded on Ω (say $v \geqslant -\alpha$ where $\alpha \geqslant 0$), $v' \geqslant -\alpha$ and hence v' has the greatest u-harmonic minorant $-u_1$ (where $u_1 \geqslant 0$ and u-harmonic on Ω). Moreover $u_1 \leqslant \alpha$ and so the canonical measure $\mu_{u_1'}$ of the harmonic function $u.u_1$ is $\leqslant \mu_u$. Consequently $\mu_{u_1'}(\mathcal{E}_\varepsilon) = 0$.

Now on X_ε, $v(x) \geqslant -\varepsilon$ and hence $v'(x) \geqslant -\varepsilon$. This implies that $\varepsilon u \geqslant R_f^{X_\varepsilon}$ on Ω where $f = u.(-v')$. Again if $V = v' + u_1$, then $V \geqslant 0$ and further

$$\varepsilon u \geqslant R^{X_\varepsilon}_{(u_1 - V)u} \geqslant R^{X_\varepsilon}_{u_1 u} - R^{X_\varepsilon}_{uV}.$$

But now since $\mu_{u_1'}(\mathcal{E}_\varepsilon) = 0$, it follows from Theorem II.2 that $R^{X_\varepsilon}_{u_1 u} = u.u_1$ and hence $\varepsilon u \geqslant u.u_1 - R^{X_\varepsilon}_{uV}$. Again

$$\frac{1}{u} \hat{R}^{X_\varepsilon}_{uV} \geqslant (u_1 - \varepsilon).$$

Now V is a u-potential and $\frac{1}{u} \hat{R}^{X_\varepsilon}_{uV} \leqslant V$ and hence $\frac{1}{u} \hat{R}^{X_\varepsilon}_{uV}$ is itself a u-potential. Now the last inequality gives that $(u_1 - \varepsilon)$ minorises a u-potential and being a u-harmonic function it satisfies $u_1 - \varepsilon \leqslant 0$, i.e., $u_1 \leqslant \varepsilon$.

Evidently the same argument holds whatever be $\varepsilon > 0$ and hence it follows that $u_1 \equiv 0$ and in turn $v' \equiv 0$. In other words v is $\geqslant 0$, which is the required result.

CONSEQUENCE :

THEOREM II.6. — *Let u and u' belong to* H^+ *and* $u' > 0$. *Then the following three statements are equivalent.*

(i) *u/u' tends to zero $\mu_{u'}$ almost everywhere following the fine filters.*

(ii) *u majorises no harmonic function $u'' > 0$ with u''/u' bounded.*

(iii) *The inf. (u, u') is a potential.*

Proof. — We shall show in order that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i) which will establish the equivalence.

If u'' is a positive harmonic function such that $u'' \leqslant u$ and $u'' \leqslant ku'$, then $(u''/u') \leqslant (u/u')$ and hence u''/u' tends to zero $\mu_{u'}$ almost everywhere. Hence from the minimum principle (note that $u''/u' \leqslant k$) it follows that $u'' \leqslant 0$, i.e., $u'' = 0$.

Assume (ii) to be valid. Let V be the greatest harmonic minorant of inf (u, u'). Then $V \leqslant u$ and $(V/u') \leqslant 1$ and hence $V = 0$. i.e., (ii) implies (iii).

Now let us assume that inf. (u, u') is a potential. For a $\varepsilon > 0$ let E_ε be the set of x in Ω where $\dfrac{u(x)}{u'(x)} \geqslant \varepsilon$, then $R_{u'}^{E_\varepsilon} \leqslant \varepsilon u$ on Ω. Assuming that $\varepsilon \leqslant 1$, we have that $\hat{R}_{u'}^{E_\varepsilon} \leqslant$ inf. (u, u') and hence $\hat{R}_{u'}^{E_\varepsilon}$ is a potential. It follows from Cor. Theorem II.2 that E_ε is thin $\mu_{u'}$ almost everywhere on Δ_1. Now by considering $\varepsilon = 1/n$ and the union of the corresponding exceptional sets (where E_ε is not thin) of $\mu_{u'}$ measure zero, we have, outside a set of $\mu_{u'}$ measure zero at every h on Δ_1, u/u' has a fine limit zero. This completely establishes the equivalence.

4. The Dirichlet Problem with the fine filters.

Axioms 1, 2, 3′ and D
Existence of a potential > 0 on Ω.
Countable base for the open sets of Ω.

Let $u > 0$ be a harmonic function on Ω. Consider the following family of functions on Ω.

$$\Sigma_u = \left\{ v :: \begin{array}{l} v \text{ is hyper-}u\text{-harmonic function on } \Omega. \\ v \text{ is lower bounded on } \Omega; \text{ that is } v \geqslant \alpha_v \text{ where} \\ \alpha_v \text{ is a real number depending on } v. \end{array} \right\}$$

Let $\mathfrak{L}$ be the family of the filters $(\mathfrak{T}_h)_{h\in\Delta_1}$. Then the minimum principle (Theorem II.5) states that the family of filters $\mathfrak{L}$ is associated to the set Σ_u of hyper-u-harmonic functions; and Σ_u is completely saturated. Corresponding to any extended real valued function f on Δ_1 (the set $\mathfrak{L}$ of filtres being identified with Δ_1) we denote respectively by $\overline{\mathcal{H}}_{f,u}$ and $\underline{\mathcal{H}}_{f,u}$ the upper and lower solutions in this Dirichlet problem [7]. Any of these functions, in case finite, is a u-harmonic function. If a function is resolutive in this problem then we say that f is u-resolutive and we denote the solution, which we call the u-solution, by $\mathcal{H}_{f,u}$.

In view of the fact that each element in Σ_u is lower bounded on Ω, we have that any function which is u-resolutive is also absolutely u-resolutive. (See [6] p. 107). Hence the functional $f \to \mathcal{H}_{f,u}(x)$ for any x in Ω on the class of the u-resolutive functions on Δ_1 is a Daniell measure $I_{x,u}$ in such a way that the u-resolutive functions are exactly the $I_{x,u}$ summable functions on Δ_1 and further this $I_{x,u}$ summability is independent of the reference to a particular point x of Ω.

Theorem II.7. — *Let A be contained in Δ_1, and φ_A the characteristic function of* A *in Δ_1. Suppose $\{E_i\}_{i\in I}$ is the family of all sets $E_i \subset \Omega$ such that E_i belongs to $\mathfrak{T}_h$ for every h in* A *(that is, E_i contains a set of $\mathfrak{T}_h$ for every h in A) this for every i in I. Then,*

$$u\overline{\mathcal{H}}_{\varphi_A, u} = \inf_{i\in I}. \hat{R}_u^{E_i}.$$

Proof. — Consider the super-u-harmonic function $(1/u)\hat{R}_u^{E_i}$. Since $\hat{R}_u^{E_i} = R_u^{E_i}$ quasi-everywhere (consequence of axiom D), there is a polar set F_i outside which $\hat{R}_u^{E_i} = R_u^{E_i}$. $\complement F_i$ is in $\mathfrak{T}_h$

[7] Our notation here is different from the one used in the classical case. In the corresponding problem in the classical case $\mathcal{H}_{f,u}$ denoted our solution multiplied by u, in which case it is a harmonic function.

for every h in Δ_1; hence $E_i \cap \complement F_i$ is in $\mathfrak{T}_h$ for all h in A and on that set $(1/u)\hat{R}_u^{E_i}$ equals 1. Hence fine $\liminf_{x \to h} (1/u)\hat{R}_u^{E_i}(x) \geqslant 1$ for every h in A. Moreover $\hat{R}_u^{E_i} \geqslant 0$; it follows that $(1/u)\, \hat{R}_u^{E_i} \geqslant \overline{\mathcal{H}}_{\varphi_A, u}$. This is true for every i in I and hence $\inf.\ \hat{R}_u^{E_i} \geqslant u_\varphi \overline{\mathcal{H}}_{A, u}$.

On the other hand if ν is such that $\nu \in \Sigma_u$ and fine lim.inf. $\nu \geqslant \varphi_A(h)$ fov every h in Δ_1 then for a given $\varepsilon > 0$ and for each h in A there is a set E_h in $\mathfrak{T}_h$ such that $\nu(x) \geqslant (1 - \varepsilon)$ for all x in E_h. (Note that any such ν is necessarily non-negative). Let $E = \bigcup_{h \in A} E_h$. Then E is an element of the family $\{E_i\}_{i \in I}$. Also $\nu(x) \geqslant (1 - \varepsilon)$ on E and so

$$\nu/(1 - \varepsilon) \geqslant (1/u)\hat{R}_u^E$$

everywhere on Ω. Hence $\nu/(1 - \varepsilon) \geqslant \inf_{i \in I}.\ (1/u)\hat{R}_u^{E_i}$. This is true for all such ν and then for all $\varepsilon > 0$, hence it follows that $u\overline{\mathcal{H}}_{\varphi_A, u} \geqslant \inf_{i \in I}.\hat{R}_u^{E_i}$. The proof is complete.

Corollary. — *Any set* $A \subset \Delta_1$ *for which* $\overline{\mathcal{H}}_{\varphi_A, u} = 0$ *has its outer* μ_u*-measure zero.*

We have, from the Corollary to Theorem II.2, that the greatest harmonic minorant of $\hat{R}_u^{E_i}$ is precisely $\int_{\Delta_1 - \mathcal{E}_{E_i}} h(x)\, d\mu_u(h)$ where $\mathcal{E}_{E_i}$ is the set on Δ_1 where E_i is thin. Now, for each E_i obviously $\Delta_1 - \mathcal{E}_{E_i}$ contains A and hence,

$$\hat{R}_u^{E_i}(x) \geqslant \int_{\Delta_1 - \mathcal{E}_{E_i}} h(x)\, d\mu_u(h) \geqq \overline{\int} \varphi_A(h) h(x)\, d\mu_u(h)$$

which again gives,

$$0 = u(x)\overline{\mathcal{H}}_{\varphi_A, u}(x) = \inf_{i \in I}.\hat{R}_u^{E_i}(x) \geqslant \overline{\int} \varphi_A(h) h(x)\, d\mu_u(h) \geqslant 0.$$

From this, one deduces the assertion of the corollary using the fact that for a given x in Ω $h(x) \geqslant \alpha > 0$ for all h in Δ_1.

Lemma II.2. — *If f is any non-negative u-resolutive function on* Δ_1 *then the canonical measure of the harmonic function* $u.\mathcal{H}_{f,u}$ *is absolutely continuous with respect to* μ_u.

Proof. — It is noted first that the constants are u-resolutive. Let $f_n = \inf. (f, n)$ where n is any positive integer. Then f_n is u-resolutive and since $u\mathcal{H}_{f_n, u} \leqslant n.u$ (n being the u-solution of the constant n), the canonical measure ν_n of the harmonic function $u.\mathcal{H}_{f_n, u}$ satisfies the inequality $\nu_n \leqslant n.\mu_u$. Hence any Borel set with μ_u measure zero has also ν_n measure zero for every n. The measures ν_n are increasing and it can be be easily proved (using the inequality $\mathcal{H}_{f, u} \geqslant \mathcal{H}_{f_n, u}$) that their total measures are uniformly bounded. Hence ν_n is vaguely convergent to a measure ν. Once again it is easily seen that ν is the canonical measure of $u.\mathcal{H}_{f, u}$ since $\mathcal{H}_{f_n, u}$ increases to $\mathcal{H}_{f, u}$. Now it follows that every Borel set of μ_u measure zero has also ν measure zero, completing the proof.

III. — AXIOM OF RESOLUTIVITY AND CONSEQUENCES

We suppose throughout in this chapter,

axioms 1, 2, 3′ and D
existence of a potential > 0
countable base for open sets of Ω.
and the axiom to be introduced below.

Axiom $\mathcal{R}_u$. — *Every finite valued function uniformly continuous on* Δ_1 *(that is the restriction to* Δ_1 *of a finite continuous function on* $\overline{\Delta}_1 \subset \Lambda$, Λ *being the base of* S^+ *already introduced) is u-resolutive.*

In what follows we shall assume that the axiom $\mathcal{R}_u$ is satisfied for the harmonic function $u > 0$, chosen once for all for the considerations below.

1. The u-harmonic measures and μ_u.

The inclusion of the specified class of continuous functions in the class of all u-resolutive functions enables us to characterise the possible u-solutions. In fact we shall see that all the possible u-solutions are exactly the integrals for the measure μ_u of the summable functions and conversely.

Definition of the measure $\lambda_{x,u}$.

Let us define the following family of Radon measures on $\bar{\Delta}_1$. For every x in Ω, $\lambda_{x,u}$ is the linear functional on the class of finite continuous functions on $\bar{\Delta}_1$, which assigns to $\tilde{f}$ the value $\mathcal{H}_{f,u}(x)$ where f is the restriction to Δ_1 of $\tilde{f}$, f is by assumption u-resolutive. It is obvious that $\lambda_{x,u}$ are positive Radon measures.

Some Properties of the measures $\lambda_{x,u}$.

By the definition it is obvious that for any fixed function f finite and continuous on $\bar{\Delta}_1$, $\int f\, d\lambda_{x,u}$ is a u-harmonic function on Ω; hence it follows that for any function φ on $\bar{\Delta}_1$, $\bar{\int} \varphi\, d\lambda_{x,u}$ is $+\infty$ or $-\infty$ identically or a u-harmonic function on Ω. A similar conclusion is true for $\underline{\int} \varphi\, d\lambda_{x,u}$. Now consider a $\lambda_{x,u}$-summable function φ, for some x in Ω, that is the upper and the lower $\lambda_{x,u}$ integrals of φ are finite and equal. Then the $\underline{\int} \varphi\, d\lambda_{y,u}$ and $\bar{\int} \varphi\, d\lambda_{y,u}$ define two u-harmonic functions for y in Ω. But $\underline{\int} \varphi\, d\lambda_{y,u} \leqslant \bar{\int} \varphi\, d\lambda_{y,u}$ and the equality holds at one point x in Ω hence these two functions are equal at all the points of Ω. In other words the $\lambda_{x,u}$-summability of any function on $\bar{\Delta}_1$ is independent of x in Ω, and moreover the $\lambda_{x,u}$-integral defines in the case of a summable function a u-harmonic function on Ω. In particular sets of outer $\lambda_{x,u}$-measure zero on $\bar{\Delta}_1$ are independent of x in Ω.

Lemma III.1. — *Let ψ_1 and ψ_2 be two lower semi-continuous functions $> -\infty$ on $\bar{\Delta}_1$ such that their restrictions to Δ_1 are equal. Then, $\int \psi_1\, d\lambda_{x,u} = \int \psi_2\, d\lambda_{x,u}$ for all x in Ω.*

Proof. — Let ψ be the restriction of ψ_1 and ψ_2 to Δ_1. ψ_1 is the limit of an increasing sequence θ_n of continuous functions on $\bar{\Delta}_1$ and

$$\int \psi_1\, d\lambda_{x,u} = \lim_{n \to \infty}. \int \theta_n\, d\lambda_{x,u}.$$

But $\int \theta_n \, d\lambda_{x,u} = \mathcal{H}_{\theta'_n, u}(x)$, θ'_n being the restriction to Δ_1 of θ_n and hence,

$$\int \psi_1 \; d\lambda_{x,u} = \lim_{n \to \infty} . \mathcal{H}_{\theta'_n, u}(x) = \overline{\mathcal{H}}_{\psi, u}(x),$$

and similarly for ψ_2. Hence,

$$\int \psi_1 \, d\lambda_{x,u} = \overline{\mathcal{H}}_{\psi,u}(x) = \int \psi_2 \, d\lambda_{x,u}.$$

Lemma III.2. — *The set $\overline{\Delta}_1 - \Delta_1$ has $\lambda_{x,u}$ measure zero (for all x in Ω).*

Since Δ_1 is a measurable set it is enough to prove that the exterior $\lambda_{x,u}$ measure of Δ_1 is the measure of the whole space which is 1. Let $\psi \geqslant 0$ be any lower semi-continuous function $\psi \geqslant 1$ on Δ_1 (that is $\psi \geqslant$ the characteristic function of Δ_1). By the previous lemma we have

$$\int \psi \, d\lambda_{x,u} \geqq \int 1 \; d\lambda_{x,u} = \overline{\mathcal{H}}_{1,u}(x) = 1.$$

This is true for any such ψ and hence the exterior $\lambda_{x,u}$ measure of Δ_1 is $\geqslant 1$. But it is obviously $\leqslant 1$, since the measure of the whole space is 1. The required result follows.

In other words none of the measures $\lambda_{x,u}$ charges $\overline{\Delta}_1 - \Delta_1$ and hence it is enough for the purposes of $\lambda_{x,u}$ integrals to consider the values of any function on Δ_1.

Theorem III.1. — *Let f be any extended real valued function on Δ_1. Then for every x in Ω.*

$$\overline{\mathcal{H}}_{f,u}(x) \leqslant \overline{\int} f \, d\lambda_{x,u}.$$

Proof. — Let $\psi \geqslant f$ be a lower bounded and lower semi-continuous function on Δ_1. By Lemma III.1,

$$\overline{\mathcal{H}}_{\psi,u}(x) = \int \psi \, d\lambda_{x,u}. \quad \text{Since } \psi \geqslant f, \; \overline{\mathcal{H}}_{\psi,u} \geqslant \overline{\mathcal{H}}_{f,u};$$

and hence for every such ψ,

$$\overline{\mathcal{H}}_{f,u}(x) \leqslant \overline{\mathcal{H}}_{\psi,u}(x) = \int \psi \, d\lambda_{x,u}$$

and hence

$$\overline{\mathcal{H}}_{f,u}(x) \leqslant \text{Inf.} \int \psi \, d\lambda_{x,u} = \overline{\int} f \, d\lambda_{x,u}.$$

Obviously all this is valid whatever be the point x (in Ω).

COROLLARY. — *Any $\lambda_{x,u}$-summable function f on Δ_1 is also u-resolutive.*

From the theorem we deduce the inequality,

$$\overline{\int} f\, d\lambda_{x,u} \geqslant \overline{\mathcal{H}}_{f,u}(x) \geqslant \underline{\mathcal{H}}_{f,u}(x) \geqslant \underline{\int} f d\lambda_{x,u}.$$

and the assertion of the corollary follows immediately.

LEMMA III.3. — *Let $A \subset \Delta_1$ and φ_A be its characteristic function. Then $\underline{\mathcal{H}}_{\varphi_A, u} = 0$ if the inner μ_u measure of A is zero.*

Proof. — Since $\varphi_A \geqslant 0$ it is enough to show that $\underline{\mathcal{H}}_{\varphi_A, u} \leqslant 0$. Let w be any upper bounded hypo-u-harmonic function on Ω and such that fine $\limsup\limits_{x \to h} w(x) \leqslant \varphi_A(h)$ for every h in Δ_1. The inner μ_u measure of A is zero and hence by an appeal to the minimum principle (Theorem II.5) we deduce that $w \leqslant 0$. Hence $\mathcal{H}_{\varphi_A, u}$ which is the supremum of all such w is itself $\leqslant 0$.

Remark. — The Lemma III.3 is valid whether or not $\mathcal{R}_u$ is true for u.

LEMMA III.4. — *The measures $\lambda_{x,u}$ are absolutely continuous with respect to μ_u on the σ-algebra of Borel subsets of Δ_1.*

Proof. — Let A be a Borel set of Δ_1 with μ_u measure zero. The characteristic function of A is u-resolutive (Cor. to Theorem III.1.) and further $\int \varphi_A\, d\lambda_{x,u} = \mathcal{H}_{\varphi_A, u}(x)$. But by the above lemma $\underline{\mathcal{H}}_{\varphi_A, u} = 0$. Hence $\int \varphi_A\, d\lambda_{x,u} = 0$, establishing the lemma.

THEOREM III.2. — *The measures $\lambda_{x,u}$ are precisely*

$$[h(x)/u(x)]\mu_u(h).$$

That is, for any $\lambda_{x,u}$ — summable function f, $f(h)h(x)$ is μ_u summable and further

$$\int f d\, \lambda_{x,u} = \frac{1}{u\,(x)} \int f(h) h(x)\, d\mu_u(h).$$

Proof. — The measure $\lambda_{x,u}$ is absolutely continuous with respect to μ_u. By Radon-Nikodym Theorem [18, p. 41] there exists a μ_u summable function $\psi_x(h)$ (naturally depending on x) on Δ_1 such that for every $\lambda_{x,u}$-summable function f on Δ_1, $f(h)\psi_x(h)$ is μ_u summable and further

$$\int f\, d\lambda_{x,u} = \int f(h)\, \psi_x(h)\, d\mu_u(h).$$

And moreover any function which is equal to $\psi_x(h)$ μ_u almost everywhere also satisfies the same conditions. Now we shall prove that for every x in Ω, $h(x)/u(x)$ is a version of $\psi_x(h)$, in equivalent terms $[h(x)/u(x)] = \psi_x(h)$ μ_u almost everywhere on Δ_1.

Let A be a Borel set of Δ_1 with $\mu_u(\mathrm{A}) > 0$. Define

$$u_{\mathrm{A}} = \int_{\mathrm{A}} h\, d\mu_u(h).$$

u_{A} is the harmonic function on Ω with the canonical measure the restriction of μ_u to A. Consider the u_{A} Dirichlet problem.

Let v be a super-u-harmonic function $\geqslant 0$ and satisfying fine $\liminf_{x \to h} v(x) \geqslant \varphi_{\mathrm{A}}(h)$, φ_{A} being the characteristic function of A in Δ_1. Hence $u.v/u_{\mathrm{A}}$ which is a super-u_{A}-harmonic function satisfies

$$\text{fine.}\liminf_{h \to x} [v(x)u(x)]/u_{\mathrm{A}}(x) \geqslant \text{fine}\liminf v(x) \geqslant \varphi_{\mathrm{A}}(h)$$

for every h in Δ_1. Hence,

$$\frac{v(x)\, u(x)}{u_{\mathrm{A}}(x)} \geqslant \overline{\mathcal{H}}_{\varphi_{\mathrm{A}}, u_{\mathrm{A}}}(x).$$

The same property is true for all such v. Hence, in the above inequality v can be replaced by the infimum of such $v's$ that is,

$$\mathcal{H}_{\varphi_{\mathrm{A}}, u} = \overline{\mathcal{H}}_{\varphi_{\mathrm{A}}, u} \geqslant \frac{u_{\mathrm{A}}}{u} \cdot \overline{\mathcal{H}}_{\varphi_{\mathrm{A}}, u_{\mathrm{A}}}(x).$$

But the canonical measure of u_{A} has zero for measure of the set $\complement \mathrm{A}$ and hence by Lemma III.3 it follows that $\underline{\mathcal{H}}_{\varphi_{\complement \mathrm{A}}, u_{\mathrm{A}}}$ is zero which in turn gives $1 = \overline{\mathcal{H}}_{\varphi_{\mathrm{A}}, u_{\mathrm{A}}}$. Hence we have,

$$\mathcal{H}_{\varphi_{\mathrm{A}}, u} \geqslant [u_{\mathrm{A}}/u] \tag{1}$$

Moreover from the corollary to Theorem III.1, we have,

$$\mathcal{H}_{\varphi_A, u}(x) = \int \varphi_A \, d\lambda_{x,u} = \int \varphi_A(h) \, \psi_x(h) \, d\mu_u(h) \qquad (2)$$

Now, the inquality (1) and the equation (2) together give

$$\int \varphi_A(h)\psi_x(h) \, d\mu_u(h) \geqslant \frac{u_A(x)}{u(x)} = \int \varphi_A(h)[h(x)/u(x)] \, d\mu_u(h)$$

that is for every Borel set A of Δ_1,

$$\int [\psi_x(h) - \{h(x)/u(x)\}] \, d\mu_u(h) \geqslant 0.$$

This results in the inequality

$$\psi_x(h) \geqslant [h(x)/u(x)] \quad \mu_u\text{-almost everywhere on } \Delta_1. \ (3)$$

But then for the function 1 (the constant function) on Δ_1

$$\int [h(x)/u(x)] \, d\mu_u(h) = 1 = \mathcal{H}_{1,u} = \int d\lambda_{x,u} = \int \psi_x(h) \, d\mu_u(h).$$

Hence,

$$\int \{\psi_x(h) - [h(x)/u(x)]\} \quad d\mu_u(h) = 0. \qquad (4)$$

Now from (3) and (4) one deduces that

$$\psi_x(h) = [h(x)/u(x)] \quad \mu_u\text{-almost everywhere on } \Delta_1.$$

Now the proof is completed easily.

Note that the $\lambda_{x,u}$ summability and the μ_u summability are equivalent for any function on Δ_1. This is true because, for any fixed x in Ω there are constants α and β such that

$$\beta \geqslant h(x) \geqslant \alpha > 0$$

for all h in Δ_1 (and in fact on $\Lambda \cap H^+$ in view of the compactness of the base Λ and the continuity of $h(x)$ for fixed x).

The following theorem proves the converse of theorem III.1 viz., it establishes the equivalence between the u-resolutivity and the μ_u-summability for functions on Λ_1.

THEOREM III.3. — *Any function f on Δ_1 is u-resolutive if and only if it is μ_u — summable and moreover*

$$\mathcal{H}_{f,u}(x) = \frac{1}{u(x)} \int h(x) f(x) \, d\mu_u(h).$$

Proof. — The μ_u summability of a function implies the $\lambda_{x,u}$ summability which in turn asserts that f is u-resolutive (Cor. to Theorem III.1) and for such functions the u-solution equals the integral $\int \frac{h(x)}{u(x)} f(h)\, d\mu_u(h)$. It remains to prove the converse.

Suppose now that f is a u-resolutive function. Further let us suppose first that f is non-negative. Let ν_f be the canonical measure (on Δ_1) in the integral representation of $u.\mathcal{H}_{f,u}$. Since ν_f is absolutely continuous with respect to μ_u (Lemma II.2), there is a μ_u summable function $\tilde{f}$ unique up to μ_u measure zero such that $\nu_f = \tilde{f}\mu_u$. That is for every ν_f summable function g, $g\tilde{f}$ is μ_u summable and

$$\int g\tilde{f}\, d\mu_u = \int g\, d\nu_f.$$

For the function $\tilde{f}$ (which is u-resolutive since it is μ_u-summable),

$$\mathcal{H}_{\tilde{f},u}(x) = \frac{1}{u(x)} \int h(x)\, \tilde{f}(h)\, d\mu_u(h).$$

Also

$$\begin{aligned} u(x)\mathcal{H}_{f,u}(x) &= \int h(x)\, d\nu_f(h) \text{ (canonical rep.)} \\ &= \int h(x)\, \tilde{f}(h)\, d\mu_u(h) \end{aligned}$$

by the defining property of $\tilde{f}$.

Hence,

$$\mathcal{H}_{f,u} = \mathcal{H}_{\tilde{f},u}.$$

Suppose it is true that $\mathcal{H}_{g,u} = 0$ for a u-resolutive function g on Δ_1 implies $g = 0$ except on a set A which is u-negligible (that is $\overline{\mathcal{H}}_{\varphi_A,u} = 0$).

Then we get that $f = \tilde{f}$ except on a set $A \subset \Delta_1$ such that $\overline{\mathcal{H}}_{\varphi_A,u} = 0$. But by the corollary to Theorem II.7 we have $\int \varphi_A(h) h\, d\mu_u(h) = 0$. Hence it follows that f equals $\tilde{f}$ μ_u-almost everywhere, and

$$\mathcal{H}_{f,u}(x) = \frac{1}{u(x)} \int f(h)\, h(x)\, d\mu_u(h).$$

The same property is true for any u-resolutive function on Δ_1

since such a function is the difference of two non-negative u-resolutive functions.

It remains to verify that $\mathcal{H}_{g,u} = 0$ for a u-resolutive function g on Δ_1 implies that $g = 0$ except on a u-negligible set.

The function g^+ [$=$ sup. $(g, 0)$] is also u-resolutive and $\mathcal{H}_{g^+,u} \geqslant 0$. But then if v is in Σ_u and fine $\liminf\limits_{x \to h} v(x) \geqslant g(h)$ for every h in Δ_1, we have $v \geqslant \overline{\mathcal{H}}_{g,u} = 0$; consequently fine $\liminf\limits_{x \to h} v(x) \geqslant g^+(h)$. This is true for any such v and hence $0 = \mathcal{H}_{g,u} \geqslant \mathcal{H}_{g^+,u}$, i.e., $\mathcal{H}_{g^+,u} = 0$, and this clearly gives the required result.

2. The extension of Fatou-Naïm-Doob theorems.

We suppose

axioms 1,2,3′ and D
existence of a potential > 0 on Ω
countable base for the open sets
$u > 0$ is a harmonic function for which $\mathcal{R}_u$ is valid.

Under the above assumptions we shall prove that any non-negative super-u-harmonic function on Ω has a finite limit following the fine filters at every point of Δ_1 except for a set of μ_u measure zero. The method of proofs of the corresponding results in the classical cases [21, 12] go through in our case as well with little change.

Define for any extended real valued function f on Δ_1 the set

$$\Lambda_{f,u} = \left\{ v : \begin{array}{l} v \text{ is hyper-}u\text{-harmonic on } \Omega. \\ v \text{ is lower bounded on } \Omega, \text{ that is } v \geqslant \alpha_v \text{ for some} \\ \quad \alpha_v \text{ a real number depending on } v. \\ \text{for every } h \text{ in } \Delta_1, \text{ there exists a set } E_h^v \text{ not thin} \\ \quad \text{at } h \text{ with fine } \limsup\limits_{x \in E_h^v,\, x \to h} v(x) \geqslant f(h). \end{array} \right\}$$

THEOREM III.4. — *Corresponding to any extended real valued function f on Δ_1 the upper u-solution satisfies,*

$$\overline{\mathcal{H}}_{f,u} = \inf_{v \in \Lambda_{f,u}} v.$$

Proof. — We have evidently

$$\overline{\mathcal{H}}_{f,u} \geqslant \inf_{v \in \Lambda_{f,u}} v.$$

Hence there is nothing to prove if $\overline{\mathcal{H}}_{f,u}$ is $\equiv -\infty$. Let us assume for the moment that $\overline{\mathcal{H}}_{f,u}$ is finite and it is u-harmonic. Then there is a sequence φ_n of u-resolutive functions on Δ_1 such that (i) $\varphi_n \geqslant f$ and (ii) for each n a positive integer and a fixed $x \in \Omega$,

$$\overline{\mathcal{H}}_{f,u}(x) \leqslant \mathcal{H}_{\varphi_n,u}(x) \leqslant \overline{\mathcal{H}}_{f,u}(x) + 1/n.$$

This is possible since $\overline{\mathcal{H}}_{f,u}$ is the upper integral of the function f for the Daniell measure (defined for a fixed x in Ω on the class of all u-resolutive functions by assigning the value of the u-solution at the point x). Moreover we may assume that φ_n form a decreasing sequence. Then $\varphi = \text{limit } \varphi_n$ is u-resolutive and satisfies $\mathcal{H}_{\varphi,u}(x) = \overline{\mathcal{H}}_{f,u}(x)$. But since

$$\mathcal{H}_{\varphi,u} \geqslant \overline{\mathcal{H}}_{f,u}$$

the equality at x implies $\overline{\mathcal{H}}_{f,u} = \mathcal{H}_{\varphi,u}$.

Let $\varepsilon > 0$. The set of points $A \subset \Delta_1$ where $\varphi > f + \varepsilon$ is such that $\underline{\mathcal{H}}_{\varphi_A,u} = 0$. (This is easy to see since $\underline{\mathcal{H}}_{\varphi - f,u} = 0$ and $\varphi \geqslant f$). Hence the inner μ_u-measure of A is also zero (Theorem III.1, Corollary).

Let now v belong to $\Lambda_{f,u}$ and w be any upper bounded sub-u-harmonic function such that for every $h \in \Delta_1$, fine $\limsup_{x \to h} w(x)$ is less than or equal to $\varphi(h) - \varepsilon$. Consider $v - w$ which is a lower bounded super-u-harmonic function. Let $h \in \Delta_1$. By the definition of $\Lambda_{f,u}$ we have a set E_h not thin at h such that fine $\limsup_{x \in E_h, x \to h} v(x) \geqslant f(h)$. From Theorem II.4, we can choose a subset E'_h of E_h, not thin at h such that

$$\text{fine} \lim_{x \in E'_h, x \to h} v(x) = \text{fine} \limsup_{x \in E_h, x \to h} v(x) \geqslant f(h).$$

Now choose a subset E''_h of E'_h, not thin at h, and for which

$$\text{fine} \lim_{x \in E''_h, x \to h} w(x) = \text{fine} \limsup_{x \in E'_h, x \to h} w(x).$$

Obviously $\underset{x \in E''_h,\, x \to h}{\text{fine lim.}}\, v(x)$ exists and is $\geqslant f(h)$. Hence,

$$\begin{aligned}\underset{x \in E''_h,\, x \to h}{\text{fine lim.}}\, [v(x) - w(x)] &= \underset{x \in E''_h,\, x \to h}{\text{fine lim.}}\, v(x) - \underset{x \in E''_h,\, x \to h}{\text{fine lim.}}\, w(x) \\ &\geqslant f(h) - \underset{x \in E'_h,\, x \to h}{\text{fine lim. sup.}}\, w(x) \\ &\geqslant f(h) - \underset{x \in \Omega,\, x \to h}{\text{fine lim. sup.}}\, w(x).\end{aligned}$$

Hence,

$$\underset{x \in E''_h,\, x \to h}{\text{fine lim.}}\, [v(x) - w(x)] \geqslant f(h) - \varphi(h) + \varepsilon \geqslant 0, \text{ if } h \in \Delta_1 - A.$$

Since the set A has inner μ_u measure zero we have by Theorem II.5, that $w \leqslant v$. This inequality is valid for all $v \in \Lambda_{f,u}$ and all such w. Hence we have,

$$\underset{v \in \Lambda_{f,u}}{\text{Inf.}}\, v \geqslant \mathcal{H}_{\varphi - \varepsilon, u} = \mathcal{H}_{\varphi, u} - \varepsilon = \overline{\mathcal{H}}_{f,u} - \varepsilon.$$

The inequality is valid for all $\varepsilon > 0$, and this proves our assertion in the case when $\overline{\mathcal{H}}_{f,u}$ is finite.

Consider finally the case when $\overline{\mathcal{H}}_{f,u} \equiv +\infty$. We shall show that each v in $\Lambda_{f,u}$ is identically $+\infty$. Let $v \in \Lambda_{f,u}$. By definition $v \geqslant k$ for some real number k. Let

$$f_k = \text{sup.}\,(f, k).$$

Then $\overline{\mathcal{H}}_{f_k,u} = +\infty$. Now if $(f_k)_n = \text{inf.}\,(f_k, n)$ then clearly $(f_k)_n \uparrow f$, and also $\mathcal{H}_{(f_k)_n,u}$ is finite and increases to $+\infty$ when n tends to infinity. From the validity of the theorem for each $(f_k)_n$ we have

$$\overline{\mathcal{H}}_{(f_k)_n,u} = \underset{w \in \Lambda_{(f_k)_n,u}}{\text{Inf.}}\, w.$$

Clearly v is in $\Lambda_{(f_k)_n,u}$ for each n and hence $v \geqslant \overline{\mathcal{H}}_{(f_k)_n,u}$ and in turn $v \equiv +\infty$. This completes the proof.

THEOREM III.5. — *Let f be a u-resolutive function on Δ_1. Then $\mathcal{H}_{f,u}$ has the fine limit f at μ_u almost every point of Δ_1.*

Proof. — Define the function $\tilde{\varphi}$ on Δ_1 by

$$\tilde{\varphi}(h) = \underset{x \to h}{\text{fine lim.sup.}}\, \mathcal{H}_{f,u}(x).$$

Let $\varphi = \sup. (\tilde{\varphi}, f)$. Then if v is a super u-harmonic function (lower bounded on Ω) and with fine lim.inf. $v(x) \geqslant f(h)$ for every h in Δ_1, then $v \geqslant \mathcal{H}_{f, u}$ and so we have fine $\limsup_{x \to h} v(x) \geqslant \varphi(h)$. Hence by Theorem III.4 we have, $v \geqslant \overline{\mathcal{H}}_{\varphi, u}$. This is true for all such v and it follows that $\overline{\mathcal{H}}_{\varphi, u} \leqslant \mathcal{H}_{f, u}$. Now,

$$\underline{\mathcal{H}}_{\varphi, u} \geqslant \underline{\mathcal{H}}_{f, u} = \mathcal{H}_{f, u} \geqslant \overline{\mathcal{H}}_{\varphi, u}.$$

This implies that φ is u-resolutive and further $\mathcal{H}_{\varphi - f, u} = 0$, and hence $\varphi = f$ μ_u almost everywhere. That is,

$$\text{fine lim.sup.}\, \mathcal{H}_{f, u}(x) \leqslant f(h) \quad \mu_u \text{ almost everywhere.}$$

By considering $-f$ which is u-resolutive we have,

$$\text{fine lim.inf.}\, \mathcal{H}_{f, u}(x) \geqslant f(h) \quad \mu_u \text{ almost everywhere.}$$

This clearly proves the assertion of the theorem.

THEOREM III.6. — *Let $v \geqslant 0$ be a super-u-harmonic function on Ω. Then v has a finite fine limit μ_u almost everywhere on Δ_1.*

Proof. — Suppose $v \geqslant 0$ is a u-potential then (Theorem II.3) v tends to zero μ_u almost everywhere on Δ_1 following the fine filters. Since any super-u-harmonic function is the sum of a u-potential and a u-harmonic function, it suffices to prove that for any u-harmonic function (non-negative) the limit exists μ_u almost everywhere.

Let $w > 0$ be a u-harmonic function. If ν is the canonical measure on Δ_1 of the harmonic function uw, then consider Radon-Nikodym decomposition of the measure ν into the sum of two measures ν_1 and ν_2 where ν_1 is absolutely continuous with respect to μ_u and ν_2 singular with respect to μ_u. Let $w_1 = (1/u)\int h\, d\nu_1(h)$ and $w_2 = (1/u)\int h\, d\nu_2(h)$. But

$$d\nu_1(h) = f(h)\, d\mu_u(h)$$

for some function f; hence w_1 is the u-solution of f. Further w_1 tends to f μ_u-almost everywhere (Theorem III.5). Now, if u' is an element in H^+ such that $u' \leqslant u$ and also $u' \leqslant uw_2$ then its canonical measure on Δ_1 is identically zero because μ_u

and ν_2 are singular to each other. That is, $u' \equiv 0$. Hence inf. (u, uw_2) is a potential. Then the Theorem II.6 asserts that w_2 tends to zero μ_u-almost everywhere on Δ_1. The assertion of the theorem follows.

3. Variation of u and conditions equivalent to $\mathcal{R}_u$.

LEMMA III.5. — *Let $0 < u' \leqslant u$ be two harmonic functions on Ω. If $f \geqslant 0$ on Δ_1, then*

$$u\,\overline{\mathcal{H}}_{f,u} \geqslant u'\,\overline{\mathcal{H}}_{f,u'}; \qquad \text{and} \qquad u\,\underline{\mathcal{H}}_{f,u} \geqslant u'\,\underline{\mathcal{H}}_{f,u'}.$$

Proof. — The first inequality has been established for particular functions, viz. the characteristic functions of measurable sets of Δ_1 in the course of the proof of Theorem III.2. The proof for any $f \geqslant 0$ is exactly similar and we omit it.

To prove the second inequality, suppose that w is a sub-u'-harmonic function upper bounded on Ω and such that fine $\limsup\limits_{x \to h} w(x) \leqslant f(h)$ for every h in Δ_1. $\underline{\mathcal{H}}_{f,u'} = \sup. w$ for all such w and it is easy to see that $\underline{\mathcal{H}}_{f,u'}$ also equals sup. w^+ for all such w. Further w^+ is upper bounded too. Hence $(w^+u')/u$ is an upper bounded sub-u-harmonic function and satisfies fine $\limsup\limits_{x \to h} (w^+u')(x)/u(x) \leqslant f(h)$ for every h in Δ_1. It follows that $(w^+u')/u \leqslant \underline{\mathcal{H}}_{f,u}$.

This is true for all such w and hence $u'\underline{\mathcal{H}}_{f,u'} \leqslant u\underline{\mathcal{H}}_{f,u}$.

Let us denote by K the intersection with Δ_1 of a compact set of $\Lambda \cap H^+$ (Λ being the already chosen compact base of S^+) and u_K the harmonic function $u\,\mathcal{H}_{\varphi_K, u}$.

Consider the following three properties.

a) The characteristic function of any such K is u-resolutive.

b) For any two such disjoint sets K_1 and K_2, $u_{K_1} + u_{K_2}$ equals $u_{K_1 \cup K_2}$.

c) For any two such disjoint sets K_1 and K_2 $(u_{K_1})_{K_2} = 0$. Then we have.

THEOREM III.7. — *The properties a), b) and c) are true in case we suppose the axiom $\mathcal{R}_u$. Conversely, the axiom $\mathcal{R}_u$ is valid in case any one of the properties a), b) or c) is satisfied.*

Proof. — Let us suppose first that $\mathcal{R}_u$ is valid. Then evidently $a)$ is true. Moreover $u_K = \int_K h\, d\mu_u(h)$ and hence $b)$ results. To prove $c)$ note that $(u_{K_1})_{K_2} \leqslant$ both u_{K_1} and u_{K_2}. The canonical measure ν of this function is hence majorised by those of u_{K_1} and u_{K_2} which are the restrictions of μ_u to K_1 and K_2. Therefore $\nu = 0$ and hence $(u_{K_1})_{K_2} = 0$.

Conversely we shall show that each of $b)$ and $c)$ implies $a)$ which in turn gives the validity of $\mathcal{R}_u$.

We assume that $b)$ is true and prove $a)$. Let us start with a K of the above type. Then $\Delta_1 - K$ is the union or the increasing limit of a sequence of sets C_n which are of the same type as K. Since for each n, $K \cap C_n$ is void we have, $u_K + u_{C_n}$ equls $u_{K \cup C_n}$. Since $u_{C_n} \nearrow u\mathcal{H}_{\varphi_{\Delta_1 - K}, u}$ and

$$u_{K \cup C_n} \nearrow u\mathcal{H}_{\varphi_{\Delta_1}, u} = u,$$

we have,

$$\overline{\mathcal{H}}_{\varphi_K, u} + \overline{\mathcal{H}}_{\varphi_{\Delta_1} - K, u} = \mathcal{H}_{\varphi_{\Delta_1}, u} = 1.$$

But always,

$$\underline{\mathcal{H}}_{\varphi_K, u} \geqslant \mathcal{H}_{1, u} + \mathcal{H}_{-\varphi_{\Delta_1} - K, u} = 1 - \overline{\mathcal{H}}_{\varphi_{\Delta_1} - K, u}.$$

So we get

$$\underline{\mathcal{H}}_{\varphi_K, u} \geqslant \overline{\mathcal{H}}_{\varphi_K, u}.$$

This is clearly the u-resolutivity of φ_K. That is $b)$ implies $a)$.

Now we assume that the condition $c)$ is true for u and prove $a)$. Let K and C_n be sets in Δ_1 as defined in the previous paragraph. Then we have, $(u_K)_{C_n} = 0$ for every n. That is, $\overline{\mathcal{H}}_{\varphi_{C_n}, u_K}$ is zero. Hence limit of $\overline{\mathcal{H}}_{\varphi_{C_n}, u_K} = 0$. But φ_{C_n} increases to $\varphi_{\Delta_1 - K}$ and that leads to $\overline{\mathcal{H}}_{\varphi_{\Delta_1} - K, u_K} = 0$. Once again since $\underline{\mathcal{H}}_{\varphi_K, u_K} \geqslant 1 - \overline{\mathcal{H}}_{\varphi_{\Delta_1} - K, u_K}$, we have $\underline{\mathcal{H}}_{\varphi_K, u_K} = 1$. Hence φ_K is u_K-resolutive and further $\mathcal{H}_{\varphi_K, u_K} = 1$. But the Lemma III.5 asserts that

$$u\underline{\mathcal{H}}_{\varphi_K, u} \geqslant u_K \underline{\mathcal{H}}_{\varphi_K, u_K}$$

and so

$$u_K = u_K \mathcal{H}_{\varphi_K, u_K} \leqslant u\underline{\mathcal{H}}_{\varphi_K, u}.$$

Hence

$$\underline{\mathcal{H}}_{\varphi_K, u} \geqslant \frac{u_K}{u} = \overline{\mathcal{H}}_{\varphi_K, u}.$$

In other words the function φ_K is u-resolutive and $c)$ gives $a)$.

Now we go to prove the last part of the theorem. Let us assume that a) is satisfied and we shall prove the validity of the axiom of resolutivity.

Consider (for fixed x in Ω) the Daniell integral $\mathrm{I}_{x,u}$ on the class $\mathfrak{V}$ of all u-resolutive functions defined by starting from the functional $f \to \mathcal{H}_{f,u}(x)$ on the class $\mathfrak{A}$ of bounded and absolutely resolutive functions on Δ_1. We note that the $\mathrm{I}_{x,u}$-summable functions are exactly the absolutely u-resolutive functions. The condition a) states that $\mathfrak{A}$ contains the characteristic functions in Δ_1 of sets K such that K is the intersection with Δ_1 of a compact set in $\Lambda \cap \mathrm{H}^+$. This implies that $\mathfrak{V}$ contains the characteristic functions of all Borel measurable sets in Δ_1. Hence every bounded Borel measurable function on Δ_1 is u-resolutive. In particular the bounded uniformly continuous functions on Δ_1 are u-resolutive. That is the axiom $\mathfrak{R}_u$ is verified and this completes the proof.

COROLLARY. — *If axiom $\mathfrak{R}_u$ is valid for $u > 0$, then $\mathfrak{R}_w$ is true for every harmonic function satisfying $0 < w \leqslant mu$ for some real number m.*

Firstly we note that $\mathfrak{R}_{mu}$ is true. Now the corollary follows by an immediate application of the condition c).

IV. — THE CASE OF « PROPORTIONALITY »

We shall suppose that the harmonic functions satisfy:

$$\left\{\begin{array}{l}\text{Axioms } 1,2,3' \text{ and D}\\ \text{Existence of a potential} > 0 \text{ on } \Omega.\\ \text{Countable base for the open sets of } \Omega.\\ \text{Hypothesis of proportionality (which we recall below).}\end{array}\right.$$

Consider the potentials of a base of S^+, such potentials have point supports if they are extreme elements of the base. The converse of this question has not yet been decided, viz., (in an equivalent form) whether for every point of Ω the potentials > 0 on Ω with that point as support are proportional, i.e., equal up to a factor. In all the known examples this is satisfied. The assumption of this property is the *proportiona-*

lity hypothesis, more precisely *for every point y of Ω all the potentials > 0 on Ω with support at y are proportional.* R. M. Hervé calls the same the case of uniqueness [17].

Our main interest now is in showing the validity of the axiom $\mathfrak{R}_u$ for all the harmonic functions $u > 0$ on Ω under the assumptions mentioned above. In this case we identify Ω homeomorphically with the subset of extreme potentials on Λ and the boundary Δ of Ω contains Δ_1. This allows us to develope a second Dirichlet problem with the trace $\mathfrak{M}_h$ on Ω of neighbourhoods of the elements h in Δ, and the lower bounded hyper-u-harmonic functions on Ω. With the help of the techniques of [5] and [19] we prove that if $\overline{\mathfrak{D}}_{\varphi_K, u}$ is the upper solution ([8]) in this Dirichlet problem corresponding to the characteristic function of a compact set $K \subset \Delta$ then $u\overline{\mathfrak{D}}_{\varphi_K, u}$ is represented by an integral with a Radon measure on K. This is the key for us to establish that $\mathfrak{T}_h$ is finer than $\mathfrak{M}_h$ for all the elements h in $\Delta_1 \subset \Delta$. Then we show that the continuous functions on Δ are resolutive for the new Dirichlet problem, which in turn gives that $\mathfrak{R}_u$ is valid.

1. A second Dirichlet problem.

THEOREM IV.1. — *Ω is homeomorphic to the subset of Λ consisting of the extreme potentials (provided with the T-topology)* ([8 bis]).

Proof. — Let E^+ be the set of all extreme potentials and positive harmonic functions. We observe that according to a theorem of R. M. Hervé (prop. 22.1, [17]), the mapping $y \to p_y$ is a continuous function of $\Omega \to E^+ \cap \Lambda$, where p_y is the potential on Ω with support at y and with T-topology on the set $E^+ \cap \Lambda$. Moreover this mapping is one-one and onto the set of extreme potentials in $E^+ \cap \Lambda$. This enables us to identify Ω with this set of extreme potentials in A, set theoretically. The continuity of the mapping $y \to p_y$ shows that the induced topology on Ω (of the T-topology

([8]) As in II.4 even here our notation is different from that one of the classical case. Any of the solutions, if finite, are u-harmonic functions; that is, there is a difference of factor u.

([8 bis]) This result and the proof do not suppose axiom D.

on $E^+ \cap \Lambda$) is less fine than the topology of Ω. (More clearly $y_n \xrightarrow{\Omega} y$ implies that $p_{y_n} \xrightarrow{T} p_y$, i.e., $y_n \xrightarrow{T} y$ with the identification.) But (according to the Scolie 22.1, [17]), the T-convergence of a sequence of potentials p_{y_n} in E^+ with support at y_n to the potential $p_y \in E^+$ with the support at y already implies $y_n \xrightarrow{\Omega} y$. Since both the topologies in question have countable bases, the required homeomorphism is established.

Let Δ be the boundary of Ω in Λ. Since $E^+ \cap \Lambda$ is compact (Corollary to Prop. 21.3 [17]) $E^+ \cap \Lambda$ is a closed subset of Λ and further it contains Ω, hence the boundary Δ of Ω contains only harmonic functions of Λ. Moreover it is known [7] that the closure of the extreme potentials contains all the extreme harmonic functions of the base Λ, in other words Δ contains Δ_1.

Notations. — Let us call by $\mathfrak{M}_h$ for each $h \in \Delta$ the trace on Ω of the filter of all neighbourhoods of h in $\overline{\Omega}$. Let Σ_u be the set of all hyper-u-harmonic functions, each lower bounded on Ω, as in the previous chapter.

For any v in Σ_u, the condition that $\liminf_{\mathfrak{M}_h} v \geqslant 0$ for all h in Δ implies that $v \geqslant 0$. To prove this, suppose on the contrary $\inf. v(x) = -k$ with $k > 0$. The function $\overline{v}$ equal to v on Ω and 0 on Δ is lower semi-continuous on the compact space $\overline{\Omega}$. Hence $\overline{v}$ attains its infimum on $\overline{\Omega}$ which is obviously $-k$ and which is attained at some point of Ω. Then $v + k$ is $\geqslant 0$ and a hyper-u-harmonic function on Ω and equals zero at some point of Ω; that implies $v \equiv -k$. This is clearly impossible since we have started with the condition that $\liminf_{\mathfrak{M}_h} v \geqslant 0$. Hence v is non-negative.

In other words Σ_u and $(\mathfrak{M}_h)_{h \in \Delta}$ (which can be identified with Δ for considering the functions on this set) are associated to each other [6]. Let us denote by $\overline{\mathfrak{D}}_{f,u}$ (respectively $\underline{\mathfrak{D}}_{f,u}$) the upper solution (resp. the lower solution) corresponding to any extended real valued function f on Δ. $\overline{\mathfrak{D}}_{f,u}$ is $+\infty$ or $-\infty$ identically or a u-harmonic function on Ω. In case f is resolutive for this problem we denote the solution which is *a* u-harmonic function by $\mathfrak{D}_{f,u}$.

2. The integral representation of $\mathfrak{D}_{\varphi_K, u}$.

Definition. — *Let A be a set contained in* Δ. *Let* $\{\omega_i\}_{i\in I}$ *be the family of all open sets in* $\overline{\Omega}$ *containing A. Then define*

$$u_A = \inf_{i\in I} R_u^{\omega_i \cap \Omega}.$$

Lemma IV.1. — *Let* φ_A *be the characteristic function of* $A \subset \Delta$. *Then* $u_A = u\,\overline{\mathfrak{D}}_{\varphi_A, u}$.

Proof. — Each $(1/u)R_u^{\omega_i\cap\Omega}$ is a super-u-harmonic function $\geqslant 0$ and equal to 1 on $\omega_i \subset \Omega$, and hence $\underline{\lim.\inf.}_{\mathfrak{M}_h}(1/u)R_u^{\omega_i\cap\Omega} \geqslant 1$ for every h in A. It follows that $\overline{\mathfrak{D}}_{\varphi_A, u} \leqslant (1/u)R_u^{\omega_i\cap\Omega}$. This is true for every i in I and hence $u\overline{\mathfrak{D}}_{\varphi_A, u} \leqslant u_A$. Let $\varepsilon > 0$ and v in Σ_u such that the $\lim.\inf._{\mathfrak{M}_h} v \geqslant \varphi_A(h)$ for every h in Δ. Note that such a v is necessarily non-negative. There is a set ω_i of the family where $v \geqslant (1 - \varepsilon)$ and hence on Ω, $[uv/(1 - \varepsilon)] \geqslant R_u^{\omega_i \cup \Omega} \geqslant u_A$. Now by varying v and $\varepsilon > 0$ we get the inequality in the reverse direction which establishes the required result.

Corollary. — *If* $A_n \subset \Delta$ *for every n (any positive integer) and* A_n *increases to A (the union of* A_n*) then* $u_A = \lim.\, u_{A_n}$.

Lemma IV.2. — *Let* $K \subset \Delta$ *be a compact set. Let* (ω_n) *be a decreasing sequence of open sets in* $\overline{\Omega}$ *such that*

$$\omega_n \supset \varpi_{n+1} \supset \omega_{n+1} \supset K$$

for each n and further such that $\bigcap_n \omega_n = K$; *such a choice is possible. Then* $u_K = \lim_{n\to\infty} R_u^{\omega_n\cap\Omega}$.

Proof. — Evidently $R_u^{\omega_n\cap\Omega}$ decreases with increasing n and $\lim.R_u^{\omega_n\cap\Omega} \geqslant u_K$. Now suppose ω is any open set in $\overline{\Omega}$ containing K, then ω contains also ω_n for some n. (If not, each ω_n intersects $\complement\omega$ (which is compact), hence the same is true of ϖ_n too. Hence $\complement\omega \cap \varpi_n \neq \emptyset$ as ϖ_n are decreasing. But $K = \cap\, \varpi_n$

and $K \cap \complement \omega = \emptyset$, and this is impossible). This implies that $R_u^{\omega \cap \Omega} \geqslant \lim. R_u^{\Omega \cap \omega_n}$. In other words, $u_K = \lim. R_u^{\omega_n \cap \Omega}$.

THEOREM IV.2. — *Let h be a minimal harmonic function in Δ, that is h in Δ_1. Then for any $A \subset \Delta$, h_A is either 0 or h identically.*

Proof. — We know that $R_u^{\omega \cap \Omega}$ is h identically or a potential on Ω. In case $R_u^{\Omega \cap \omega_i}$ is identically h for every ω_i (open neighbourhoods of A in $\overline{\Omega}$) then $h_A \equiv h$. On the other hand if there is some $\omega \supset A$ for which $R_u^{\omega} \cap \Omega$ is a potential then $h_A (\geqslant 0)$ is zero since it is harmonic and minorises a potential.

THEOREM IV.3. — *Let K be a compact set in Δ. Then u_K is represented by an integral $\int h(x)\, d\lambda(h) = u_K(x)$ with a Radon measure λ on Δ supported by K.*

Proof. — We shall take a decreasing sequence of open sets ω_n in $\overline{\Omega}$ such that (i) $\cap\, \omega_n = K$ and (ii) $\omega_n \supset \varpi_{n+1}$ for every n. Let x_0 be a fixed point of Ω and δ a fixed regular neighbourhood of x_0. We may assume without loss of generality that

$$\overline{\delta} \cap \varpi_1 = \emptyset.$$

Define the function ψ_x on ϖ_1 by setting,

$$\begin{aligned} \psi_x(y) &= p_y(x) \qquad \text{for} \qquad y \in \varpi_1 \cap \Omega \\ \psi_x(h) &= h(x) \qquad \text{for} \qquad h \in \Delta \cap \varpi_1; \end{aligned}$$

where p_y is the potential on Λ with support at y (for all $x \in \Omega$). Then ψ_{x_0} is a continuous function and > 0. Hence there are two numbers α and β such that $0 < \alpha \leqslant \psi_{x_0} \leqslant \beta$, on ϖ_1.

Consider now $R_u^{\Omega \cap \omega_n}$ for any ω_n. Since $\omega_n \cap \Omega$ is the limit of an increasing sequence of compact sets $K_{n,p}$, we have that $R_u^{\Omega \cap \omega_n}$ equals the limit of the increasing sequence of potentials $\hat{R}_u^{K_{n,p}}$ (as p tends to infinity), quasi-everywhere on Ω.

Suppose $\nu_{n,p}$ is the measure on the set of extreme potentials in Λ (or equivalently a measure on Ω) in the integral representation of $\hat{R}_u^{K_{n,p}}$. Then (Lemma 22.1, [17]) the measure $\nu_{n,p}$ does not charge Ω-$K_{n,p}$ and further

$$\int p_y(x)\, d\nu_{n,p}(y) = \hat{R}_u^{K_{n,p}}(x) \quad \text{for every } x \in \Omega.$$

Now since $K_{n,p} \subset \omega_n$,

$$\int p_y(x_0)\, d\nu_{n,p}(y) \geq \alpha \int d\nu_{n,p}(y).$$

Hence,

$$\alpha \int d\nu_{n,p} \leqslant \int p_y(x_0)\, d\nu_{n,p}(y) = \hat{R}_u^{K_n,p}(x_0) \leqslant u(x_0).$$

This means that the measures $\nu_{n,p}$ for different p (whose supports are all contained in $\varpi_n \subset \varpi_1$) have their total measures bounded. Hence we may choose a subsequence which is vaguely convergent to a measure λ_n and the measure λ_n has support contained in ϖ_n. The measure also satisfies,

$$\alpha \int d\lambda_n \leqslant u(x_0).$$

The functions ψ_x are continuous on ϖ_n for all $x \in \Omega \cap \complement \varpi_n$. It follows that for x in $\Omega \cap \complement \varpi_n$ quasi-everywhere

$$R_u^{\Omega \cap \omega_n}(x) = \int_{\nu \in \overline{\Omega}} \psi_x(\nu)\, d\lambda_n(\nu). \left(\text{or same} \int_{p \in \Lambda} p\, d\lambda_n(p)\right)$$

[Actually it can be proved without much difficulty that

$$R_u^{\Omega \cap \omega_n}(x) = \int_{\nu \in \overline{\Omega}} \psi_x(\nu)\, d\lambda_n(\nu)$$

quasi-everywhere on Ω.]

Now the measures λ_n (with support contained in ϖ_1) have uniformly bounded total measures and hence we can choose a subsequence λ_{n_i} which is convergent vaguely to a measure λ; and the support of λ is K. Again it is easily seen that

$$\begin{aligned} u_K = \lim.\, R_u^{\Omega \cap \omega_n} &= \lim.\, R_u^{\omega_{n_i} \cap \Omega} \\ &= \lim. \int_\Lambda \nu\, d\lambda_{n_i}(\nu) \\ &= \int_\Lambda h\, d\lambda(h) = \int_K h\, d\lambda(h). \end{aligned}$$

This completes the proof.

3. The comparison of filters.

THEOREM IV.4. — *Let h be in Δ_1. Then $\mathfrak{T}_h$ is finer than $\mathfrak{M}_h$.*

Proof. — Let ω' be any open neighbourhood in $\overline{\Omega}$ of h. Let $\omega = \Omega \cap \omega'$. Then ω belongs to $\mathfrak{M}_h$ and such ω form a base for $\mathfrak{M}_h$. We want to show that $\Omega - \omega$ is thin at h which allows to conclude that $\mathfrak{T}_h$ is finer than $\mathfrak{M}_h$.

Let $K \subset \complement \omega' \cap \Delta$. Then K is compact. Suppose $h_K = 0$. Then there is an open set $\delta' \subset \overline{\Omega}$ and containing K and such that R_h^δ (where $\delta = \delta' \cap \Omega$) is a potential. Now $\delta' \cup \omega'$ is open in $\overline{\Omega}$ and contains Δ, hence $B = \complement(\delta' \cup \omega')$ is contained in Ω and is compact. Hence $\hat{R}_h^B$ is a potential. Now,

$$\hat{R}_h^{\Omega-\omega} \leqslant \hat{R}_h^B + \hat{R}_h^\delta$$

as $B \cup \delta \supset \Omega - \omega$. Hence $\hat{R}_h^{\Omega-\omega}$ is itself a potential and this implies that $\Omega - \omega$ is thin at h, which is exactly the required result.

Hence it suffices to prove that $h_K = 0$. Assume on the contrary that this is not true. Then $h_K = h$ by Theorem IV.2.

Supose now for every point $P \in K$, $h_{\{P\}} = 0$. Then there is an open neighbourhood N'_P of P in $\overline{\Omega}$ such that $N_P = \Omega \cap N'_P$ satisfies $R_h^{N_P}$ is a potential. We may choose a finite number of points $P_1, P_2, \ldots P_m$ in K such that $\bigcup_{i=1}^{m} N'_{P_i}$ contains K. Hence it follows that $R_h^{U \cap \Omega}$ is a potential where $U = \bigcup_{i=1}^{m} N'_{P_i}$. U is an open set in $\overline{\Omega}$ containing K and that R_h^U is a potential is impossible since $R_h^{U \cap \Omega} \geqslant h_K = h$ (by our assumption). Hence there is at least one point h' in K such that $h_{\{h'\}} \neq 0$ and so $h_{\{h'\}} = h$.

But by Theorem IV.2, $h_{\{h'\}}$ is an integral with a Radon measure supported by h'. In other words $h = \alpha\, h'$ where α is some (non-zero) positive real number. But this is impossible since h and h' belong to the same base and they cannot be proportional.

We have finally proved that $h_K = 0$ and this completes the proof of the theorem.

Corollary 1. — *For any extended real valued function f on Δ, if $\overline{\mathcal{H}}_{f,u}$ and $\underline{\mathcal{H}}_{f,u}$ are the upper and lower u-solutions corresponding to the restriction of f to Δ_1, then*

$$\overline{\mathfrak{D}}_{f,u} \geqslant \overline{\mathcal{H}}_{f,u} \geqslant \underline{\mathcal{H}}_{f,u} \geqslant \underline{\mathfrak{D}}_{f,u}.$$

This follows because any v in Σ_u which satisfies

$$\liminf_{\mathfrak{M}_h} v \geqslant f(h)$$

at every h in Δ also satisfies fine lim.inf. $v \geqslant f(h)$, for all $h \in \Delta_1$.

Corollary 2. — *For any compact set K in* Δ,

$$u_K = \int_{\Delta_1 \cap K} h \, d\mu_u(h).$$

We have already remarked that $u_K = \lim. R_u^{\omega_n \cap \Omega}$ where ω_n are decreasing open sets in $\overline{\Omega}$ with (i) $\omega_n \supset \overline{\omega}_{n+1}$ and (ii) $K = \bigcap_n \omega_n$. Now it is obvious that u_K is also the decreasing limit of u_n, u_n being the greatest harmonic minorant of $R_u^{\omega_n \cap \Omega}$. But by Theorem II.2, Cor., we have,

$$u_n = \int_{\Delta_1 - \mathcal{E}_n} h \, d\mu_u(h)$$

where $\mathcal{E}_n$ is the set on Δ_1 where $\Omega \cap \omega_n$ is thin. It is obvious that $\Delta_1 - \mathcal{E}_n$ contains $K \cap \Delta_1$; and in fact

$$K \cap \Delta_1 = \bigcap_n (\Delta_1 - \mathcal{E}_n).$$

Hence

$$\lim. u_n = \lim. \int \varphi_{\Delta_1 - \mathcal{E}_n}(h) h \, d\mu_u(h) = \int \varphi_{\Delta_1 \cap K}(h) h \, d\mu_u(h)$$

where $\varphi_{\Delta_1 - \mathcal{E}_n}$ and $\varphi_{\Delta_1 \cap K}$ are respectively the characteristic functions of $\Delta_1 - \mathcal{E}_n$ and $K \cap \Delta_1$, as $\varphi_{\Delta_1 - \mathcal{E}_n}$ decreases and tends to $\varphi_{\Delta_1 \cap K}$ when n tends to infinity. That is we have,

$$u_K = \int_{\Delta_1 \cap K} h \, d\mu_u(h).$$

4. The resolutivity of continuous functions on Δ.

Theorem IV.5. — *The axiom $\mathcal{R}_u$ is valid for all harmonic functions $u > 0$ on Ω.*

Proof. — We shall prove the theorem in an equivalent form, viz., the property *a*) of the Theorem III.7 is satisfied.

Let K be a compact set contained in Δ. Let K_n be an increasing sequence of compact sets in Δ such that their union is $\Delta - u$. We have (Corollary to Theorem IV.4),

$$u . \overline{\mathfrak{D}}_{\varphi_{K_n}, u} = u_{K_n} = \int_{\Delta_1 \cap K_n} h \, d\mu_u(h).$$

Hence with the Lemma IV.1 and its corollary we deduce,

$$u\overline{\mathfrak{D}}_{\varphi_{\Delta-}} = u_{\Delta-\mathrm{K}} = \lim.\ u_{\mathrm{K}_n} = \int_{\Delta_1-\mathrm{K}} h\, d\mu_u(h).$$

That is for every compact set $\mathrm{K} \subset \Delta$,

$$\overline{\mathfrak{D}}_{\varphi_{\mathrm{K}},u} + \overline{\mathfrak{D}}_{\varphi_{\Delta-\mathrm{K}},u} = 1.$$

From this we deduce that

$$\underline{\mathfrak{D}}_{\varphi_{\mathrm{K}},u} \geqslant \overline{\mathfrak{D}}_{\varphi_{\mathrm{K}},u} \qquad \text{since} \qquad \underline{\mathfrak{D}}_{\varphi\mathrm{K},u} \geqslant 1 - \overline{\mathfrak{D}}_{\varphi_{\Delta-\mathrm{K}},u}.$$

That is φ_{K} is resolutive for the second Dirichlet problem. But now from the corollary 1 to Theorem IV.4 it follows that

$$\overline{\mathcal{H}}_{\varphi_{\mathrm{K}\cap\Delta_1},u} = \mathfrak{D}_{\varphi_{\mathrm{K}},u} = \underline{\mathcal{H}}_{\varphi_{\mathrm{K}\cap\Delta_1},u}.$$

This is clearly the u-resolutivity of $\varphi_{\Delta_1\cap\mathrm{K}}$ and this completes the proof of the theorem.

Corollary. — *For any f on Δ*

$$\overline{\mathfrak{D}}_{f,u}(x) = \frac{1}{u(x)}\overline{\int} f(h)h(x)\, d\mu_u(h).$$

The u-resolutivity of any f on Δ valid is simultaneously for both the problems and the solutions $\mathfrak{D}_{f,u}$ and $\mathcal{H}_{f,u}$ are equal.

The u-resolutivity for the second Dirichlet-problem of the functions φ of φ_{K}-type and the equality

$$\mathfrak{D}_{\varphi,u} = \mathcal{H}_{\varphi,u} = \frac{1}{u}\int \varphi h\, d\mu_u(h)$$

for such functions imply the same for finite continuous functions. The integral representation of $\overline{\mathfrak{D}}_{f,u}$ is deduced now by the standard method of considering first lower bounded and l.s.c. f on Δ etc.

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