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# ON THE ISOMETRIES OF REFLEXIVE ORLICZ SPACES

by Gunter **LUMER** (1) (Seattle).

## I. Introduction.

The problem we consider is that of determining the form of all isometries of an arbitrary Orlicz space. A partial result on Orlicz space isometries was obtained in recent years by J. Lamperti [4]. It was also Lamperti who suggested the problem to us. In this paper we solve the question for any reflexive Orlicz space over a non atomic measure space (2). The method used is of a more general nature, and can be applied to the case of discrete or not purely atomic measure spaces. However for the latter cases additional new argument is needed. Also the results are not exactly the same, since other isometries than the analogous to the $L^p$ case may occur and we intend therefore to publish this material separately later.

The best known examples of reflexive Orlicz spaces are the Lebesgue $L^p$ spaces, for $1 < p < \infty$, and their isometries were determined by Banach [1] (at least for $L^p$ on the real line). In that case an isometry U is of the form $(\mathrm{U}f)(.) = u(.)f(\mathrm{T}.)$, where $f$ is any function in $L^p$, T a measurable set isomorphism, $u(.) = \mathrm{U}\,1(.)$ a fixed function in $L^p$.

The general Orlicz space situation is of a less explicit nature

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than are the $L^p$ cases, but for the situation treated here, the end result is essentially the same.

The ideas of the proof are closely related ot those used by the author to develop a unified approach to results of Banach [1], Stone [9] and Kadison [2] on isometries of $L^p$, $C(X)$ and $C^*$ algebras. Although the systematic use of semi-inner-products and hermitian operators [5] plays a central role, the paper is essentially self-contained.

## 2. Semi-inner-products.

The concept of a semi-inner-product was introduced in [5]. However we shall need here very little of the general theory, and proceed thus to state explicitly all the facts we shall actually use. A vector space E (which we shall consider complex unless mentioned otherwise) will be called a semi-inner-product space if to each pair $x, y \in E$ there corresponds a complex number denoted by $[x,y]$, and the following holds:

(*i*) $[x,y]$ linear in $x$.
(*ii*) $[x,x]$ real $> 0$ if $x \neq 0$.
(*iii*) $|[x,y]|^2 \leq [x,x]\,[y,y]$.

$[x, y]$ is called the semi-inner-product of $x$ and $y$. If E is a semi-inner-product space, then $([x,x])^{1/2}$ actually defines a norm on E. Conversely, given a normed vector space E, it always admits a semi-inner-product, such that $[x, x] = \|x\|^2$. There are however in general infinitely many semi-inner-products compatible with the same norm. Given an operator H on a semi-inner-product space E, we call it bounded if it is bounded with respect to the norm $([x,x])^{1/2}$.
An operator H on E will be called hermitian, if $[Hx,x]$ is real valued for every $x$ in the domain of H (in particular if H is bounded, for every $x \in E$).

In [5] it was shown that if H is a bounded hermitian operator on the semi-inner-product space E, then H is a (bounded of course) hermitian operator on any semi-inner-product space obtained by providing E with another semi-inner-product inducing the same norm.

We finish this section with a simple but useful lemma, going back, with different notation to McShane [7], and whose existence in the literature was called to our attention recently by C. A. McCarthy.

LEMMA 1. — *If* E *is any semi-inner-product space,* $x, y \in \mathrm{E}$, $||x|| = ([x, x])^{1/2}$, *then:*

$$||y|| \frac{d^-}{dt}\Big|_{t=0^-} \{||y + tx||\} \leq re\,[x, y] \leq ||y|| \frac{d^+}{dt}\Big|_{t=0^+} \{||y + tx||\}$$

*where* $\frac{d^-}{dt}, \frac{d^+}{dt}$, *denote left hand and right hand derivatives with respect to the real variable t. In particular, if the norm is differentiable, then*

$$[x, y] = ||y|| \frac{d}{dt}\Big|_{t=0} \{||y + tx|| + i||y - itx||\}$$

PROOF. — Suppose $y \neq 0$. Then

$$||y + tx||\,||y|| \geq |[y + tx, y]| = |||y||^2 + t[x, y]| \geq (||y||^4 + 2t||y||^2\, re\,[x, y])^{1/2}.$$

Thus $(||y + tx|| - ||y||)/t \geq ||y||^{-1}\, re\,[x, y] + o(t)$ for $t > 0$, and the right (left) hand side derivative exists by the convexity of the function $||y + tx||$. Similarly for $t \to 0^-$. If the norm is differentiable, from above $re\,[x, y] = ||y|| \frac{d}{dt}\Big|_{t=0} \{||y + tx||\}$, and the rest follows from $re\,[-ix, y] = im\,[x, y]$.

## 3. Semi-inner-products on Orlicz spaces.

From now on, $(\Omega, \Lambda, \mu)$ will denote a measure space, $\Lambda$ being the $\sigma$-ring of « measurable » subsets of $\Omega$, $\mu$ being countably additive, complete, and $\sigma$-finite. Let F denote the set of all measurable complex valued functions on $\Omega$.

From here on, and until further notice, E will denote the Banach function space (whose elements are functions in F) with unit sphere given by $\mathrm{S} = \{f \in \mathrm{F} : \int_\Omega \Phi(|f|) \leq 1\}$, where the integral is taken with respect to the measure $\mu$, $\mu(\Omega) < \infty$. Here $\Phi(t)$ is a non negative real valued convex function of the real variable $t \geq 0$. It shall be understood also that $\Phi$

is everywhere finite (i. e. does not «jump to $+\infty$»). It is well known that in these conditions S is in fact the unit sphere of a Banach space [6].

Since the function $\Phi$ is convex, it possesses a derivative except at most for a countable set of values of $t$ which we shall denote by $\Delta$. We shall say that a function $f \in F$ «avoids $\Delta$» if the $\mu$-measure of the set $\{x \in \Omega : |f(x)| \in \Delta\}$ is 0.

Lemma 2. — *Suppose $f$, $g \in L^\infty(\Omega, \Lambda, \mu)$, $\frac{g}{\|g\|}$ avoids $\Delta$; and let any semi-inner-product be given on* E; *then*

$$[f, g] = C(g) \int_\Omega f \Phi'\left(\frac{|g|}{\|g\|}\right) sgn\ g$$

*where*

$$sgn\ g = \begin{cases} \frac{|g|}{g} & where \quad g \neq 0 \\ 0 & where \quad g = 0 \end{cases}$$

$$C(g) = \left(\int_\Omega |g|\, \Phi'\left(\frac{|g|}{\|g\|}\right)\right)^{-1} \|g\|^2.$$

Proof. — We shall suppose first $\|g\| = 1$, and $t$ sufficiently small whenever needed. Then we have, since $\mu(\Omega) < \infty$

$$\int_\Omega \Phi(|g|) = 1, \qquad \int_\Omega \Phi\left(\frac{|g + tf|}{\|g + tf\|}\right) = 1.$$

We shall write

$$\delta_t = \frac{|g + tf|}{\|g + tf\|} - |g|$$

$\delta_t \in L^\infty$ for any fixed $t$, and $\|\delta_t\|_\infty$ is uniformly bounded for $t$ small. Here of course $\|\ \|_\infty$ denotes the usual « essential supremum » norm. One has explicitly :

$$\delta_t = (\|g + tf\|)^{-1} \{(-\|g + tf\| + 1)|g| + t \operatorname{re} f \operatorname{sgn} g + o(t)\}.$$

On the other hand since $\Phi(t) = \int_0^t \varphi(s)\, ds$, where $\varphi$ is non negative, monotone increasing with discontinuities on the set $\Delta$, it follows that

$$\varphi(|g|)\, \delta_t \leq \Phi\left(\frac{|g + tf|}{\|g + tf\|}\right) - \Phi(|g|) \leq \varphi(|g| + \delta_t)\, \delta_t$$

and also since $g$ avoids $\Delta$, we have that $\varphi(|g| + \delta_t) \to \varphi(|g|)$

almost everywhere as $t \to 0$. But we have (identically in $t$):

$$\int_\Omega \left\{ \Phi\left( \frac{|g+tf|}{\|g+tf\|} \right) - \Phi(|g|) \right\} = 0$$

and dividing first by $t > 0$, and then by $t < 0$, and passing to the limit we obtain:

$$\left.\frac{d^+}{dt}\right|_{t=0^+} \{\|g+tf\|\} \int_\Omega |g|\varphi(|g|) + \int_\Omega \operatorname{re} f\varphi(|g|) \operatorname{sgn} g = 0,$$
$$\left.\frac{d^-}{dt}\right|_{t=0^-} \{\|g+tf\|\} \int_\Omega |g|\varphi(|g|) + \int_\Omega \operatorname{re} f\varphi(|g|) \operatorname{sgn} g = 0.$$

Thus using Lemma 1, and replacing $f$ by $-if$ and finally $g$ by $g/\|g\|$, we obtain

$$[f, g] = C(g) \int_\Omega f\varphi\left( \frac{|g|}{\|g\|} \right) \operatorname{sgn} g = C(g) \int_\Omega f\Phi'\left( \frac{|g|}{\|g\|} \right) \operatorname{sgn} g$$

where we may substitute $\Phi'$ for $\varphi$ since $\frac{|g|}{\|g\|}$ avoids $\Delta$.

Lemma 3. — *Suppose* $g \in L^\infty$, $g \neq 0$, $\frac{|g|}{\|g\|}$ *avoids* $\Delta$. *Let* $f \in E$, *and any semi-inner-product be given on* E *(compatible with the norm), then if* E *has an absolutely continuous norm* [6, p. 12]

$$[f, g] = C(g) \int_\Omega f\Phi'\left( \frac{|g|}{\|g\|} \right) \operatorname{sgn} g.$$

Proof. — If E has an absolutely continous norm, then for $f \in E$, there is a sequence $f_n \in L^\infty$ such that $f_n \to f$ in the E-norm [6]. Since $\Phi'\left( \frac{|g|}{\|g\|} \right)$ is bounded $f\Phi'\left( \frac{|g|}{\|g\|} \right)$ has a finite integral, hence by Lemma 2

$$[f_n, g] = C(g) \int_\Omega f_n\Phi'\left( \frac{|g|}{\|g\|} \right) \operatorname{sgn} g \to C(g) \int_\Omega f\Phi'\left( \frac{|g|}{\|g\|} \right) \operatorname{sgn} g$$

but clearly also $[f_n, g] \to [f, g]$.

Next let us remove the restriction concerning « $g$ avoids $\Delta$ ».

Lemma 4. — *Suppose* E *has absolutely continous norm,* $f \in E$, $g \in L^\infty$, $g \neq 0$, *then there exists a semi-inner-product on* E *compatible with the norm, such that*

$$[f, g] = C(g) \int_\Omega f\varphi\left( \frac{|g|}{\|g\|} \right) \operatorname{sgn} g$$

*where*

$$\varphi(t) = \begin{cases} \Phi(t) & \textit{for } t \textit{ not in } \Delta \\ \dfrac{1}{2}\left\{\left(\dfrac{d^+}{dt}\Phi\right)(t) + \left(\dfrac{d^-}{dt}\Phi\right)(t)\right\} & \textit{for } t \in \Delta. \end{cases}$$

PROOF. — Let us assume for the moment that given $g \in L^\infty$ there exist two sequences of functions $g'_n$, $g''_n \in L^\infty$, such that $g'_n \to g$, $g''_n \to g$ (as $n \to \infty$) pointwise and in the norm of E; $\operatorname{sgn} g'_n = \operatorname{sgn} g''_n = \operatorname{sgn} g$;

$$\varphi\left(\frac{|g|}{\|g\|}\right) = \frac{1}{2}\left\{\varphi\left(\frac{|g'_n|}{\|g'_n\|}\right) + \varphi\left(\frac{|g''_n|}{\|g''_n\|}\right)\right\} + \rho_n, \qquad \|\rho_n\|_\infty = O\left(\frac{1}{n}\right);$$

also $g'_n$, $g''_n$ are uniformly bounded in the $L^\infty$ norm and all avoid $\Delta$. Then we have: $C(g'_n) \to C(g)$, $C(g''_n) \to C(g)$. Let us now consider E provided with any semi-inner-product compatible with the norm (which always exists). Then for $f \in E$, Lemma 3 implies that:

$$C(g)\int_\Omega f\varphi\left(\frac{|g|}{\|g\|}\right)\operatorname{sgn} g$$
$$= \frac{1}{2}\left\{(C(g)/C(g'_n)\,[f, g''] + C(g)/C(g''_n)\,[f, g''_n]\right\} + O\left(\frac{1}{n}\right).$$

Thus

$$\left|C(g)\int_\Omega f\varphi\left(\frac{|g|}{\|g\|}\right)\operatorname{sgn} g\right|$$
$$\leq \frac{1}{2}\left\{|C(g)/C(g'_n)|\,\|g'_n\| + |C(g)/C(g''_n)|\,\|g''_n\|\right\}\|f\| + O\left(\frac{1}{n}\right),$$

$$C(g)\int_\Omega |g|\varphi\left(\frac{|g|}{\|g\|}\right)$$
$$= \frac{1}{2}\left\{C(g)/C(g'_n)[g, g'_n] + C(g)/C(g''_n)[g, g''_n]\right\} + O\left(\frac{1}{n}\right)$$

and

$$\left|\|g'_n\|^2 - [g, g'_n]\right| = \left|[g_n - g, g'_n]\right| \leq \|g'_n - g\|\,\|g'_n\| \to 0.$$

It follows that the conditions for a semi-inner-product are satisfied.

To complete the proof we shall show how $g'_n$, $g''_n$ are obtained. Consider the sequence of sets $\Omega_m \subset \Omega$ such that $|g(\Omega_m)| \in \Delta$, $|g|$ being constant on such $\Omega_m$. If $\chi_m$ is the characteristic function of $\Omega_m$, then it is easy to find numbers $\varepsilon'_{nm}$, such that if

$$g_n = \begin{cases} g + \varepsilon'_{nm}\chi_m & \text{on } \Omega_m \\ g & \text{on } \Omega - \bigcup_1^\infty \Omega_m \end{cases}$$

and similarly for $g''_n$, then all above conditions are satisfied (here of course the fact that $\Delta$ is countable plays a role).

## 4. Hermitian operators.

The task being eventually to determine the bounded hermitian operators on various classes of Orlicz spaces, we derive some general properties, assuming absolutely continuous norm.

LEMMA 5. — *Let* H *be any bounded hermitian operator on* E, $f'$, $f'' \in L^\infty(\Omega, \Lambda, \mu)$ *not identically zero, and with supports* $\Omega'$, $\Omega''$ *such that* $\mu(\Omega' \cap \Omega'') = 0$. *Then*

$$\int_{\Omega''} Hf'\varphi\left(\frac{|f'|}{\|f' + f''\|}\right) \operatorname{sgn} f'' = \int_{\Omega'} Hf''\varphi\left(\frac{|f''|}{\|f' + f''\|}\right) \operatorname{sgn} f'.$$

PROOF. — It is clear that $\|f' + f''\| = \|e^{i\alpha}f' + e^{i\beta}f''\|$ where $\alpha$, $\beta$ are reals; applying now Lemma 4, to

$$[H(e^{i\alpha}f' + e^{i\beta}f''), \quad e^{i\alpha}f' + e^{i\beta}f''] = \text{real}$$

we obtain

$$\int_{\Omega'} Hf'\varphi\left(\frac{|f'|}{\|f' + f''\|}\right) \operatorname{sgn} f' + \int_{\Omega''} Hf''\varphi\left(\frac{|f''|}{\|f' + f''\|}\right) \operatorname{sgn} f''$$
$$+ e^{i(\alpha-\beta)} \int_{\Omega''} Hf'\varphi\left(\frac{|f''|}{\|f' + f''\|}\right) \operatorname{sgn} f''$$
$$+ e^{-i(\alpha-\beta)} \int_{\Omega'} Hf''\varphi\left(\frac{|f'|}{\|f' + f''\|}\right) \operatorname{sgn} f' = \text{real}.$$

Since $\alpha$ and $\beta$ are arbitrary the above conclusion readily follows.

An immediate consequence is

LEMMA 6. — *If* $\Omega'$, $\Omega'' \in \Lambda$, *are (almost) disjoint, i. e.* $\mu(\Omega' \cap \Omega'') = 0$, $\chi'$, $\chi''$ *their characteristic functions, then* $\int_{\Omega'} H\chi'' = 0$ *if and only if* $\int_{\Omega''} H\chi' = 0$.

LEMMA 7. — *If* $h \in L^\infty(\Omega, \Lambda, \mu)$ *is real valued, the operator* $H_h$ *defined by* $H_h f = hf$ *for all* $f \in E$ *is a bounded hermitian operator; and* $||H_h|| = ||h||_\infty$.

PROOF. — We shall prove the hermiticity of $H_h$ directly since our formula for the semi-inner-products is not proved for all $f, g \in E$. We shall construct a special semi-inner-product in the following way: given $f \in E$, $f = uf_r$ where $|u| \equiv 1$, $f_r$ is real valued. Using the real Banach space $E_r$ generated by the real valued functions in E, we obtain a functional $f_r^*$ such that $(f_r, f_r^*) = ||f_r||^2$, $||f_r^*|| = ||f_r||$. We extend $f_r^*$ to E by setting $f^*(g_1 + ig_2) = f_r^*(g_1) + if_r^*(g_2)$ where $g_1$, $g_2$ are real and imaginary components of $g \in E$ (and since $|g_1| \leq |g|$ implies $||g_1|| \leq ||g||$, $g_1, g_2 \in E$). Then

$$|f_0^*(g)| = e^{i\alpha} f_0^*(g) = f_0^*(e^{i\alpha} g) = f_r^*(\text{re } e^{i\alpha} g) \leq ||\text{re } e^{i\alpha} g|| \, ||f_r^*|| \leq ||g|| \, ||f||.$$

Next define $f^* \in E^*$, by setting $f^*(g) = f_0^*(u^{-1} g)$, where $u$ is the fixed function defined above. Then we have

$$f^*(f) = f_0^*(f_r) = ||f_r||^2 = ||f||^2, \qquad ||f^*|| = ||f_0^*|| = ||f||.$$

There is thus a semi-inner-product such that

$$[Hf, f_h] = f^*(hf) = f_0^*(n\ {}^1 huf_r) = \text{real}.$$

The rest is easily shown.

The following lemma will be needed later on.

LEMMA 8. — *Let* $\Omega_0$ *be any measurable subset of* $\Omega$, *and* P *the projection of* E *on the subspace* $E_0$ *of functions in* E *vanishing outside* $\Omega_0$, *then* $PH = PHP$ *is a hermitian operator on* $E_0$.

PROOF. — Cleary $||P|| = 1$; and for any $f \in E_0$, I the identity operator, $t$ real, $||(I + itPHP)f|| = ||P(I + itH)Pf|| \leq (1 + o(t))||f||$ since H is hermitian; (see[5], p. 39); and it then follows easily that $||I + itPHP|| = 1 + o(t)$, which proves PHP is hermitian.

## 5. Hermitians and isometries when the measure space is non-atomic.

THEOREM 9 (3). — *Suppose* H *is a bounded hermitian operator on* E, E *is reflexive, and the measure space* $(\Omega, \Lambda, \mu)$ *contains no* $\mu$*-atoms, then either there exists a real valued function*

(3) A recent letter from C. A. McCarthy shows that McCarthy independently knew, or at least had conjectured, Theorem 9.

$h \in L^\infty(\Omega, \Lambda, \mu)$ *such that* $Hf = hf$, for $f \in E$, and $\|H\| = \|h\|_\infty$, *or else* $E = L^2(\Omega, \Lambda, \mu)$.

Proof. — Let $\Omega'$ be any measurable subset of $\Omega$, and $\chi'$ its characteristic function. Suppose now that $H\chi'$ is not identically zero (almost everywhere) on $\Omega - \Omega'$, then there exists a measurable set (of positive measure) $\Omega''$ on which $\int H\chi' \neq 0$, and $\Omega'' \subset \Omega - \Omega'$. Let $\chi''$ be the characteristic function of $\Omega''$, and $\alpha$ any real non-negative number.

We have from Lemma 5 [4] :

$$\int_{\Omega'} H\alpha\chi''\varphi\left(\frac{\chi'}{\|\alpha\chi'' + \chi'\|}\right) = \int_{\Omega''} H\chi'\varphi\left(\frac{\alpha\chi''}{\|\alpha\chi'' + \chi'\|}\right)$$

$$\alpha\int_{\Omega'} H\chi''\varphi\left(\frac{\chi'}{\|\chi'' + \chi'\|}\right) = \alpha\int_{\Omega''} H\chi'\varphi\left(\frac{\chi''}{\|\alpha\chi'' + \chi'\|}\right)$$

hence :

$$\left\{\varphi\left(\frac{\alpha}{\|\alpha\chi'' + \chi'\|}\right) - \alpha\varphi\left(\frac{1}{\|\alpha\chi'' + \chi'\|}\right)\right\}\int_{\Omega''} H\chi' = 0$$

and a contradiction is reached unless

$$\varphi\left(\frac{\alpha}{\|\alpha\chi'' + \chi'\|}\right) = \alpha\varphi\left(\frac{1}{\|\alpha\chi'' + \chi'\|}\right)$$

but since the measure space is non atomic we may without changing the situation replace $\Omega''$ by subsets of arbitrary small measure so that ($\varphi$ may be supposed continuous from the left) we reach a contradiction, or else

$$\varphi\left(\frac{\alpha}{\|\chi'\|}\right) = \alpha\varphi\left(\frac{1}{\|\chi'\|}\right) \quad \text{for all} \quad \alpha > 0.$$

From this it would follow $\Phi(t) = Ct^2$ and $E = L^2(\Omega, \Lambda, \mu)$. Now if we exclude $E = L^2$, we conclude $H\chi' = 0$ on $\Omega - \Omega'$.

Next suppose $f$ is a step function, and $\Omega'$ is the « support of one step of $f$», then by the previous argument, since, the support of $f - f(\Omega')1$ is in $\Omega - \Omega'$, $H(f - f(\Omega')1) = 0$ on $\Omega'$, so that on $\Omega'$, $Hf = (H1)f(\Omega')$, where 1 denotes of course the constant function 1. We conclude by linearity $Hf = hf$, where $h = H1$. The step functions are dense, and the proof is complete.

[4] Since E is reflexive the norm is absolutely continuous, [6]. Theorem 9 remains valid if the hypothesis « E is reflexive » is replaced by « E has an absolutely continuous norm ».

THEOREM 10. — *Let* U *denote an isometry from* E *onto* E. *Suppose same hypothesis as in Theorem* 9, *and that* $E \neq L^2(\Omega, \Lambda, \mu)$, *then there exists a measurable set isomorphism* T *(i. e. not necessarily a point transformation in* $\Omega$), *and a fixed function* $u \in E$, *such that*

$$(Uf)(.) = u(.)f(T.) \quad \text{for all } f \in E.$$

PROOF. — By Lemma 7, if $h \in L^\infty$, $H_h f = hf$ defines a hermitian operator. On the other hand given a semi-inner-product on E, then $[U^{-1}f, U^{-1}g]$ defines a new semi-inner-product compatible with the norm, and it follows that if H is a bounded hermitian operator, the same holds for $UHU^{-1}$. Hence by Theorem 9, for each real valued $h \in L^\infty$, there exists a real valued $\hat{h} \in L^\infty$ such that $UH_hU^{-1}f = \hat{h}f, f \in E$; i. e.

$$UhU^{-1}f = \hat{h}f, \quad f \in E.$$

From above relation follows also at once that the correspondence « ^ » is multiplicative as well as linear, and one-one onto (in the real $L^\infty$). In particular « ^ », when applied to $h = \chi'$ where $\chi'$ is the characteristic function of a generic measurable set $\Omega' \in \Lambda$, gives $\hat{\chi}' = \hat{\chi}'^2$. $\hat{\chi}'$ is the characteristic function of a set $\Omega''$ which we shall denote $T^{-1}\Omega'$ (notice that in general $||h||_\infty = ||H_h|| = ||UH_hU^{-1}|| = ||\hat{h}||_\infty$). It is immediate that $T^{-1}$ is a set isomorphism, and we have

$$U\chi'g = \hat{\chi}'Ug, \quad g \in E$$

so that if $g$ vanishes on $\Omega'$, then $Ug$ vanishes on $T^{-1}\Omega'$, from this it follows, as done at the end of Theorem 9, that for any step function $f$, $(Uf)(T^{-1}_{.}) = (U1)(T^{-1}.)f(.)$, hence if $U1 = u$

$$(Uf)(.) = u(.)f(T.)$$

for all step functions; but these are dense in the present situation, [6].

We shall now connect the previous results with the usual Orlicz space definitions (and notations) in the literature. Here we follow [6], (compare also [8], [10] Chapter 4, and [3]).

Reflexivity assumed, the question reduces in terminology of [6] to the isometries of $L_{M\Phi}$ and $L_\Phi$. Theorem 10 applies directly to $L_{M\Phi}$ (with the same assumptions on the measure space). The result also follows readily for $L_\Phi$, since in our situation $L_\Phi^* = L_{M\Phi}$, and if U is an isometry (H a hermitian) on $L_\Phi$, then $U^*$ is an isometry a her ($H^*$mitian) on $L_{M\Phi}$.

### 6. $\sigma$-finite measure spaces.

So far we have assumed that $\mu(\Omega) < \infty$. One can see, however, that this restriction may now be dropped, in Theorems 9 and 10, and substituted by the assumption that « $\mu$ is $\sigma$-finite ». One starts by extending Theorem 9:

Let thus $\Omega_n$ be an increasing sequence of subsets of finite measure. If H is a bounded hermitian on E, Lemma 8 shows that the restriction $H_n$ of H to the $E_n$-space corresponding to $\Omega_n$, is hermitian, hence of the form $h_n f$. The rest of the argument is clear (since we have absolute continuity of the norm).

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