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ON CAPACITABILITY AND MEASURABILITY

by **Maurice SION** (1) (Vancouver).

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1. — INTRODUCTION

The definition of capacitability of a set as equality of its inner and outer capacities bears an obvious resemblance to the definition of measurability of a set as equality of its inner and outer measures, which occurs in many developments of measure theory in a topological space. From this point of view, measurability is just capacitability when the capacity is a measure.

In non-topological spaces as well as in many topological situations such an approach to measure and measurability is not at all suitable. There, a much more satisfactory point of view is that of Caratheodory measure and the corresponding notion of measurability (see definitions 3. 1 and 3. 2). In this context, however, a measure need not be a capacity and, even if it is, measurability and capacitability are unrelated. Thus, in general, theorems about capacitability do not imply corresponding theorems about measurability nor vice-versa. In the case of analytic sets, this is rather frustrating, for the arguments used in the proofs of the capacitability and measurability of an analytic set for various classes of capacities and measures are very similar (see [2, 4, 6, 8]). As a matter of fact, similar arguments are also used to prove other proper-

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ties of analytic sets, e.g. that they are Lindelöf (see [8]). Is there a central notion that ties all these facts together?

An attempt to isolate such a notion was made by Choquet in [4]. His definition of an abstract capacity goes a long way toward including that of a Caratheodory measure. It does not quite do it and, as he points out himself, his general capacitability theorem does not yield all the results it should even on capacitability, e.g. theorem 3. 2 in [8] does not follow from it. Only the idea of the proof is applicable.

In this paper, we solve the problem by introducing the notion of outer content (3. 7), and the more abstract ones of outer family (5. 3) and capacitance (5. 5). First, in section 3, we study the relation between measurability and capacitability and delineate that portion of measure theory which can be included in the theory of capacities. In section 4, we discuss Newtonian capacity as a measure. For this measure, very important sets are not measurable but are capacitable. The definition is new, but it coincides with the standard one on capacitable sets and should replace the inner capacity in most problems where the latter is used. In section 5, we abstract the notions of analytic set, outer measure and capacity and prove a central theorem (5. 4) which yields various properties of analytic sets as special cases.

2. — NOTATION

2. 1 *dmn f* denotes the domain of *f*.

2. 2 ω denotes the set of natural numbers.

2. 3 0 is both the empty set and the smallest element in ω.

2. 4 H is a σ-field in X *iff* H is a family of subsets of X closed under countable unions and complements with respect to X.

2. 5 μ is additive on H iff

$$\mu\left(\bigcup_{n\in\omega} A_n\right) = \sum_{n\in\omega} \mu A_n$$

whenever $A_n \in H$ and $A_n \cap A_m = 0$ for $n,\ m \in \omega,\ n \neq m$.

2. 6 $A \sim B = \{x : x \in A \text{ and } x \notin B\}$.

3. — MEASURABILITY AND CAPACITABILITY

In measure theory there are two big streams of thought: additivity and approximability. Caratheodory's approach to measurability is concerned primarily with additivity properties of a function which is assumed to be only subadditive to begin with. To be precise, let us introduce the following definitions.

DEFINITION 3. 1. — *μ is a measure on X iff μ is a function on the family of all subsets of X such that*

(i) $$\mu 0 = 0$$

and

(ii) $0 \leqq \mu A \leqq \sum_{n \in \omega} \mu B_n$ *whenever* $A \subset \bigcup_{n \in \omega} B_n \subset X$.

DEFINITION 3. 2. — *A is μ-measurable iff $A \in dmn\mu$ and for every $T \in dmn\mu$,*

$$\mu T = \mu(T \cap A) + \mu(T \sim A).$$

Then, for any measure μ, the family of μ-measurable sets forms a σ-field on which μ is additive. The idea of starting with an additive function on a σ-field avoids the problem completely and is not really more general since one can, and usually does, extend the function to an (outer) measure μ such that the family of μ-measurable sets includes the original σ-field. This measure possesses the property incorporated in the following definition.

DEFINITION 3. 3. — *μ is an outer measure on X iff μ is a measure on X and for every $A \subset X$ there exists a μ-measurable set B such that $A \subset B$ and $\mu A = \mu B$.*

An essential property of outer measures that will concern us later on is the following (see [6], p. 51).

THEOREM 3. 4. — *If μ is an outer measure on X and $A_n \subset A_{n+1} \subset X$ for $n \in \omega$ then*

$$\mu\left(\bigcup_{n \in \omega} A_n\right) = \lim_n \mu A_n.$$

The other stream of thought in measure theory is concerned with approximation from below, i.e. one picks a priori a family F and considers those sets which satisfy the following definition.

DEFINITION 3. 5. — *A is* (μ, F) — *capacitable iff* $A \in dmn\mu$ and

$$\mu A = \sup_{\substack{C \subset A \\ C \in F}} \mu C$$

In practice, another family G is also chosen a priori and one considers sets A for which

$$\inf_{\substack{A \subset \alpha \\ \alpha \in G}} \mu\alpha = \sup_{\substack{\alpha \subset A \\ \alpha \in F}} \mu\alpha$$

However, by defining

$$\nu A = \inf_{\substack{A \subset \alpha \\ \alpha \in G}} \mu\alpha$$

if $\nu\alpha = \mu\alpha$ for $\alpha \in F$, the problem is reduced to one about (ν, F)-capacitability.

Comparison of definitions 3. 2 and 3. 5 shows that in general measurability and capacitability are unrelated. Yet, in measure theory the two streams of thought usually run side by side and very frequently they are allowed to become confused, e.g. measurability is defined as capacitability. This occurs usually in topological spaces where F is taken to be the family of closed or compact or compact $\mathcal{G}_\delta$ sets.

One reason for not separating the two concepts is that the families F considered in capacitability theorems consist usually of measurable sets and the following elementary result is well known.

THEOREM 3. 6. — *For any measure* μ, *if* F *is a family of* μ*-measurable sets,* $\mu A < \infty$ *and* A *is* (μ, F)*-capacitable, then* A *is* μ*-measurable.*

Thus, it has seemed so far that in situations where one has a measure, capacitability almost automatically implies measurability. If one has capacitability but not measurability, then it is felt the function μ involved must not be a measure. In section 4, we give an example, important in potential theory, where we have a measure μ and a family F such that non-measurable sets are (μ, F)-capacitable. Thus, the feeling that in measure theory the two concepts always go hand in hand is not justified.

The separation of the two concepts becomes all the more important if one wants to understand the relation between measure theory and the theory of capacities. Only that part of measure theory concerned with approximation from below can be incorporated in the theory of capacities. Thus, only when measurability is a consequence of capacitability can we expect it to follow from some result in the theory of capacities. This is precisely the case for analytic sets and explains why the proofs of their measurability and capacitability are so similar. That the results do not follow from one another is due to the definition of capacity. In order to have the theory of capacities applicable to measure theory, the definition of a capacity should be broad enough to include as wide a class of measures as possible. Choquet's treatment in [2], which we followed in [8], is too restrictive and even his definition of an abstract capacity in [4] is not broad enough, mainly because it is tied to a family H chosen a priori. For this reason we introduce the following definition, where such a tie-in is omitted.

DEFINITION 3. 7. — *μ is an outer content on X iff μ is a function on the family of all subsets of X such that if $A_n \subset A_{n+1} \subset X$ for $n \in \omega$ then*

$$-\infty \leqq \mu A_n \leqq \mu A_{n+1} \leqq \lim_n \mu A_n = \mu\Big(\bigcup_{n\in\omega} A_n\Big) \leqq \infty .$$

Then, an outer measure is an outer content, but a measure in general is not. This difficulty is frequently solved by passing to an outer measure, as we do in the proof of 5. 10. An interesting measure, which is an outer content but not an outer measure, is discussed in the next section. We defer theorems about outer contents until section 5 where the notion is abstracted to that of a capacitance.

4. — NEWTONIAN CAPACITY AS A MEASURE

In this section we shall consider an application of the point of view of section 3 to classical potential theory. We refer to any standard text, e.g. [1, 5], for background to the subject. Although we restrict ourselves to the Newtonian potential in Euclidean 3-space, workers in the field will readily see the extension to more general situations. The main point here is the definition of the capacity of any set before inner and outer capacities. It is a measure for which not all compact sets are measurable and, since it coincides with inner capacity on all absolutely measurable sets (in particular analytic sets), we see that there are non-measurable sets which are capacitable. The significance of this example goes beyond this fact. It brings capacities back into measure theory where we can apply known results rather than just known techniques. Theorem 4. 5 and corollary 4. 6 are good examples of this. Throughout this section, X is Euclidean 3-space.

DEFINITIONS 4. 1.

1. K *is the family of all compact sets in* X.

2. M′ *is the set of all Radon measures on* X, *i.e. measures* μ *on* X *such that.*

(i) *open sets are* μ*-measurable,*

(ii) *if* C *is compact then* $\mu C < \infty$,

(iii) *for any* $A \subset X$ *and* $\varepsilon > 0$ *there exists an open* A′ *such that*

$$A \subset A' \text{ and } \mu A' \leqq \mu A + \varepsilon.$$

It then follows that

(iv) *if* A *is* μ*-measurable then* $\mu A = \sup\limits_{\substack{C \subset A \\ C \in K}} \mu C$.

3. A *is absolutely measurable iff* A *is* μ*-measurable for every* $\mu \in M'$.

4. *For any* $\mu \in M'$, *the potential* $U_\mu(x) = \int \frac{1}{|x - t|} d\mu t$.

5. $M = \{\mu : \mu \in M' \text{ and } U_\mu \leqq 1\}$.

6. *For any* $A \subset X$,

(i) *the capacity* $\emptyset A = \sup\limits_{\mu \in M} \mu A$

(ii) *the outer capacity* $\emptyset^* A = \inf_{\substack{A \subset B \\ B \text{ open}}} \emptyset B$

(iii) *the inner capacity* $\emptyset_* A = \sup_{\substack{C \subset A \\ C \in K}} \emptyset C$.

7. A *is capacitable iff* $\emptyset^* A = \emptyset_* A$.
8. *For any* $T \subset X$ *and* $\alpha \subset X$, $\mu_T(\alpha) = \mu(T \cap \alpha)$.
9. $M_T = \{\mu : \mu \in M$ *and* $\mu(X \sim T) = 0\}$.

The following theorems follow readily from the above definitions and elementary measure theory.

THEOREM 4. 2. — $\emptyset$ *is a measure on* X.

PROOF. — For any family F of measures, if

$$\nu A = \sup_{\mu \in F} \mu A$$

then ν is a measure. Measurability, however, is not preserved in general.

THEOREM 4. 3. — *If* A *is absolutely measurable,* $\emptyset A = \sup_{\mu \in M_A} \mu A$

PROOF. — Clearly, $\emptyset A \geq \sup_{\mu \in M_A} \mu A$.

On the other hand, if $\mu \in M$, since A is absolutely measurable, μ_A is also a Radon measure and $\mu A \leq \mu$, so that $\mu_A \in M_A$. But $\mu_A(A) = \mu_A$. Hence the desired result.

Theorem 4. 3 shows in particular that $\emptyset_*$ is the standard inner capacity.

THEOREM 4. 4. — *If* A *is absolutely measurable then* $\emptyset A = \emptyset_* A$, *i.e.* A *is* $(\emptyset, K)$-*capacitable.*

PROOF. — Since A is absolutely measurable, for any $\mu \in M$ we have

$$\mu A = \sup_{\substack{C \subset A \\ C \in K}} \mu C.$$

Therefore

$$\emptyset A = \sup_{\mu \in M} \sup_{\substack{C \subset A \\ C \in K}} \mu C = \sup_{\substack{C \subset A \\ C \in K}} \emptyset C = \emptyset_* A.$$

THEOREM 4. 5. — *If* $\emptyset A_n = 0$ *for* $n \in \omega$ *then* $\emptyset\left(\bigcup_{n \in \omega} A_n\right) = 0$.

Corollary 4. 6. — *If* A_n *is absolutely measurable and* $\emptyset_* A_n = 0$ *for* $n \in \omega$ *then* $\emptyset_*\left(\bigcup_{n\in\omega} A_n\right) = 0$.

Theorem 4. 7. — $\emptyset$ *is an outer content on* X.

The next theorems depend more deeply on properties of the potentials U_μ. These properties are well known and we omit their proofs since they are beyond the scope of this paper. For them, we refer the reader to [1, 2, 5].

Theorem 4. 8. — A *is* $\emptyset$*-mesurable iff* $\emptyset A = 0$ *or* $\emptyset(X \sim A) = 0$.

Proof. — We use the fact that for any compact α there exists $\mu \in M_\alpha$ with $\mu\alpha = \emptyset\alpha$ and that for any compact disjoint α, β with $\emptyset\alpha > 0$ and $\emptyset\beta > 0$ we have $\emptyset(\alpha \cup \beta) < \emptyset\alpha + \emptyset\beta$.

Let A be $\emptyset$-measurable. If $\emptyset_* A > 0$ and $\emptyset_*(X \sim A) > 0$ then there exist compact $\alpha \subset A$ and $\beta \subset X \sim A$ such that $\emptyset\alpha > 0$ and $\emptyset\beta > 0$ and

$$\emptyset(\alpha \cup \beta) = \emptyset((\alpha \cup \beta) \cap A) + \emptyset((\alpha \cup \beta) \sim A) = \emptyset\alpha + \emptyset\beta,$$

which is impossible. Let $\emptyset_* A = 0$. For any compact C, let $\mu \in M_C$, $\mu C = \emptyset C$. Then

$$\mu C = \emptyset C = \emptyset(C \cap A) + \emptyset(C \sim A) \geqq \mu(C \cap A) + \mu(C \sim A) \geqq \mu C.$$

Since μ is an outer measure and $\mu C < \infty$ we conclude that A is μ-measurable. If $\mu A > 0$ there is a compact $\alpha \subset A$ with $\mu\alpha > 0$, i.e. $\emptyset_* A > 0$. Hence $\mu A = 0$,

$$\emptyset C = \mu C = \mu(C \sim A) \leqq \emptyset(C \sim A) \leqq \emptyset C,$$
$$\emptyset(C \cap A) = \emptyset C - \emptyset(C \sim A) = 0.$$

Taking for C a sequence of compact spheres covering the whole space, we conclude $\emptyset A = 0$.

Remark. — 4. 8 shows that $\emptyset$ is not an outer measure even though it is an outer content.

Theorem 4. 9. — *If* A *is compact then* $\emptyset A = \emptyset^* A$.

Theorem 4. 10. — $\emptyset^*$ *is an outer content.*

Theorem 4. 11. — *If* A *is analytic then* A *is capacitable, i.e.* A *is* ($\emptyset^*$, K)*-capacitable.*

Theorem 4. 11, which is due to Choquet [2], follows from 4. 9, 4. 10 and 5. 9 of the next section.

5. — ANALYTIC SETS AND ABSTRACT APPROXIMATION

Let us first introduce the classical notions so that we may have them for ready reference.

Definitions 5. 1.

1. $H_\sigma = \{A : A = \bigcup_{n \in \omega} B_n$ *for some* $B_n \in H\}$.
2. $H_\delta = \{A : A = \bigcap_{n \in \omega} B_n$ *for some* $B_n \in H\}$.
3. $H_{\sigma\delta} = (H_\sigma)_\delta$.
4. *Borelian* H *is the smallest family* B *such that* $H \subset B = B_\sigma = B_\delta$.
5. *Souslin* H *is the family of all* A *such that*

$$A = \bigcup_{s \in S} \bigcap_{n \in \omega} h(s_0, \ldots, s_n)$$

where S *is the set of sequences of natural numbers and* $h(s_0, \ldots, s_n) \in H$ *for every* $s \in S$ *and* $n \in \omega$.

6. K(X) *is the family of all closed compact sets in* X.
7. *A is analytic iff* A *is the continuous image of a set in* $K_{\sigma\delta}(X)$ *for some* X.
8. A *is Lindelöf in* X *iff any open covering of* A *in* X *can be reduced to a countable subcovering.*
9. *For any set valued function f and any set* A,

$$f[A] = \bigcup_{x \in A \cap dmn f} f(x).$$

It is well known that if $A \in$ Souslin K(X) then A is analytic. The converse is known to hold only if $X \in K_\sigma(X)$ (see [3, 7]). For this reason, results about Souslin K(X) sets are not as strong as corresponding results for analytic sets. On the other hand, the notion of a set in Souslin H holds for any family H whereas that for an analytic set requires a topology. Thus, in general, one notion does not include the other. We remedy this situation by introducing a definition of abstract analytic set which generalizes both of the above concepts.

Definition 5. 2. — *A is (f, F)-analytic iff F is closed under finite intersections and for $i, j \in \omega$ there exist $d(i, j) \in \mathrm{F}$ such that:*
(i) $d(i, j) \subset d(i, j+1)$,
(ii) *if* $\mathrm{D} = \bigcap_{i\in\omega} \bigcup_{j\in\omega} d(i, j)$ *then f is on* D *and* $\mathrm{A} = f[\mathrm{D}]$.

Clearly, any set is (f, F)-analytic for some f and F. The point of the above definition is that there exist f and F satisfying conditions imposed by the hypotheses of theorems below so that analytic and Souslin sets are (f, F)-analytic.

As we pointed out in the introduction, the arguments used to prove various properties of analytic sets are very similar, even when no numerical valued function such as a measure or a capacity is involved. What do all these results have in common? Essentially this: they all prove that if the analytic set has a certain property P then there exists a compact set inside it which also has property P. Thus, they are all concerned with approximation from below with respect to a property P. We may well call it P-capacitability. We isolate this notion in theorem 5. 4 and then derive from it known results about Souslin and analytic sets. The conditions to be imposed on P are stated in the following definition.

Definition 5. 3. — *P is an outer family iff, for every sequence α, if $\alpha_n \subset \alpha_{n+1}$ for $n \in \omega$ and $\bigcup_{n\in\omega} \alpha_n \in \mathrm{P}$ then $\alpha_n \in \mathrm{P}$ for some $n \in \omega$.*

We then have the following central result.

Theorem 5. 4. — *Let P be an outer family, A be (f, F)-analytic, $\mathrm{A} \in \mathrm{P}$. Then, for every $n \in \omega$, there exists $\alpha_n \in \mathrm{F}$ such that $\alpha_{n+1} \subset \alpha_n$, $f[\alpha_n] \in \mathrm{P}$ and $\bigcap_{n\in\omega} \alpha_n \subset dmnf$.*

Proof. — Let $\mathrm{D} = dmnf$. Then

$$\mathrm{D} = \bigcap_{i\in\omega} \bigcup_{j\in\omega} d(i, j)$$

where $d(i, j) \subset d(i, j+1) \in \mathrm{F}$ for $i, j \in \omega$ and $= f\mathrm{A}[\mathrm{D}]$. By recursion, we shall define a sequence s of natural numbers so that for every $n \in \omega$

$$(*) \qquad f\left[\bigcap_{i=0}^{n} d(i, s_i)\right] \in \mathrm{P}.$$

To this end, we note that since

$$\bigcup_{j \in \omega} f[d(0, j)] = A \in P$$

there exists $s_0 \in \omega$ so that $f[d(0, s_0)] \in P$. For any $n \in \omega$, having defined $s_0, \ldots, s_n$ to satisfy $(*)$ above, let

$$\alpha_n = \bigcap_{i=0}^{n} d(i, s_i).$$

Then, since

$$\bigcup_{j \in \omega} f[\alpha_n \cap d(n+1, j)] = f[\alpha_n] \in P$$

we can choose $s_{n+1} \in \omega$ so that

$$f[\alpha_n \cap d(n+1, s_{n+1})] \in P$$

i.e. $(*)$ is satisfied with « n » replaced by « $n+1$ ».

Since F is closed under finite intersections, we have

$$\alpha_{n+1} \subset \alpha_n \in F, \qquad f[\alpha_n] \in P \qquad \text{and} \qquad \bigcap_{n \in \omega} \alpha_n \subset D.$$

In order to get a similar result for Souslin sets, we need a stronger condition on the family P. To this end we introduce the following.

Definition 5. 5. — *P is a capacitance in X iff P is an outer family of subsets of X such that if $\alpha \in P$ and $\alpha \subset \beta \subset X$ then $\beta \in P$.*

We then have the following.

Theorem 5. 6. — *Let P be a capacitance in X, H a family of subsets of X closed under finite unions and intersections, $A \in P \cap$ Souslin H. Then, for every $n \in \omega$, there exists $\alpha_n \in H \cap P$ such that $\alpha_{n+1} \subset \alpha_n$ and $\bigcap_{n \in \omega} \alpha_n \subset A$.*

Proof. — Let

$$S = \{s : s \text{ is on } \omega \text{ to } \omega\}$$
$$S_n = \{s : s \text{ is on } \{0, \ldots, n\} \text{ to } \omega\}$$
$$S' = \bigcup_{n \in \omega} S_n.$$

For $s \in S$, $n \in \omega$, let s/n be the restriction of s to $\{0, \ldots, n\}$. Since $A \in$ Souslin H, we have

$$A = \bigcup_{s \in S} \bigcap_{n \in \omega} h(s/n)$$

where $h(s') \in H$ for every $s' \in S'$. We shall define F and f so that A is (f, F)-analytic. For $s, t \in S \cup S'$, let $s \prec t$ iff $dmn\ t \subset dmn\ s$ and $s_i \leqq t_i$ for every $i \in dmn\ t$,

$$\bar{t} = \{s : s \in S \cup S' \quad \text{and} \quad s \prec t\}$$
$$F = \bigcup_{t \in S'} \{\bar{t}\}.$$

Then F is closed under finite intersections and if we let for $i, j \in \omega$

$$d(i, j) = \{s : s \in S \cup S' \quad \text{and} \quad s_n \leqq j \text{ for } n = 0, \ldots, i\}$$

we see that $d(i, j) \subset d(i, j + 1) \in F$ and

$$S = \bigcap_{i \in \omega} \bigcup_{j \in \omega} d(i, j).$$

For $s \in S$, let

$$f(s) = \bigcap_{n \in \omega} h(s/n).$$

Then A is (f, F)-analytic and by 5. 4, for every $i \in \omega$, there exists $t^i \in S'$ such that $\bar{t}^{i+1} \subset \bar{t}^i$, *i.e.* $t^{i+1} \prec t^i$, $f[\bar{t}^i] \in P$ and

$$\bigcap_{i \in \omega} \bar{t}^i \subset S.$$

Let

$$p_n = \inf_{i \in \omega} t_n^i \qquad \text{for} \qquad n \in \omega.$$

Then $p \in S$, and for each $n \in \omega$ there exists $i \in \omega$ such that

$$(*) \qquad (p_0, \ldots, p_n) = (t_0^i, \ldots, t_n^i).$$

One easily checks (see [6], p. 49) that

$$f[\bar{p}] = \bigcap_{n \in \omega} \bigcup_{s \in \bar{p}} h(s/n).$$

Let

$$\alpha_n = \bigcap_{k=0}^{n} \bigcup_{s \in \bar{p}} h(s/k)$$

so that $\alpha_{n+1} \subset \alpha_n \in \mathrm{H}$ and $\bigcap_{n \in \omega} \alpha_n = f[\bar{p}] \subset \mathrm{A}$.

Since, for every $i \in \omega$,

$$f[\bar{t}^i] \subset \bigcap_{n \in \omega} \bigcup_{s \in \bar{t}} h(s/n),$$

in view of (*) above we conclude for every $n \in \omega$ there exists $i \in \omega$ so that $f[\bar{t}^i] \subset \alpha_n$. Since $f[\bar{t}^i] \in \mathrm{P}$ we have $\alpha_n \in \mathrm{P}$.

Using the same arguments, theorems 5. 4 and 5. 6 can be generalized as follows.

Theorem 5. 7. — *Let* P *be a relation such that for every* α *and* β, $\{\beta' : (\alpha, \beta') \in \mathrm{P}\}$ *and* $\{\alpha' : (\alpha', \beta) \in \mathrm{P}\}$ *are outer families,* A *be* (f, F)*-analytic and* B *be* (g, G)*-analytic. If* (A, B) $\in$ P *then, for every* $n \in \omega$, *there exist* $\alpha_n \in \mathrm{F}$ *and* $\beta_n \in \mathrm{G}$ *such that:* $\alpha_{n+1} \subset \alpha_n$, $\beta_{n+1} \subset \beta_n$, $\bigcap_{n \in \omega} \alpha_n \subset dmnf$, $\bigcap_{n \in \omega} \beta_n \subset dmng$ and

$$(f[\alpha_n],\ g[\beta_n]) \in \mathrm{P},$$

Theorem 5. 8. — *Let* P *be a relation such that for every* α *and* β, $\{\beta' : (\alpha, \beta') \in \mathrm{P}\}$ *and* $\{\alpha' : (\alpha', \beta) \in \mathrm{P}\}$ *are capacitances in* X *and* X′ *respectively*; H *and* H′ *be families of subsets of* X *and* X′ *respectively closed under finite unions and intersections*; A $\in$ *Souslin* H, B $\in$ *Souslin* H′.

If (A, B) $\in$ P *then, for every* $n \in \omega$, *there exist* $\alpha_n \in \mathrm{H}$, $\beta_n \in \mathrm{H}'$ *such that:* $\alpha_{n+1} \subset \alpha_n$, $\beta_{n+1} \subset \beta_n$, $\bigcap_{n \in \omega} \alpha_n \subset \mathrm{A}$, $\bigcap_{n \in \omega} \beta_n \subset \mathrm{B}$ *and* $(\alpha_n, \beta_n) \in \mathrm{P}$.

We conclude this section with a few applications of 5. 4, 5. 6 and 5. 8. Theorem 5. 9 is a rewording of a theorem in [4], 5. 10 is a classical result (see [6]), 5. 11 combines two results in [8]. The use of an outer content makes these results applicable directly to measures as well as capacities. Theorem 5. 12 was proved in [8], and 5. 13 is a slight generalization of a theorem in [7].

Theorem 5. 9. — *Let* μ *be an outer content on* X, H *a family of subsets of* X *closed under finite unions and countable inter-*

sections and such that whenever $\alpha_{n+1} \subset \alpha_n \in H$ *for* $n \in \omega$ *we have*

$$\mu\left(\bigcap_{n\in\omega} \alpha_n\right) = \lim_n \mu\alpha_n,$$

and $0 \in H$. *If* $A \in$ *Souslin* H *then* A *is* (μ, H)*-capacitable.*

Proof. — If $\mu A = -\infty$ then $\mu A = \mu 0$. Otherwise, let $a < \mu A$ and

$$P = \{\alpha : \mu\alpha > a\}.$$

Then, by 5. 6, for $n \in \omega$ there exists $\alpha_n \in H$ such that $\alpha_{n+1} \subset \alpha_n$, $\mu\alpha_n > a$ and $\bigcup_{n\in\omega} \alpha_n \subset A$. The conditions on H guarantee that

$$\bigcap_{n\in\omega} \alpha_n \in H \qquad \text{and} \qquad \mu\left(\bigcap_{n\in\omega} \alpha_n\right) \geq a.$$

Theorem 5. 10. — *Let* μ *be a measure on* X, H *a family of* μ*-measurable sets. If* $A \in$ *Souslin* H *then* A *is* μ*-measurable.*

Proof. — For $T \subset X$, $\alpha \subset X$ let $\mu_T(\alpha) = \mu(T \cap \alpha)$ and

$$\mu_T^*(\alpha) = \inf_{\alpha \subset \beta} \mu_T(\beta)$$
$$\beta \text{ is } \mu_T\text{-measurable.}$$

Then α is μ-measurable iff α is μ_T^*-measurable for every T with $\mu T < \infty$. Given such a T, let $\varnothing = \mu_T^*$. Then $\varnothing$ is an outer measure on X and the elements of H are $\varnothing$-measurable. Let $H' =$ Borelian H. Then $A \in$ Souslin H' and by 5. 9, for $\varepsilon > 0$ there exists $\alpha \in H'$ such that $\alpha \subset A$ and

$$\varnothing(A \sim \alpha) = \varnothing A - \varnothing\alpha < \varepsilon.$$

Since α is $\varnothing$-measurable and ε is arbitrary, A is $\varnothing$-measurable.

Theorem 5. 11. — *Suppose* μ *is an outer content on* X *and for every* $\alpha \in K(X)$ *and* $\varepsilon > 0$ *there exists an open* α' *such that* $\alpha \subset \alpha'$ *and* $\mu\alpha' \leqslant \mu\alpha + \varepsilon$. *If* A *is analytic in* X *then* A *is* $(\mu, K(X))$*-capacitable.*

Proof. — Let $D \in K_{\sigma\delta}(X')$, g be continuous on D and $A = g(D)$. Let $F = K(X')$ and, for $x \in D$, $f(x) = \{g(x)\}$. Then A is (f, F)-analytic. Given $a < \mu A$, let

$$P = \{\alpha : \mu\alpha > a\}.$$

Then, by 5. 4, for $n \in \omega$ there exists $\alpha_n \in \mathrm{F}$ such that $\alpha_{n+1} \subset \alpha_n$, $\mu(f[\alpha_n]) > a$ and $\bigcap_{n \in \omega} \alpha_n \subset \mathrm{D}$.

Let
$$\beta = \bigcap_{n \in \omega} \alpha_n.$$

Then $\beta \in \mathrm{K}(\mathrm{X}')$ and hence $f[\beta] \in \mathrm{K}(\mathrm{X})$. If α' is open and $f[\beta] \subset \alpha'$ then for some $n \in \omega$, $f[\alpha_n] \subset \alpha'$. Thus, $\mu\alpha' > a$ and hence $\mu(f[\beta]) \geqq a$.

THEOREM 5. 12. — *If* A *is analytic then* A *is Lindelöf.*

PROOF. — Let G be an open covering of A,

$\mathrm{P} = \{\alpha : \alpha \subset \mathrm{A}$ and no countable subfamily of G covers $\alpha\}$.

Introducing f and F as in 5. 11 so that A is (f, F)-analytic, if $\mathrm{A} \in \mathrm{P}$ then by 5. 4 there exist $\alpha_n \in \mathrm{F}$ such that

$$\alpha_{n+1} \subset \alpha_n, \qquad f[\alpha_n] \in \mathrm{P} \qquad \text{and} \qquad \bigcap_{n \in \omega} \alpha_n \subset dmn\, f.$$

But, $f\left[\bigcap_{n \in \omega} \alpha_n\right]$ is compact so that a finite subfamily of G covers it and hence covers $f[\alpha_n]$ for some $n \in \omega$. This contradiction shows that $\mathrm{A} \notin \mathrm{P}$.

THEOREM 5. 13. — *Let* A *and* B *be disjoint sets in Souslin* K(X). Then there exist disjoint sets A′ and B′ in Borelian K(X) such that $\mathrm{A} \subset \mathrm{A}'$ and $\mathrm{B} \subset \mathrm{B}'$.

PROOF. — Let

$\mathrm{P} = \{(\alpha, \beta) : \alpha \subset \mathrm{X},\ \beta \subset \mathrm{X}$ and there are no disjoint sets α', β' in Borelian K(X) such that $\alpha \subset \alpha'$ and $\beta \subset \beta'\}$.

Then P satisfies the conditions of theorem 5. 8 and if $(\mathrm{A}, \mathrm{B}) \in \mathrm{P}$ there exist α_n, β_n in K(X) such that $\alpha_{n+1} \subset \alpha_n$, $\beta_{n+1} \subset \beta_n$, $\bigcap_{n \in \omega} \alpha_n \subset \mathrm{A}$, $\bigcap_{n \in \omega} \beta_n \subset \mathrm{B}$ and $(\alpha_n, \beta_n) \in \mathrm{P}$. Let

$$\alpha' = \bigcap_{n \in \omega} \alpha_n, \qquad \beta' = \bigcap_{n \in \omega} \beta_n.$$

Since $\alpha', \beta' \in \mathrm{K}(\mathrm{X})$ and $\alpha' \cap \beta' = 0$, we have $\alpha_n \cap \beta' = 0$, for some $n \in \omega'$, and therefore $\alpha_n \cap \beta_m = 0$ for some $m \in \omega$, hence $(\alpha_{n+m}, \beta_{n+m}) \notin \mathrm{P}$.

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