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On the representation of certain functionals by measures on the Choquet boundary

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ON THE REPRESENTATION OF CERTAIN FUNCTIONALS
BY MEASURES ON THE CHOQUET BOUNDARY

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1. Introduction.

M. Hervé [6] has recently published a simple proof of Choquet's theorem on the representation of the points of a compact convex metrizable subset of a locally convex real linear topological space as barycentres of measures carried by the extreme points of the set. F. F. Bonsall [5] has shown that, by a use of the Hahn-Banach theorem, the discussion can be made still more simple and that a restatement of the problem then allows the convexity condition to be dropped. The present paper shows that further pursuit of these ideas provides new information about the Choquet boundary, as defined by Bishop and de Leeuw [4]. It is then possible to give a simple direct proof of a result of these authors: that, in the presence of a separability condition (stated in § 4), the Choquet boundary is a $G_δ$ set and every probability Radon measure admits balayage onto it. These methods are also shown, in § 5, to lead to a proof of one of Bauer's main theorems in his theory [3] of an abstract Dirichlet problem. The effect of an additional equicontinuity condition is also considered in § 5.

We consider here only real-valued functions, remarking that the passage to the complex case is known [4] to be a straightforward matter.

I am indebted to Professor F. F. Bonsall for showing me his work before publication.
2. Construction of functionals; the Choquet theorem.

We consider a compact Hausdorff space $X$, the set $C$ of all real continuous functions on $X$, and a linear manifold $L$ of $C$ that contains the constant functions. We denote by $M$, $M^+$, and $P$ respectively the spaces of Radon measures, non-negative Radon measures, and probability Radon measures on $X$, and by $R$ the set of real numbers. For each $x \in X$ we define the set

$$M_x \equiv M_x(L) = \{ \mu \in M^+ | \mu(g) = g(x) \text{ for all } g \in L \}.$$

The unit atomic measure at $x$, denoted by $\delta_x$, belongs to $M_x$. Also since $1 \in L$ we have $\mu(1) = 1$ for all $\mu \in M_x$, so that $M_x \subseteq P$.

For each $f \in C$ and each $\sigma \in P$ we define

$$f^*(\sigma) = \inf \{ \sigma(g) | g \in L, g \geq f \}.$$

Each $f \in C$ is bounded, $L$ contains the constants, and so $f^*(\sigma)$ is well-defined and finite. Evidently

$$f^*(\sigma) \leq \max_{y \in X} f(y) \leq |f|,$$

and

$$g^*(\sigma) = \sigma(g) \text{ whenever } g \in L.$$

We adopt the convenient abuse of notation of writing $f^*(x)$ for $f^*(\delta_x)$, so that $x \mapsto f^*(x)$ is precisely the upper semi-continuous real function on $X$ defined by

$$f^*(x) = \inf \{ g(x) | g \in L, g \geq f \}.$$

**Lemma.** — For each $\sigma \in P$ the map $f \mapsto f^*(\sigma)$ of $C$ into $R$ is a sublinear functional on $C$.

Let $p(f) = f^*(\sigma)$ for all $f \in C$. Suppose $f_1, f_2 \in C$, let $\epsilon > 0$, and choose $g_1, g_2 \in L$ such that

$$g_r \geq f_r, \quad \sigma(g_r) < p(f_r) + \epsilon \quad (r = 1, 2).$$

Then $f_1 + f_2 \leq g_1 + g_2 \in L$ and so

$$p(f_1 + f_2) \leq \sigma(g_1 + g_2) = \sigma(g_1) + \sigma(g_2) < p(f_1) + p(f_2) + 2\epsilon.$$
But $\varepsilon > 0$ was arbitrary and so

$$p(f_1 + f_2) \leq p(f_1) + p(f_2)$$

for all $f_1, f_2 \in C$. One proves similarly that

$$p(\lambda f) = \lambda p(f)$$

for all real $\lambda \geq 0$ and $f \in C$.

Now choose $h \in C$ and let $W_h$ be the set of all points $x \in X$ that satisfy the condition that for at least one $\nu \in M_x(L)$ we have

$$\nu(h) > h(x).$$

**Theorem 1.** — For each $h \in C$.

(3) $W_h = \{ x \in X | h^*(x) > h(x) \}$;

the set $W_h$ is consequently an $F_\sigma$ set. Moreover, given $\tau \in P$, we can find $\mu \in P$ (depending on $h$ and $\tau$) such that

(i) $\mu(g) = \tau(g)$ for all $g \in L$;

(ii) $\mu(W_h) = 0$.

Let $x \in W_h$ and let $\nu \in M_x$ be such that $\nu(h) > h(x)$. Then if $g \in L$, $g \geq h$ we have

$$g(x) = \nu(g) \geq \nu(h) > h(x),$$

so that $g(x) - h(x) \geq \nu(h) - h(x) > 0$ and hence $h^*(x) > h(x)$. This proves that

(4) $W_h \subseteq \{ x \in X | h^*(x) > h(x) \}$.

Now take a measure $\sigma \in P$ and write $p(f) = f^*(\sigma)$, for all $f \in C$. Then by the lemma and the Hahn-Banach theorem there exists a linear functional $\nu \equiv \nu_x$ on $C$ that satisfies

$$\nu(f) \leq p(f) \quad \text{for all } f \in C,$$

and

$$\nu(h) = p(h).$$

By (1) the functional $\nu$ is continuous. For $g \in L$ we have, by (2), $\nu(g) \leq p(g) = \sigma(g)$ and also $-g \in L$, so that $-\nu(g) = \nu(-g) \leq \sigma(-g) = -\sigma(g)$, whence in fact

(5) $\nu(g) = \sigma(g)$ for all $g \in L$. 

8.
Next, (1) implies that for $f \in C$ with $f \leq 0$ we have $v(f) \leq 0$ and hence $v(-f) \geq 0$, so that $v \geq 0$, and thus $v \in M^+$. Now take $x \in X$ with $h^*(x) > h(x)$ and let $\varepsilon_x$ in the above construction, so that now $v \in M_x$, and $p(f) = f^*(x)$ for all $f \in C$. Then $v(h) = p(h) = h^*(x) > h(x)$ and therefore $x \in W_h$. So we have

$$\{x \in X | h^*(x) > h(x)\} \subseteq W_h,$$

which with (4) establishes (3).

Next $h^*$, and hence $(h^* - h)$, is upper semi-continuous and hence

$$F_n = \left\{ x \in X | h^*(x) - h(x) \geq \frac{1}{n} \right\}$$

is closed. Therefore $W_h = \bigcup_{n=1}^{\infty} F_n$ is an $F_\sigma$ set.

For the last part let $\mu = v_x$ as above. Then $\mu \geq 0$, and (5) provides the proof of relation (i) of theorem 1 and in particular the fact that $\mu(1) = \tau(1)$, so that $\mu \in P$.

To prove that $\mu(W_h) = 0$ it is enough to show that $\mu(F_n) = 0$ for all $n \geq 1$. Suppose there is an exceptional $n$ with $\mu(F_n) = \delta > 0$. Then if $g \geq h$, $g \in L$ we have $g \geq h^*$ and consequently

$$\tau(g) - \mu(h) = \mu(g) - \mu(h) \geq \int_{F_n} (g - h) \, d\mu \geq \frac{\delta}{n}.$$

On the other hand

$$\mu(h) = h^*(\tau) = \inf \{ \tau(g) | g \geq h, g \in L \},$$

which contradicts (6) and completes the proof that $\mu(W_h) = 0$.

**Corollary (Choquet).** — Let $X$ be a compact convex metrizable set in a locally convex real linear topological space. Then the set $E$ of extreme points of $X$ is a $G_\delta$ set. Moreover, for each $a \in X$ there exists a probability Radon measure $\mu$ on $X$ such that

$$\mu(g) = g(a), \quad \text{for all} \quad g \in L,$$
where \( L \) is now the set of restrictions to \( X \) of real continuous affine functions, and,

\[
\mu\left( \bigcap E \right) = 0.
\]

For the proof we take \( h \) to be the strictly convex real continuous function on \( X \) constructed by Hervé [6]. Then it is clear that \( W_h \cap E = \emptyset \) (see § 3). But Hervé shows that if \( x \in E \) then \( h^*(x) = h(x) \) so that, by (3), we have \( W_h = \bigcap E \) for this \( h \). On taking \( \tau = \varepsilon_x \) in theorem 1 we obtain therefore a \( \mu \in \mathcal{P} \) satisfying (j) and (jj). In § 4 we present a generalization of this argument.


Now let \( A(L) \) denote the smallest uniformly closed subalgebra of \( C \) that contains \( L \). Evidently

\[
M_x(L) \ni M_x(A(L)) \quad \text{for all} \quad x \in X.
\]

The Choquet boundary of the space \( X \) for the class of functions \( L \) is by definition the set

\[
\delta_L X = \{ x \in X | M_x(L) = M_x(A(L)) \}.
\]

The Weierstrass-Stone theorem, together with a simple measure-theoretic argument like that used to prove proposition 1 below, implies that this definition is equivalent to the slightly different one given by Bishop and de Leeuw [4]. If \( L \) separates the points of \( X \) then \( A(L) = C \) and so, in this case,

\[
\delta_L X = \{ x \in X | M_x(L) = (\varepsilon_x) \}.
\]

PROPOSITION 1. — For each linear subspace \( L \) of \( C \) that contains the constants, we have

\[
\delta_L X = \bigcap_{h \in A(L)} \bigcup W_h = \bigcap_{g \in L} \bigcup W_{ig}.
\]

We emphasize here that \( W_f \), for \( f \in C \), depends on \( f \) and on \( L \).

Suppose \( h \in A(L), \ x \in W_h \). Then there is a \( \psi \in M_x(L) \) with \( \psi(h) > h(x) \), so that \( \psi \in M_x(A(L)) \) and hence \( x \in \delta_L X \).

This shows that

\[
W_h \cap \delta_L X = \emptyset \quad \text{for all} \quad h \in A(L).
\]
Conversely suppose that \( a \in \partial L \), let \( v \in M_a(L) \setminus M_a(A(L)) \), and let \( \text{supp} \ v \) denote the support of \( v \). Then we can find \( b \in \text{supp} \ v \), with \( b \neq a \), together with a function \( g_1 \in L \) such that \( g_1(b) \neq g_1(a) \). For otherwise we should have
\[
g(x) = g(a) \quad \text{for all} \quad x \in \text{supp} \ v, \quad g \in L,
\]
which would imply
\[
h(x) = h(a) \quad \text{for all} \quad x \in \text{supp} \ v, \quad h \in A(L),
\]
and hence \( v \in M_a(A(L)) \), contrary to hypothesis.

Now define
\[
g(x) = g_1(x) - g_1(a) \quad (x \in X),
\]
so that \( g \in L \). Then the continuous non-negative function \( h = |g| \) is strictly positive at the point \( b \in \text{supp} \ v \) and so
\[
v(h) > 0 = h(a),
\]
so that \( a \in W_h = W_{|g|} \). So we have proved that
\[
\partial L \subset \bigcup_{g \in L} W_{|g|}
\]
which with (8) yields the desired formula (7).

By theorem 1 we now have.

**Corollary 1.** — **Under the same conditions**

(9) \[
\partial L \subset \{ x \in X | h^*(x) = h(x) \quad \text{for all} \quad h \in A(L) \} \cup \{ x \in X | g^*(x) = |g(x)| \quad \text{for all} \quad g \in L \}.
\]

Now write \( F = \partial L \) and consider the restriction map
\[
g \mapsto \tilde{g} = g|F
\]
from \( L \) into the space \( R(F) \) of real continuous functions on \( F \), letting \( \tilde{L} = \{ \tilde{g} | g \in L \} \).

**Corollary 2.** — **If \( L \) separates the point of \( X \) then for each \( u \in R(F) \), \( x \in \partial L \), we have**

(10) \[
u(x) = \inf \{ \nu(x) | \nu \in \tilde{L} \}, \quad \nu \geq u\]
\[
u(x) = \sup \{ \nu(x) | \nu \in \tilde{L} \}, \quad \nu \leq u\].
The first equality follows from the proof of (9), applied to the pair \((F, L)\) in place of \((X, L)\), and the obvious fact that \(\partial F \supseteq \partial X\). The same reasoning applied to \(-u\) then yields the second part.

\section{Measures on the boundary for separable \(L\).

In this section we suppose that \(L\) is separable.

\textbf{Proposition 2.} — If \(L\) is a separable linear subspace of \(C\) that contains the constants then there exists a function \(h \in A(L)\) such that

\begin{equation}
\partial X = \bigcap W_h.
\end{equation}

Let \((g_m)_{m \geq 1}\) be a countable dense set in \(L\), and let \((r_n)_{n \geq 1}\) be an enumeration of the rationals, and let

\[ h = \sum_{m,n \geq 1} \frac{1}{2^{m+n}} \frac{h_{mn}}{1 + \|h_{mn}\|^2}, \]

where \(h_{mn}(x) = |g_m(x) - r_n| \quad (m, n \geq 1; \ x \in X),\)

so that \(h \in A(L)\). We show that this \(h\) satisfies (11).

First if \(a \in X, \ \nu \in M_a(L), \ g \in L, \ r \in \mathbb{R}\) then

\[ \nu(|g - r|) = \int |g(x) - r| \nu(dx) \geq \int (g(x) - r) \nu(dx) = |g(a) - r|, \]

and hence in particular

\begin{equation}
\nu(h_{mn}) \geq h_{mn}(a) \quad (m, n \geq 1). \tag{12}
\end{equation}

Now suppose \(a \in \partial X\) and let \(\nu \in M_a(L) \setminus M_a(A(L))\). Then as in the proof of proposition 1 we can find \(b \in \text{supp} \ \nu\), with \(b \neq a\), and \(p \geq 1\) such that \(g_p(b) \neq g_p(a)\). We therefore have

\[ \int |g_p(x) - g_p(a)| \nu(dx) = \delta > 0. \]

But we can find a rational \(r_q\) such that

\[ h_{pq}(a) = |g_p(a) - r_q| < \frac{1}{2} \delta. \]

Then

\[ \nu(h_{pq}) = \int |g_p(x) - r_q| \nu(dx) \geq \int (|g_p(x) - g_p(a)| - |g_p(a) - r_q|) \nu(dx) > \delta - \frac{1}{2} \delta = \frac{1}{2} \delta. \]
Hence \( \nu(h_{pq}) > h_{pq}(a) \) which together with (12) shows that \( \nu(h) > h(a) \), so that \( a \in W_h \). We have thus shown that \( \delta_L X \subseteq W_h \). But \( W_h \cap \delta_L X = \emptyset \) and so (11) is proved.

By theorem 1 we now have the

**Corollary (Bishop and de Leeuw).** — If \( L \) is a separable linear subspace of \( C \) that contains the constants then the Choquet boundary \( \delta_L X \) is a \( G_\delta \) set. Moreover, for each \( \tau \in P \) we can find \( \mu \in P \) such that

(i) \( \mu(g) = \tau(g) \) for all \( g \in L \);

(ii) \( \mu\left( \bigcup \delta_L X \right) \)

5. The boundary when is lattice.

We shall not require \( L \) to be separable in this section.

In his paper [3] Bauer has shown that the theory of the Choquet boundary becomes specially satisfactory when \( L \) is a lattice. We show here that corollary 2 to proposition 1 makes possible a direct proof of one of Bauer's results, and then consider the effect of an additional equicontinuity condition.

**Theorem 2 (Bauer).** — If \( L \) is a linear subspace of \( C \) that contains the constants, separates the points of \( X \), and is a lattice for the natural partial ordering, then \( \delta_L X \) is a closed set and the restriction map \( f \rightarrow \overline{f} \equiv f|_{\delta_L X} \) from \( L \) into \( R(\delta_L X) \) is an isometric linear and lattice isomorphism onto a dense subset of \( R(\delta_L X) \) (and hence actually onto \( R(\delta_L X) \) if \( L \) is complete). Moreover, given \( \tau \in M \), we can find a unique \( \mu \equiv \mu_\tau \in M \) satisfying

(i) \( \mu(g) = \tau(g) \) for all \( g \in L \);

(ii) \( \text{supp } \mu \subseteq \delta_L X \).

The map \( \tau \rightarrow \mu_\tau \) in \( M \) is linear and it maps \( M^+ \) isometrically into itself.

For this we use Bauer's maximum principle [2], which we need only in the following weak form: if \( L \) is a linear subspace of
C that contains the constants and separates the points of X then for each $f \in L$ there is a point $a \in \partial L_X$ such that

$$f(a) = \max_{x \in X} f(x).$$

Now let $F = \overline{\partial L_X}$ and consider the restriction map $f \mapsto \tilde{f} = f|F$ from $L$ into $R(F)$. This is linear and order-preserving. The maximum principle applied to $f$ and to $-f$ shows that it is also an isometry. Now if also $L$ is a lattice for the natural partial ordering then the restriction map preserves the lattice structure. For let $f, g \in L$, $h = f \wedge g$, and let $h$ and $u \in R(F)$ be compared, where

$$u(x) = \min\{f(x), g(x)\} \quad (x \in F).$$

Following e.g. Kadison [7], we have $h \leq f$, $h \leq g$ and hence $h \leq u$. If for some $x \in \partial L_X$ we have $h(x) < u(x)$ then by corollary 2 to proposition 1 we can find $k \in L$ such that $k \leq u$ and $h(x) < h(k) \leq u(x)$. Then $k \leq \tilde{f}$, $k \leq \tilde{g}$ and the maximum principle implies that $k \leq f$, $k \leq g$; whence $k \leq f \wedge g = h$, which contradicts the inequality $h(x) < h(k)$. Since $\partial L_X = F$ we must therefore have $h = u$; that is, the restriction of $f \wedge g$ to $F$ is equal to $\min(\tilde{f}, \tilde{g})$. Likewise the restriction of $f \vee g$ to $F$ is $\max(\tilde{f}, \tilde{g})$.

The set $\tilde{L}$ is thus a linear sublattice of $R(F)$ that contains the constants and separates points and hence, by the Weierstrass-Stone theorem, it lies densely in $R(F)$. Any continuous linear functional on $\tilde{L}$ is therefore representable by a unique Radon measure on $F$. The map $\tilde{f} \mapsto \tau(f)$ is such a functional, and so we find $\mu \equiv \mu_\tau \in M$ to satisfy (i) and (ii). The remaining properties of the map $\tau \mapsto \mu_\tau$ are immediate, if we assume that $F = \partial L_X$.

We complete the proof by showing that $F = \partial L_X$. For this let $x \in F$, $\nu \in M_\nu(L)$, $g \in L$ and let $H = \{y \in X|g(y) \leq g(x)\}$. Adapting a construction of Bishop and de Leeuw we write, for any Borel set $E$, $\tau(E) = \nu(E \cap H)$, $\sigma(E) = \nu(E \setminus H)$, so that $\tau, \sigma \in M^+$, $\tau + \sigma = \nu$. Then $\mu_\tau + \mu_\sigma = \mu_\nu$, and we have: $\mu_\nu = \nu_x$ because $L$ is dense in $R(F)$, $\mu_\tau \geq 0$, $\mu_\sigma \geq 0$, and consequently $\mu_\tau = \tau(1)\nu_x$, $\mu_\sigma = \sigma(1)\nu_x$. Therefore

$$\tau(g) = \tau(1)g(x), \quad \sigma(g) = \sigma(1)g(x),$$
which implies that \( g(y) = g(x) \) in \( \text{supp } \tau \cup \text{supp } \sigma = \text{supp } \nu \). Thus every \( g \in L \) takes the constant value \( g(x) \) on \( \text{supp } \nu \); but \( L \) separates points, and hence \( \text{supp } \nu = x \), \( \nu = \varepsilon_x \), \( x \in \partial L X \), and the proof is complete.

Now suppose that \( L \) is complete and meets the conditions of theorem 2 and let \( \mu_x \) denote the measure constructed in that theorem for the special case \( \tau = \varepsilon_x \), where \( x \in X \). Suppose further that the functions \( f \in L \) with \( ||f|| \leq 1 \) are equicontinuous at each point of \( \bigcup \partial L X \) and let \( \varphi \neq K \subseteq \bigcup \partial L X \) with \( K \) compact. For each \( u \in R(\partial L X) \) the map \( x \rightarrow \mu_x(u) \) from \( X \) into \( R \) is, by theorem 2, the unique function \( \bar{u} \) in \( L \) whose restriction to \( \partial L X \) is \( u \). If \( \mu^k(u) \) denotes the restriction of \( \bar{u} \) to \( K \) then, by the maximum principle,

\[
||\mu^k(u)||_{R(K)} \leq ||u||_{R(\partial L X)}
\]

and hence by Ascoli's theorem the map \( u \rightarrow \mu^k(u) \) from \( R(\partial L X) \) into \( R(K) \) is a compact linear operator. If now \( E \in B \) (the class of Borel subsets of \( \partial L X \)) then by a theorem of Bartle, Dunford and Schwartz [1] the map \( x \rightarrow \mu_x(E) \) restricted to \( K \) is an element \( \mu^k(E) \), of \( R(K) \). Moreover the map

\[
\mu^k : B \rightarrow R(K)
\]

is a vector-valued regular Borel measure with conditionally compact range and we have

\[
\mu^k(u) = \int_{\partial L X} u(x) \mu^k(dx) \quad \text{for all } u \in R(\partial L X)
\]

where the integral exists as a strong integral in the sense of [1].

*Note added in proof, 7 December 1962.*

Mokobodzki and Choquet (see Séminaire Brelot-Choquet-Deny (Théorie du Potentiel) 6e année, 1962, no 12) have shown that further improvements in the use of the Hahn-Banach theorem to study barycentres are possible: If in the present context \( L \) separates points and \( \hat{L} \) denotes the set of all \( \nu \in C \) of the form

\[
\nu = \inf (g_1, g_2, \ldots, g_n),
\]

where all the \( g_i \) are in \( L \), and if for \( \sigma, \tau \in P \) we write \( \sigma \preceq \tau \) whenever \( \sigma(\nu) \geq \tau(\nu) \) for all \( \nu \in \hat{L} \) then \( \sigma \preceq \tau \) implies that
\( \sigma(g) = \tau(g) \) for all \( g \in L \). The relation \( \leq \) is a partial ordering, and by Zorn's lemma each element of \( P \) is dominated by a maximal element of \( P \). A modification of the construction in theorem 1 that uses \( f(\sigma) \equiv \inf \{ \nu(\sigma) \mid \nu \in \hat{L}, \ \nu \geq f \} \) in place of \( f^*(\sigma) \) provides for each \( \tau \in P \) and \( h \in C \) a \( \mu \geq \tau \) with \( \mu(W_h) = 0 \). It follows that the maximal elements of \( P \) are precisely those \( \mu \in P \) for which \( \mu(W_h) = 0 \) for all \( h \in C \).

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