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A SIMPLEX WITH DENSE EXTREME POINTS

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1. — Introduction.

Let $L$ be a locally convex linear topological space, and let $C$ be a compact convex subset of $L$. The Krein-Milman theorem [3] asserts that $C$ is the closed convex hull of the set $E(C)$ of extreme points of $C$. It follows that for every $x \in C$ there exists a positive measure $\mu_x$ of mass 1 on $E(C)$ such that

$$x = \int_{E(C)} y d\mu_x(y).$$

This representation is of little interest in the case where $C = \overline{E(C)}$, and according to a result due to Klee [2] this is the rule rather than the exception.

Recently Choquet [1] has shown that if $C$ is metrizable the measures $\mu_x$ may be chosen so as to be supported by $E(C)$ itself, and furthermore that these measures are uniquely determined if and only if $C$ is a simplex (i.e. such that the intersection of any two positive homothetic images of $C$ is either empty, a single point or a positive homothetic image of $C$).

The question is raised by Choquet whether the situation $C = \overline{E(C)}$ can arise when $C$ is a simplex. It is the object of this note to construct an example which shows that the answer is affirmative. The ideas governing the construction
are closely related to the ideas of [4] where a simple example of a convex set with dense extreme points is exhibited. In § 2 we perform the actual construction of the simplex $S$ and observe that $S = E(S)$, and in § 3 we prove that $S$ really is a simplex.

2. — Construction of the example.

In the Hilbert space $\ell^p$ of sequences

$$x = (\xi_1, \xi_2, \ldots, \xi_n, \ldots)$$

we denote by $e_j$ the unit vector having the coordinates $\xi_i = \delta_{ij}$. Further, we denote by $E_n$ the subspace spanned by $e_1, e_2, \ldots, e_n$ and by $P_n$ the projection on $E_n$.

We first construct a sequence of simplexes $S_n$ with the following properties:

(i) $S_n \subset E_n$ for every $n$.

(ii) $S_n \subset S_m$ and $E(S_n) \subset E(S_m)$ for $n < m$.

(iii) $P_n S_m = S_n$ for $n < m$.

(iv) for every $\varepsilon > 0$ there exists an $n$ such that every point of $S_n$ has distance at most $\varepsilon$ from $E(S_n)$.

The construction of the simplexes $S_n$ falls in groups as follows:

a) The first group consists of one simplex

$$S_1 = \{x | 0 \leq \xi_1 \leq 2^{-1}; \ x \in E_1 \}.$$

b) Assume that $S_1, S_2, \ldots, S_{n_p}$ have been constructed, $S_{n_p}$ being the last simplex in the $p$'th group. Choose points $y_1, y_2, \ldots, y_{q_p}$ in $S_{n_p}$ such that every point of $S_{n_p}$ has distance at most $2^{-p}$ from the set $\{y_1, y_2, \ldots, y_{q_p}\}$.

For $n_p < k \leq n_p + q_p = n_{p+1}$ we define

$$z_k = y_k - n_p + 2^{-k} e_k,$$

whereupon we define $S_k$ as the convex hull of the set

$$S_{n_p} \cup \{z_{n_p+1}, \ldots, z_k\}.$$

With this construction it is clear that the sets $S_n$ are simplexes satisfying (i), (ii), (iii) and (iv).
Now define
\[ T_n = P_n^{-1}(S_n) = \{ x | P_n x \in S_n \} \]
and
\[ S = \bigcap_{n=1}^{\infty} T_n \]

It then follows that
(iii') \( T_n \supset T_m \) for \( n < m \).

(iii') \( P_n T_n = S_n \) for \( n < m \).

(iv') The set \( \bigcup_{n=1}^{\infty} E(S_n) \) is dense in \( S \).

Thus, to prove that \( S = \overline{E(S)} \) it suffices to prove that \( E(S_n) \subset E(S) \) for all \( n \). The proof of this is exactly the same as in [4], but it is so short that we may as well repeat it here:

Let \( z \in E(S_n) \) and let \( y \neq 0 \). Then there exists \( m \geq n \) so that \( P_m y \neq 0 \), and by (ii) \( z \in E(S_m) \). Therefore, the segment
\[ \{ x \mid x = z + tP_m y ; \quad -1 \leq t \leq 1 \} \subset S_m, \]
and consequently
\[ \{ x \mid x = z + ty ; \quad -1 \leq t \leq 1 \} \subset S. \]

Hence, \( z \in E(S) \).

Finally, let us note for completeness that \( S \) is compact and convex.

3. — Proof that \( S \) is a simplex.

We must prove that every set of the form
\[ A = S \cap (qS + a) \quad \text{with} \quad q > 0 \]
containing at least two points is itself of the form
\[ A = rS + b \quad \text{with} \quad r > 0. \]

Now since
\[ A = \bigcap_{n=1}^{\infty} T_n \cap (q \bigcap_{n=1}^{\infty} T_n + a) \]
\[ = \bigcap_{n=1}^{\infty} (T_n \cap (qT_n + a)) \]
each of the sets \(T_n \cap (qT_n + a)\) contains at least two points, and therefore
\[
P_n(T_n \cap (qT_n + a)) = S_n \cap (qS_n + a_n),
\]
where \(a_n = P_n a\), is non-empty for every \(n\) and contains at least two points for sufficiently large \(n\).
Since \(S_n\) is a simplex, we have
\[
S_n \cap (qS_n + a_n) = r_n S_n + b_n \quad \text{with} \quad r_n \geq 0
\]
for every \(n\) and \(r_n > 0\) for sufficiently large \(n\). Now, for \(m > n\) we have
\[
P_n(S_m \cap (qS_m + a_m)) \subseteq P_n S_m \cap P_n(qS_m + a_m),
\]
i.e.
\[
P_n(r_m S_m + b_m) \subseteq S_n \cap (qS_n + a_n)
\]
or
\[
r_m S_n + P_n b_m \subseteq r_n S_n + b_n
\]
from where it follows that
1) \(r_m \leq r_n\).
2) \(P_n b_m \subseteq r_n S_n + b_n\) (since \(0 \in S_n\)).

By the construction all points of \(S_n\) have all their coordinates non-negative, and hence, writing
\[
b_n = (\beta_{n1}, \beta_{n2}, \ldots, \beta_{nm}, 0, \ldots)
\]
we get
3) \(\beta_{ni} \geq \beta_{ni}\) for all \(i\).

From 1) it follows that
\[
r_n \rightarrow r \ (\geq 0) \quad \text{for} \quad n \rightarrow \infty
\]
and from 3) that
\[
\beta_{ni} \rightarrow \beta_i \ (\text{for} \ n \rightarrow \infty) \quad \text{for all} \ i.
\]

It is easily seen that the sequence
\[
b = \{\beta_1, \beta_2, \ldots\}
\]
belongs to \(l^\prime\) and that
\[
b_n \rightarrow b \quad \text{for} \quad n \rightarrow \infty
\]
whence \(b \in A\).

We shall complete our proof by showing that
\[
A = rS + b.
\]

First, since \(r \leq r_m\) for every \(m\), we have
\[
rS + b_m \subseteq rT_m + b_m \subseteq r_m T_m + b_m = T_m \cap (qT_m + a_m) = T_m \cap (qT_m + a)
\]
for every $m$, and since

$$T_m \cap (qT_m + a) \subset T_n \cap (qT_n + a) \quad \text{for} \quad m > n$$

we have

$$rS + b_m \subset T_n \cap (qT_n + a) \quad \text{for} \quad m > n.$$ 

Since $T_n$ is closed, it follows that

$$rS + b \subset T_n \cap (qT_n + a) \quad \text{for every} \quad n,$$

whence $rS + b \subset A$.

Secondly, since

$$r_n \geq r_m \quad \text{for} \quad m > n,$$

we have

$$r_nT_n + b_m \supseteq r_nT_m + b_m$$

$$= T_m \cap (qT_m + a)$$

$$\supseteq A \quad \text{for every} \quad m > n.$$

It follows that

$$r_nT_n + b \supseteq A \quad \text{for every} \quad n,$$

hence also that

$$r_nT_m + b \supseteq r_nT_m + b \supseteq A \quad \text{for} \quad m > n,$$

whence $r_nS + b \supseteq A$ for all $n$.

From here, finally, it follows that

$$rS + b \supseteq A,$$

and the proof is completed.

**BIBLIOGRAPHY**


