M. S. Narasimhan

Variations of complex structures on an open Riemann surface


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1. Introduction.

The purpose of this paper is to study the local properties of variations of complex structures on a relatively compact subdomain of an open Riemann surface.

Let $M$ be an open Riemann surface and $M_1$ a relatively compact subdomain of $M$. Let $\mathcal{S}(t)$ be a family of complex structures on $M$ (or on a neighbourhood of $M_1$) which depends holomorphically on $t$, $t$ being in a neighbourhood $U_1$ of $t_0$ in $\mathbb{C}^n$. We suppose that $\mathcal{S}(t_0)$ is identical with the given structure on $M$. Consider the family of complex structures $\mathcal{S}(t, M_1)$ induced on $M_1$ by $\mathcal{S}(t)$. The family $\mathcal{S}(t, M_1)$ defines a complex analytic structure on $M_1 \times U_1$; we denote by $\mathcal{S}(M_1 \times U_1)$ the complex analytic manifold (or structure) thus defined. The projection $\pi_1 : M_1 \times U_1 \rightarrow U_1$ defines a family of deformations of complex structures in the sense of Kodaira-Spencer.

We first prove that for every sufficiently small Stein neighbourhood $U$ of $t_0$, $\mathcal{S}(M_1 \times U)$ is a Stein manifold (Theorem 1). We then show that the restriction of the family

$$\pi_1 : \mathcal{S}(M_1 \times U_1) \rightarrow U_1$$

to a sufficiently small neighbourhood $U$ of $t_0$ is complex ana-
lytically homeomorphic to the family $\pi : \Omega \rightarrow \pi(\Omega) \subset C^n$, where $\Omega$ is an open Stein submanifold of the product complex manifold $M \times C^n$ and $\pi : M \times C^n \rightarrow C^n$ is the canonical projection of $M \times C^n$ onto $C^n$ (Theorem 2). This result may be viewed as a sort of local triviality (« semi-triviality ») or a local imbedding theorem.

We prove also an analogue of Theorem 2 for differentiable variations of complex structures (Theorem 3).

The proofs use the theory of linear elliptic partial differential equations and some tools from functional analysis.

We now give a rough sketch of the proofs. We show that there exists a sufficiently small neighbourhood $U_2$ of $t_0$ such that any function which is holomorphic (upto the boundary) on any fibre over a point of $U_2$ can be extended to a holomorphic function on the whole fibre system restricted to $U_2$. From this it follows easily that we can separate points on the fibre system by holomorphic functions and that there exist $(m + 1)$ holomorphic functions which form a local coordinate system at a given point. To prove the holomorph-convexity, we first prove, by considering variations of complex structures on a disc, that the fibre system, restricted to a small Stein neighbourhood of $t_0$, is « locally holomorphically convex ». Then, by solving a problem analogous to the first Cousin problem with the help of currents, the holomorph-convexity is proved.

Once Theorem 1 is proved, Theorem B on Stein manifolds assures the vanishing of certain cohomology groups; we then prove theorem 2, adopting a method of Kodaira-Spencer.

Theorem 3 (differentiable case) is proved by solving the following problem: given Cousin data on $\mathcal{H}(t, M_1)$ which depend differentiably on the parameter, to find solutions of the (first) Cousin problem such that the solutions also depend differentiably on the parameter. The proof is inspired by a proof (unpublished) by L. Schwartz of some results concerning Cousin problems on a compact Riemann surface with varying complex structures and by some considerations in Kodaira-Spencer [2].

The author is thankful to Professor L. Schwartz for suggesting the use of Lemma 1, which simplifies the earlier demonstration of the author using power series expansions.
2. Statement of the theorems.

Let $M$ be an open Riemann surface. Let $\Theta$ be the holomorphic tangent bundle of $M$. Let $\mathcal{E}(\Theta \otimes \overline{\Theta}^*)$ denote the space of $\mathcal{C}^\infty(0, 1)$ forms with coefficients in $\Theta$, endowed with the natural topology [5]. If $\tilde{\mu} \in \mathcal{E}(\Theta \otimes \overline{\Theta}^*)$ and $z$ a local coordinate system then $\tilde{\mu}$ is of the form $\mu(z) \frac{dz}{dz_\beta}$. If we define $|\tilde{\mu}| = |\mu|$ locally, then $|\tilde{\mu}|$ is intrinsically defined as a function on $M$. If $\tilde{\mu} \in \mathcal{E}(\Theta \otimes \overline{\Theta}^*)$ with $|\tilde{\mu}| < 1$ then locally the forms $dz + \mu(z) d\overline{z}$ define a $(1, 0)$ form for a complex structure and thus $\tilde{\mu}$ defines a complex structure on $M$.

Let $t_0 \in \mathbb{C}^n$ and $U_0$ be an open set in $\mathbb{C}^n$ containing $t_0$. 't' will denote a point in $U_0$.

For our purposes a holomorphic family $\mathcal{F}(t)$ of complex structures on $M$ will be, by definition, a holomorphic function $\tilde{\mu}(t)$ defined in $U_0$ with values in $\mathcal{E}(\Theta \otimes \overline{\Theta}^*)$ such that $|\tilde{\mu}(t)| < 1$ and $\tilde{\mu}(t_0) = 0$. We then have on $M \times U_0$ an almost complex structure defined locally by the forms $dz + \mu(t, z) d\overline{z}$, $dt_1, \ldots, dt_n$ where $t_1, \ldots, t_n$ are the coordinate function in $\mathbb{C}^n$. This almost complex structure is integrable since $\tilde{\mu}(z, t)$ is holomorphic in $t$. Hence we have a complex structure on $M \times U_0$ (see also proposition 1). We denote $M \times U_0$ endowed with this complex structure by $\mathcal{F}(M \times U_0)$. The projection $\pi_1 : S(M \times U_0) \rightarrow U_0$ is holomorphic and we have a holomorphic family of deformations of complex structures in the sense of Kodaira-Spencer [2].

If $M_1$ is a subdomain of $M$ and $V$ a neighbourhood of $t_0$ in $\mathbb{C}^n$ with $V \subset U_0$, we denote the manifold $M_1 \times V$ with the complex structure induced from $\mathcal{F}(M \times U_0)$ by $\mathcal{F}(M_1 \times V)$. We denote by $\mathcal{F}(t)$ the complex analytic structure on $M$ defined by $\tilde{\mu}(t)$.

We have

Theorem 1. — Let $\mathcal{F}$ be a holomorphic family of complex structures on an open Riemann surface $M$. Let $M_1$ be a relatively compact subdomain of $M$. Then there exists a neighbourhood $V$ of $t_0$ such that for every Stein neighbourhood $U$ of $t_0$ contained in $V$, $\mathcal{F}(M_1 \times U)$ is a Stein manifold.
Theorem 2. — Let $\mathcal{F}$ be a holomorphic family of complex structures on an open Riemann surface $M$ and $M_1$ a relatively compact subdomain of $M$. Then there exist a neighbourhood $U$ of $t_0$, an open Stein submanifold $\Omega$ of the product manifold $M \times \mathbb{C}^n$, a complex analytic homeomorphism $\Phi$ of $\mathcal{F}(M_1 \times U)$ onto $\Omega$ and a complex analytic homeomorphism $\varphi$ of $U$ onto $\pi(\Omega)$ ($\pi$ denoting the projection $M \times \mathbb{C}^n \rightarrow \mathbb{C}^n$) such that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathcal{F}(M_1 \times U) & \xrightarrow{\Phi} & \Omega \\
\downarrow{\pi_1} & & \downarrow{\pi} \\
U & \xrightarrow{\varphi} & \pi(\Omega).
\end{array}
$$

Remark. — In Theorems 1 and 2 and as well in Theorem 3, if the boundary of $M_1$ is smooth it is sufficient to assume that the variation is given only up to the boundary of $M_1$.

Let $U_0$ be an open subset in $\mathbb{R}^n$ and $t_0 \in U_0$. A differentiable family of complex structures we mean differentiable function $\tilde{\mu}(t)$ defined in $U_0$ with values in $\mathcal{E}(\Theta \otimes \Theta^*)$ such that $|\tilde{\mu}(t)| < 1$ and $\tilde{\mu}(t_0) = 0$. (By differentiable we always mean « indefinitely differentiable ».) For a subdomain $M_1$ of $M$ we denote by $\mathcal{F}(t, M_1)$, $t \in U_0$, the surface $M_1$ endowed with the complex structure defined by $\tilde{\mu}(t)$.

We have then

Theorem 3. — Let $\mathcal{F}(t)$ be a family of complex structures on $M$ depending differentiably on $t$, $t$ being in a neighbourhood of $t_0$ in $\mathbb{R}^n$. Let $M_1$ be a relatively compact subdomain of $M$. Then there exist a neighbourhood $U$ of $t_0$ and a differentiable map $\Phi$ of $M_1 \times U$ into $M$ which maps each fibre $\mathcal{F}(t, M_1)$, $t \in U$, biholomorphically into $M$.

3. Some lemmas in functional analysis and potential theory.

Some of the lemmas stated in this section are more or less well-known. We state them here for convenience of reference. We denote by $U_0$ an open set in $\mathbb{C}^n$ or $\mathbb{R}^n$ according as we consider holomorphic or differentiable variations. $t_0$ is a point of $U_0$. 
Let $E$ and $F$ be two complete barrelled locally convex topological vector spaces. We shall say that a family of continuous linear operators $T_t : E \to F$, $t \in U_0$, depends holomorphically (resp. differentiably) on $t \in U_0$ if $t \to T_t$ is a holomorphic (resp. differentiable) function of $U_0$ with values in $\mathcal{L}(E, F)$, where $\mathcal{L}(E, F)$ denotes the space of continuous linear operators of $E$ into $F$ endowed with the topology simple convergence. We remark that if $T_t$ depends holomorphically (resp. differentiably) on $t$ and $f(t)$ is a holomorphic (resp. differentiable) function with values in $E$ then $t \to T_t f(t)$ is a holomorphic (resp. differentiable) function with values in $F$.

**Lemma 1.** — Let $E$ and $F$ be two Banach spaces and $T_t : E \to F$ depend holomorphically (resp. differentiably) on $t$. Assume that $T_t$ is an isomorphism. Then there exists a neighbourhood $U_0$ of $t_0$ such that $T_t$ is an isomorphism for each $t \in U_0$ and the operators $T_t^{-1} : F \to E$ depend holomorphically (resp. differentiably) on $t \in U_0$.

This lemma is a special case of implicit function theorem in Banach spaces and is proved easily.

**Lemma 2.** — Let $E$ and $F$ be two Banach spaces and $T_t : E \to F$ depend holomorphically (resp. differentiably) on $t$. Assume that $T_t$ admits of a right inverse. Then there exists a neighbourhood $U_0$ of $t_0$ such that for $t \in U_0$, $T_t$ admits of a right inverse depending holomorphically (resp. differentiably) on $t$.

**Proof.** — We recall that a right inverse for $T_t$ is a continuous linear map $S_t : F \to E$ such that $T_t \circ S_t$ is the identity map of $F$. Now we apply Lemma 1 to the operators $T_t \circ S_t(F) : S_t(F) \to F$ and Lemma 2 follows.

Let $D$ be a relatively compact open subset of $\mathbb{C}$. Let $\alpha$ be a fixed real number with $0 < \alpha < 1$. Let $f$ be a complex valued function satisfying a Hölder condition of order $\alpha$ on $D$. Put

$$
||f||_{\alpha, D} = \sup_{D} |f| + \sup_{z_1, z_2 \in D, z_1 \neq z_2} \left| \frac{f(z_1) - f(z_2)}{|z_1 - z_2|^\alpha} \right|.
$$
We denote the space of these functions by $H_{\alpha}(D)$.

If $f$ is a function which is once differentiable such that its partial derivatives satisfy in $\overline{D}$ a Hölder condition of order $\alpha$ put

$$
||f||_{1,\alpha,D} = \text{Sup} \ |f| + \left\| \frac{\partial f}{\partial z} \right\|_{0,\alpha,D} + \left\| \frac{\partial f}{\partial \bar{z}} \right\|_{0,\alpha,D}.
$$

Let $H_{1,\alpha}(D)$ denote the space of such functions.

Let now $D$ be a disc $|z| < R$, $0 < R < \infty$ in the plane. The operator $\frac{\partial}{\partial z}$ is a continuous linear operator from the Banach space $H_{1,\alpha}(D)$ (with the norm $||f||_{1,\alpha}$) to the Banach space $H_{\alpha}(D)$ (with the norm $||f||_{0,\alpha}$).

**Lemma 3.** — Let $D$ be a disc in the plane. The operator

$$
\frac{\partial}{\partial z} : H_{1,\alpha}(D) \to H_{\alpha}(D)
$$

admits of a right inverse.

This lemma is classical. For instance convolution with $\frac{1}{\pi z}$ yields a right inverse [1].

Let $M_0$ be a relatively compact subdomain of an open Riemann surface $M$ such that $M_0$ is bounded by a finite number of disjoint analytic Jordan curves. We shall say, for brevity, that $M_0$ has an analytic boundary. We shall denote by $\partial M_0$ boundary of $M_0$ in $M$.

Let $D_1, \ldots, D_k, D_{k+1}, \ldots, D_n$ be a covering of $\overline{M_0}$ by coordinate discs $D_i$ in $M$ with $\overline{D_i}$ compact and contained in a coordinate disc such that the following conditions are satisfied:

i) $\overline{D_i}$ is contained in $M_0$ for $i = 1, \ldots, k$.

ii) if $z_j$ is the coordinate function in $D_j$ mapping $D_j$ onto $|z| < \varepsilon$, then $z_j$ maps for $j = k + 1, \ldots, n$, $D_j \cap M_0$ onto the « semi-disc » $\{|z| < \varepsilon, \text{Im} \ z > 0\}$ and $D_j \cap \partial M_0$ onto $\{-\varepsilon < \text{Re} \ z < \varepsilon\}$.

Let $D_i'$ denote the covering of $M_0$ formed by $D_1, \ldots, D_k, D_{k+1} \cap M_0, \ldots, D_n \cap M_0$. Let $\{D_i\}$ be a shrinking of the covering $\{D_i'\}$.

Let $H_{1,\alpha}(M_0)$ denote the Banach space of complex valued functions in $M_0$ which are once differentiable in $M_0$ and whose first partial derivatives satisfy a Hölder condition of order $\alpha$. 
in every compact subset contained in a coordinate neighbourhood of $M_0$ (e.g. $D'_i$) with the norm

$$||f||_{1, \alpha} = \sup_{i=1, \ldots, n} ||f||_{1, \alpha, D'_i}.$$  

Let $H^\alpha_\sigma(M_0)$ denote the space of $(0, 1)$ forms whose coefficients satisfy a Hölder condition of order $\alpha$ in every compact set contained in a coordinate neighbourhood of $M_0$. If $f \in H^\alpha_\sigma(M_0)$ and $f = f_i \, dz^i$ in $D'$ ($z^i$ being the coordinate function in $D'_i$) define

$$||f||_{0, \alpha, M_0} = \sup_i ||f_i||_{0, \alpha, D'_i}$$

with this norm $H^\alpha_\sigma(M_0)$ becomes a Banach space.

**Lemma 4.** — *Let $M_0$ be a relatively compact subdomain of $M$ with analytic boundary. Then the operator

$$d^\tau : H_{1, \alpha}(M_0) \rightarrow H^\alpha_\sigma(M_0)$$

admits of a right inverse.*

**Proof.** — We give a sketch of the proof of this lemma. Let $M_1$ be a relatively compact subdomain of $M$, with analytic boundary, containing $\bigcup_{i=1, \ldots, n} D'_i$. We first remark that we can find a continuous linear map $\rho : H^\alpha_\sigma(M_0) \rightarrow H^\alpha_\sigma(M_1)$ such that $\gamma \circ \rho = \text{identity map of } H^\alpha_\sigma(M_0)$, where $\gamma : H^\alpha_\sigma(M_1) \rightarrow H^\alpha_\sigma(M_0)$ denotes the restriction map. [The question being local at the boundary, locally the extension is given by reflection at the $x$-axis. For details see e.g. [4, Th. 2. 4]]. On $M_1 \times M_1$ there exists (H. Behnke-K. Stein, Math. Ann. 120, p. 436) a meromorphic differential $K(z, d\zeta)$, holomorphic for $z \neq \zeta$ such that in a coordinate disc around $z = \zeta$ we have,

$$K(z, d\zeta) = \frac{-1}{4\pi(z - \zeta)} + \text{regular function } d\zeta.$$

We may then estimate the potential

$$T_1 \tilde{f} = 2i \int_{M_1} K(z, d\zeta) \wedge \tilde{f}(\zeta), \quad \tilde{f} \in H^\alpha_\sigma(M_1)$$

on compact subsets of $D'_i$ using the estimate on a disc.
for the potential with the kernel $\frac{1}{\pi(x - \zeta)}$ [1]. If $f \in H^a(M_0)$ let $Tf$ denote the restriction of $T(x(P))$ to $M_0$. Then $d_\xi Tf = f$ and

$$||Tf||_{1, \alpha, M_0} \leq C_1 ||f||_{0, \alpha, M_1} \leq C_2 ||f||_{0, \alpha, M_0}$$

with positive constants $C_1$ and $C_2$. This proves Lemma 4.

The next lemma will be required only for the holomorphic tangent bundle of $M$. But we shall prove it for a general holomorphic line bundle.

Let $L$ be a holomorphic line bundle on $M$. Let $H_{1, \alpha}(M_0, L)$ denote the Banach space of sections of $L$ in $M_0$ which are once differentiable in $M_0$ and whose first partial derivatives satisfy a Hölder condition of order $\alpha$. Let $H^a(M_0, L)$ denote the Banach space of Hölder continuous $(0, 1)$ forms in $M_0$ with coefficients in $L$ (we introduce norms on $H_{1, \alpha}(M_0, L)$ and $H^a(L)$ as on $H_{1, \alpha}(M_0)$ and $H^a(M_0)$).

**Lemma 5.** — Let $L$ be a holomorphic line bundle on $M$. Let $M_0$ be a relatively compact subdomain with analytic boundary. Then the operator

$$d_\xi : H_{1, \alpha}(M_0, L) \to H^a(M_0, L)$$

admits a right inverse.

**Proof.** — Since $M$ is an open Riemann surface, every holomorphic line bundle on $M$ is holomorphically trivial. This follows for example from the exact sequence.

$$H^1(M, O) \to H^1(M, O^*) \to H^2(M, \mathbb{Z})$$

remarking that $H^1(M, O) = 0$, $H^2(M, \mathbb{Z}) = 0$. (Here $O$ denotes the sheaf of germs of holomorphic functions and $O^*$ the sheaf of germs of non-vanishing holomorphic functions.) Since $L$ is holomorphically trivial on $M$ there exist topological isomorphisms

$$\psi_1 : H_{1, \alpha}(M_0, L) \to H_{1, \alpha}(M_0)$$

$$\psi_2 : H^a(M_0, L) \to H^a(M_0)$$
such that the following diagram is commutative:

\[
\begin{array}{ccc}
H_{1,\alpha}(M_0, L) & \xrightarrow{\phi_1} & H_{1,\alpha}(M_0) \\
\downarrow d_{\bar{z}} & & \downarrow d_{\bar{z}} \\
\bar{H}_{\alpha}(M_0, L) & \xrightarrow{\phi_2} & \bar{H}_{\alpha}(M_0)
\end{array}
\]

Since \( d_{\bar{z}} : H_{1,\alpha}(M_0) \to \bar{H}_{\alpha}(M_0) \) admits of a right inverse it follows that \( d_{\bar{z}} : H_{1,\alpha}(M_0, L) \to \bar{H}_{\alpha}(M_0, L) \) admits of a right inverse.

4. Variation of complex structures on a disc.

**Proposition 1.** — Let \( D \) be a disc in the plane. Let \( \mu(t) = \mu(z, t) \) be a holomorphic function defined in a neighbourhood of \( t_0 \) in \( \mathbb{C}^n \) with values in \( H_{\alpha}(D) \) with \( \mu(t_0) = 0 \). Then there exist a neighbourhood \( U' \) of \( t_0 \) and a \( C^1 \) function \( \zeta(z, t) \) defined in \( D \times U' \) such that

i) \( \frac{\partial \zeta(z, t)}{\partial \bar{z}} - \frac{\partial \zeta(z, t)}{\partial z} = 0, \quad i = 1, \ldots, m. \)

ii) there exist positive constants \( K_1 \) and \( K_2 \) such that one has

\( K_1 |z_1 - z_2| \leq |\zeta(z_1, t) - \zeta(z_2, t)| \leq K_2 |z_1 - z_2| \) for \( z_1, z_2 \in \overline{D} \) and all \( t \in U' \).

iii) If \( F(t) = F(z, t) = \frac{1}{\zeta(z, t) - \zeta(z_0, t)} \), \( z_0 \in D \), the function \( t \to F(t) \) is a holomorphic function in \( U' \) with values in \( \mathcal{D}'(D) \), where \( \mathcal{D}'(D) \) denotes the space of distributions in \( D \); moreover for each fixed \( t \), \( F(z, t) \) is holomorphic outside \( z_0 \) for the complex structure defined by \( dz + \mu(z, t) \bar{dz} (|\mu| < 1) \).

**Proof.** — There exists a constant \( C_1 > 0 \) such that for \( f, g \in H_{\alpha}(D) \) one has \( ||fg||_{0, \alpha} \leq C_1 ||f||_{0, \alpha} ||g||_{0, \alpha} \). Hence the operator of multiplication by \( \mu(z, t) \) is a holomorphic function of \( t \) with values in \( L_{\alpha}(H_{\alpha}, H_{\alpha}) \). It follows that the operators

\[
T_t = \frac{\partial}{\partial \bar{z}} - \mu(z, t) \frac{\partial}{\partial z} : H_{1,\alpha}(D) \to H_{\alpha}(D)
\]

depend holomorphically on \( t \). Now \( T_{t_0} = \frac{\partial}{\partial \bar{z}} \). By Lemma 3
Tₜ admits a right inverse. Hence by lemma 2 there exists a neighbourhood U" of t₀ and continuous linear operators

\[ S_\iota : H_\alpha \to H_{1,\iota} \]

depending holomorphically on \( t \in U" \) such that \( T_\iota \circ S_\iota = \text{Identity map of } H_\alpha \). Now \( \mu(t) \) is a holomorphic function with values in \( H_\alpha \). Hence \( f(t) = S_\iota(\mu(t)) \) is a holomorphic function with values in \( H_{1,\iota} \). Let

\[ \zeta(z, t) = z + f(z, t). \]

\( \zeta(z, t) \) is of class \( C^1 \). Moreover

\[ \frac{\partial \zeta}{\partial z} = \frac{\partial f(z, t)}{\partial z} = \mu(z, t) \frac{\partial f}{\partial z} + \mu(z, t) \]
\[ = \mu(z, t) \left( 1 + \frac{\partial f}{\partial z} \right) \]
\[ = \mu(z, t) \frac{\partial \zeta}{\partial z}, \]

so that \( \zeta(z, t) \) satisfies

\[ \begin{cases} 
\frac{\partial \zeta(z, t)}{\partial z} - \mu(z, t) \frac{\partial \zeta}{\partial z} = 0, \\
\frac{\partial \zeta(z, t)}{\partial t} = 0, \quad i = 1, \ldots, m.
\end{cases} \tag{1} \]

To prove ii) we remark that there exists a constant \( k > 0 \) (depending only on \( D \)) such that for each \( f \in H_{1,\iota} \) one has

\[ |f(z_1) - (z_2)| \leq k|z_1 - z_2| ||f||_{1,\iota}, \quad z_1, z_2 \in \overline{D}. \]

(This is proved easily applying the mean value theorem.) Since \( f(t_0) = 0 \) we can choose a relatively compact neighbourhood \( U' \) of \( t_0 \) with \( \overline{U'} \subset U" \) such that for \( t \in U' \), \( ||f(t)||_{1,\iota} \leq \frac{\epsilon}{k} \), given \( \epsilon \) with \( 0 < \epsilon < 1 \). It is evident that there exists a constant \( K_2 \) such that

\[ |\zeta(z_1, t) - \zeta(z_2, t)| \leq K_2|z_1 - z_2|, \quad t \in U', \quad z_1, z_2 \in \overline{D}. \]

On the other hand

\[ |\zeta(z_1, t) - \zeta(z_2, t)| = |\{ z_1 + f(z_1, t) \} - \{ z_2 + f(z_2, t) \}| \]
\[ \geq |z_1 - z_2| - |f(z_1, t) - f(z_2, t)| \]
\[ \geq (1 - \epsilon)|z_1 - z_2|. \]

This completes the proof of ii).
To prove iii), we note that for $t$ fixed $1/\zeta(z, t) - \zeta(z_0, t)$ is a locally summable function in $D$ (see ii); and since

$$|F(z, t)| \leq K_1^{-1} |z - z_0|,$$

$t \in U'$, we see, by Lebesgue's dominated convergence theorem, that $t \to F(t)$ is a continuous function with values in $\mathcal{D}'(D)$. To prove that $F(t)$ is a holomorphic function with values in $\mathcal{D}'(D)$ it is sufficient to prove that $h(t) = \langle F(t), \varphi \rangle$ is a holomorphic function of $t$ for each $\varphi \in \mathcal{D}(D)$. $\mathcal{D}(D)$ denotes the space of $C^\infty$ functions with compact supports in $D$; $\langle F(t), \varphi \rangle$ denotes the scalar product between $F(t)$ and $\varphi$. As was noted earlier $h(t)$ is a continuous function. Let $t_i = (t_i^1, \ldots, t_i^n) \in U'$. We shall show that $h(t^1, t_i^1, \ldots, t_i^n)$ is differentiable at $t_i^1$ as a function of $t^1$.

Let

$$\psi(t^1) = \{ h(t^1, t_i^1, \ldots, t_i^n) - h(t_i^1, t_i^1, \ldots, t_i^n) \} / (t^1 - t_i^1).$$

Then

$$\psi(t^1) = \int_{K} \frac{1}{t^1 - t_i^1} \times \left\{ \left[ \zeta(z, t_i^1) - \zeta(z_0, t_i^1) \right] - \left[ \zeta(z, t^1, t_i^1, \ldots, t_i^n) - \zeta(z_0, t^1, \ldots, t_i^n) \right] \right\} \varphi \, dx \, dy$$

where $K$ is the support of $\varphi$.

We assert that there exists a constant $K_2$ such that for $t_i^1$ in a sufficiently small neighbourhood of $t_i$ we have

$$\left| \left[ \zeta(z, t_i^1, \ldots, t_i^n) - \zeta(z_0, t_i^1, \ldots, t_i^n) \right] - \left[ \zeta(z, t^1, t_i^1, \ldots, t_i^n) - \zeta(z_0, t^1, t_i^1, \ldots, t_i^n) \right] \right| \leq K_2 / |z - z_0|.$$

In fact, consider the function with values in $H_{1, \alpha}$ defined in a neighbourhood of $t_i^1$:

$$g(t^1) = \begin{cases} \frac{\zeta(t_i^1, \ldots, t_i^n) - \zeta(t^1, t_i^1, \ldots, t_i^n)}{t^1 - t_i^1} & \text{for } t^1 \neq t_i^1 \\ \left. \frac{d}{dt} \zeta(t_i^1, t_i^1, \ldots, t_i^n) \right|_{t_i} & \text{for } t^1 = t_i^1 \end{cases}$$

Since $\zeta(t)$ is a holomorphic function with values in $H_{1, \alpha}$,
$g(t')$ is a continuous function and hence in a neighbourhood of $t'$, $\|g(t')\|_{t', \alpha} \leq K$. Using the inequality

$$|f(z_1) - f(z_2)| \leq k|z_1 - z_2| \|f\|_{t', \alpha}$$

we obtain (A). From (A) and the first inequality in ii) we see that the integrand is majorised by $K|z - z_0|$ for all $t'$ in a sufficiently small neighbourhood of $t_i$. By Lebesgue's theorem we see that $\lim_{t \to t_i'} \varphi(t')$ exists and is equal to

$$\int K \int \left\{ \frac{d}{dt} (\zeta(t', \ldots, t_m)(z))_{t = t_i} - (\frac{d}{dt} \zeta(t', \ldots, t_m)(z_0))_{t = t_i} \right\} dx dy.$$ 

Similary we show that the other derivatives exist. This proves that $h(t)$ is holomorphic.

From the first inequality in ii) we see that $\zeta(z, t) - \zeta(z_0, t) = 0$ for $z \neq z_0, t \in U'$. The second assertion in iii) follows immediately from this fact. This completes the proof of Proposition 1.

**Remark 2.** — Using i) and ii) we can show easily that if $U$ is a polydisc contained in $U'$ the map $(t, z) \to (t, \zeta(t, z))$ maps $\mathcal{S}(\mathbb{C}^m)$ (endowed with the complex structure defined by $dz + \mu(z, t) dz, dt, \ldots, dt_m, |\mu(z, t)| < 1$), biholomorphically onto a bounded domain of holomorphy in $\mathbb{C}^{m+1}$. Proposition 1 is also valid if we replace the disc by a bounded plane domain with a smooth boundary. Thus Theorems 1 and 2 are immediate consequences of Proposition 1 in the case of plane domains.

5. Elementary kernels for elliptic differential operators depending holomorphically on a parameter.

Let $M$ be an open Riemann surface and let $\mathcal{D}(M), \mathcal{E}(M)$, $\mathcal{D}'(M), \mathcal{E}'(M)$ denote the space of $\mathcal{C}^\infty$ forms of type $(p, q)$ with compact supports, $\mathcal{C}^\infty$ forms of type $(p, q)$, currents of type $(p, q)$ and currents of type $(p, q)$ with compact supports respectively, each endowed with the usual topology [5].
Let $\mathcal{I}(t, M)$ be a family of complex structures depending holomorphically on $t$. Define $T_f: \mathcal{D}'(M) \rightarrow \mathcal{D}'(M)$ by

$$T_f = d_f - \langle \tilde{\mu}(t), d_z \rangle,$$

where $\langle \tilde{\mu}(t), d_z \rangle$ denotes the current of type $(0, 1)$ obtained by contracting $\tilde{\mu}(t)$ (which is of type $(-1, 1)$) and $d_z$ (which is of type $(1, 0)$). We remark that a function $f$ is holomorphic for the structure $\mathcal{I}(t, M)$ if and only if $T_f = 0$. A function $f$ defined on $U_0 \times M$ is holomorphic for the structure $\mathcal{I}(U_0 \times M)$ if and only if it satisfies the system of differential equations:

$$\begin{cases}
T_f(z, t) = 0,
\frac{\partial f(z, t)}{\partial t} = 0, & i = 1, \ldots, m.
\end{cases}$$

**Proposition 2.** — Let $M_0$ be a relatively compact subdomain of $M$, with analytic boundary. Then there exists a neighbourhood $U_0$ of $t_0$ and for $t \in U_0$ continuous linear operators $S_t: \mathcal{E}(M_0) \rightarrow \mathcal{E}'(M_0)$ depending holomorphically on $t$ such that for $f \in \mathcal{E}'(M_0)$ one has $T_S f = f$. Let $f(z, t) = \langle \tilde{\mu}(t), d_z \rangle$. We shall show that $S_t: \mathcal{E}' \rightarrow \mathcal{E}$ is continuous by Banach’s closed graph theorem. We prove that $S_t: \mathcal{D}' \rightarrow \mathcal{E}$ depends holomorphically on $t$. Let $\varphi \in \mathcal{D}'$. Then the current $F(z, t) = S_t \varphi$ satisfies the system of differential equations:

$$\begin{cases}
T_t F = \varphi,
\frac{\partial F}{\partial t} = 0, & i = 1, \ldots, m.
\end{cases}$$
Since this system is elliptic, \( F(z, t) \) is a \( C^\infty \) function in \( M_0 \times U_3 \). It follows that \( t \to S_t \) is a holomorphic function with values in \( \mathcal{E} \).

Let \( T_t : \mathcal{D}' \to \mathcal{D}' \) be the transpose of the differential operator \( T_t \). Then \( T_t \) is a linear elliptic differential operator with \( C^\infty \) coefficients. Let \( S_t : \mathcal{E}' \to \mathcal{E}' \) be the transpose of \( S_t : \mathcal{D} \to \mathcal{E} \). Then \( S_t : \mathcal{E}' \to \mathcal{E}' \) depends holomorphically on \( t \).

By the hypo-ellipticity of \( S_t' \), \( S_t' \) maps \( \mathcal{D} \) into \( \mathcal{E} \) and \( S_t : \mathcal{D} \to \mathcal{E} \) is continuous. As we proved for \( S_t \), we prove, using the hypo-ellipticity of the system \( \left\{ S_t', \frac{\partial}{\partial t^i}, \ldots, \frac{\partial^m}{\partial t^m} \right\} \) that \( S_t : \mathcal{D} \to \mathcal{E} \) depends holomorphically on \( t \). By taking transposes we obtain \( S_t : \mathcal{E}' \to \mathcal{E}' \) depending holomorphically on \( t \) and coinciding on \( \mathcal{D} \) with the \( S_t \) originally given.

This proves Proposition 2.

6. A result on the prolongation of holomorphic functions.

**Proposition 3.** — Let \( M_0 \) be a relatively compact subdomain of \( M \) with analytic boundary. Then there exists a neighbourhood \( U_2 \) of \( t_0 \) with the following property: if \( f(z) \) is a function which is holomorphic for the structure \( \mathfrak{g}(t_1) \) in \( M_1, t_1 \in U_2 \), then there exists a function \( F(z, t) \) in \( M_0 \times U_2 \) which is holomorphic for the structure \( \mathfrak{g}(M_0 \times U_2) \) such that \( F(z, t_1) = f(z) \).

**Proof.** — Let \( S_t : H_{t_0}(M_0) \to H_{t_0}(M_0) \) be right inverses for \( T_t \) depending holomorphically on \( t \in U_2 \). Now \( f \in H_{t_0} \). Define

\[
F(z, t) = f - S_t T_t f, \quad (t \in U_2).
\]

We then have

\[
T_t (f - S_t T_t f) = T_t f - T_t S_t T_t f = T_t f - T_t f = 0;
\]
since $T_i f = 0$, $F(z, t_1) = f(z)$. $F(z, t)$ satisfies the system of differential equations

$$\begin{align*}
T_i F(z, t) &= 0, \\
\frac{\partial F}{\partial t^i} &= 0, \\
&i = 1, \ldots, m.
\end{align*}$$

Hence $F(z, t)$ is holomorphic for the structure $\mathcal{F}(M_0 \times U_2)$.

7. Proof of Theorem 1.

We now proceed to prove Theorem 1. Let $M_0$ be a relatively compact sub-domain of $M$ with analytic boundary such that $\overline{M}_0 \subset M_0$. Let $O_1, \ldots, O_1, \ldots, O_k$ be a finite number of coordinate discs for the structure $\mathcal{F}(t_0)$ with $\overline{O}_i \subset M_0$ and $U_i O_i \supset \overline{M}_1$.

Let $z'$ be the coordinate function in $O_i$. Then $\hat{\mu} = \mu_i dz' \otimes \frac{\partial}{\partial z'}$.

Let $V_i, i = 1, \ldots, k$, be neighbourhoods of $t_0$ such that functions $\zeta_i(z', t), z' \in O_i, t \in V_i$ can be defined satisfying conditions i), ii), iii) of Proposition 1. (By an obvious abuse of notation we use the letter $z'$ to denote a point on the Riemann surface and as well its image by the coordinate function $z'$.)

Let $U_3$ and $U_2$ be neighbourhoods of $t_0$ given in Proposition 2 and 3. Let $V$ and $V'$ relatively compact neighbourhoods of $t_0$ such that $\overline{V} \subset V'$ and $\overline{V}' \subset \bigcap_{i=1,\ldots,k} V_i \cap U_2 \cap U_3$. Let $U$ be a Stein neighbourhood of $t_0$ contained in $V$.

We first show that holomorphic functions on $\mathcal{F}(M_1 \times U)$ separate points. Since $(\mathcal{F}, \ldots, t^m)$ are holomorphic functions on $\mathcal{F}(U \times M_1)$ we have only to consider the case when the points are on the same fibre. Let then $(z_1, t_1), (z_2, t_1)$ be two points $t_1 \in U$, $z_1, z_2 \in M_1$, $z_1 \neq z_2$. Now there exists a function $f(z)$ holomorphic in $M_0$ for $\mathcal{F}(t_1)$ with $f(z_1) \neq f(z_2)$. [This is shown, for example, by taking an open set slightly larger than $M_0$ and using the fact every open Riemann surface is a Stein manifold.] By Proposition 3 there exists a function $F(z, t)$ holomorphic for the structure $\mathcal{F}(M_0 \times U)$ such that $F(z_1, t_1) = f(z)$. Hence $F(z_1, t_1) \neq F(z_2, t_1)$.

Next let $(z_1, t_1), z_1 \in M_1, t_1 \in U$ be a point in $M_1 \times U$. We shall show that there exist $(m + 1)$ functions in $M_1 \times U$ which
are holomorphic on \( \mathcal{J}(M_1 \times U) \) and which form a local coordinate system at \((z_1, t_1)\). Let \( f(z) \) be a function holomorphic for \( \mathcal{J}(t) \) in \( M_0 \) which forms a local coordinate system at \( z_1 \) in \( M_0 \). Let \( F(z, t) \) be an extension of \( f(z) \) to \( M_0 \times U \) as a function holomorphic for the structure \( \mathcal{J}(M_0 \times U) \). Suppose \( z_1 \in O_i \). With respect to the coordinate system \((z^i, t^1, \ldots, t^m)\) the Jacobian of \((F, t^1, \ldots, t^m)\) at \((z_1, t_1)\) is

\[
\left| \frac{\partial F(z, t)}{\partial z^i} \right|_{(z_1, t_1)}^2 (1 - |\mu_i(z, t)|^2)
\]

or

\[
\left| \frac{\partial f(z)}{\partial z^i} \right|^2 (1 - |\mu_i(z, t)|^2).
\]

But \( \left( \frac{\partial f}{\partial z^i} \right)_{z^i = z_1} \neq 0 \); for if it were zero, then \( \frac{\partial f}{\partial z^i} = \mu_i \frac{\partial f}{\partial z^i} \) would be zero so that the Jacobian of \( f \) at \( z_1 \) would be zero. Thus the Jacobian of \((F, t^1, \ldots, t^m)\) at \((z_1, t_1)\) is different from zero.

Finally we show that given infinite discrete set of points \( \{z_n, t_n\}_n \), \( z_n \in M_1, t_n \in U \) there exists a holomorphic function \( \Psi \) on \( \mathcal{J}(M_1 \times U) \) such that the sequence \( \{\Psi(z_n, t_n)\}_n \) is not bounded. Now either the sequence \( \{t_n\}_n \) contains an infinite discrete subset or the sequence \( \{z_n\}_n \) contains an infinite discrete subset. If \( \{t_n\}_n \) contains an infinite discrete subset we can find, since \( U \) is Stein, a holomorphic function \( F(t) \) in \( U \) such that \( \{F(t_n)\}_n \) is not bounded. Then \( \Psi(z, t) = F(t) \) is holomorphic on \( \mathcal{J}(M_1 \times U) \) and \( \Psi(z_n, t_n) \) is not bounded. If \( \{z_n\}_n \) contains an infinite discrete subsequence \( \{z_k\}_k \) let \( z_0 \in M_1 \subset M_0 \) be an adherent point of \( \{z_k\}_k \), \( z_0 \in M_1 \). Suppose \( z_0 \in O_i \). Consider the currents of degree 0, \( F(t) = 1/\zeta_i(z, t) - \zeta_i(0, t) \) in \( O_i \). Since \( F(t) \) is a holomorphic function with values \( \mathcal{D}'(O_i) \) and \( T_i: \mathcal{D}'(O_i) \to \mathcal{D}'(O_i) \) depends homorphically on \( t \), \( T_i F(t) \) is a holomorphic function \( t \) with values in \( \mathcal{D}'(O_i) \). But the supports of \( T_i F(t) \) are at \( z_0 \) and hence \( T_i F(t) \) is a holomorphic function with values in \( \mathcal{D}'(M_0) \). Let \( S_i \) be the operators given by Proposition 2, in \( U_2 \). Let \( \Psi(t) = S_i(T_i F(t)) \). Then

\[
\begin{cases}
T_i \Psi(t) = (T_i F(t)), \\
\frac{\partial \Psi(t)}{\partial t^i} = 0, \quad i = 1, \ldots, m.
\end{cases}
\]
Since $z_0 \in M_1$, $\Psi(t)$ defines a function in $M_1 \times U$ satisfying in $M_1 \times U$

\[
\begin{align*}
T_i \Psi(z, t) &= 0, \\
\frac{\partial}{\partial t^i} \Psi(z, t) &= 0, \quad i = 1, \ldots, m;
\end{align*}
\]

that is $\Psi(z, t)$ is holomorphic on $\mathcal{S}(M_1 \times U)$. It remains to show that $\{\Psi(z_k, t_k)\}$ is not bounded. Let $O'_i$ be a relatively compact neighbourhood of $z_0$ such that $\overline{O'_i} \subset O_i$. We may suppose that all $z_k$ belong to $O'_i$. On $O_i \times V'$ the currents $G(z, t) = F_i(z) - \Psi_i(z)$ satisfies the system of differential equations

\[
\begin{align*}
T_i G(z, t) &= 0 \\
\frac{\partial}{\partial t^i} G(z, t) &= 0, \quad i = 1, \ldots, m.
\end{align*}
\]

Hence $G(z, t)$ is a $C^\infty$ function in $O_i \times V'$ and is hence bounded on $O_i \times U$. For $z \in M_1 \times O_i$, $t \in U$

$\Psi(z, t) = F(z, t) - G(z, t)$.

Hence

$\Psi(z_k, t_k) = F(z_k, t_k) - G(z_k, t_k) = \frac{1}{\zeta(z_k, t_k) - \zeta(z_0, t_k)} - G(z_k, t_k)$

So

$|\Psi(z_k, t_k) + G(z_k, t_k)| \geq K^{-1} |z_k - z_0|.$

by Proposition 1. Since $G(z_k, t_k)$ is bounded and $z_0$ is adherent to $z_k$, it follows that $\Psi(z_k, t_k)$ is not bounded. This completes the proof of Theorem 1.

8. Proof of Theorem 2.

The proof is essentially same as the proof of Theorem 5.1 in Kodaira-Spencer [2], once we have Theorem 1. Still we give the complete proof since some changes are required in our case. It is sufficient to prove the theorem without the requirement that $\Omega$ be Stein. For, once we have a $\Phi$ with $\Omega$ an open subset of $M \times \mathbb{C}^n$ we could restrict $\Phi$ to $\mathcal{S}(M_1 \times U)$ where $U$ is a sufficiently small Stein neighbourhood of $t_0$ and obtain Theorem 2 (since $\mathcal{S}(M_1 \times U)$ is Stein by Theorem 1).
Thus it is enough to show that there exist a neighbourhood $U'$ of $t_0$, an analytic homeomorphism $\Phi : \mathcal{F}(M_1 \times U') \to \Omega$ where $\Omega$ is an open submanifold of $M \times \mathbb{C}^n$, and an analytic homeomorphism $\varphi : U' \to \pi(\Omega)$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathcal{F}(U' \times M_1) & \xrightarrow{\Phi} & \Omega \\
\downarrow \pi_1 & & \downarrow \\
U' & \xrightarrow{\varphi} & \pi(\Omega)
\end{array}
$$

We make the following inductive assumption:

$A_{p-1}$: If the dimension of $U_0$ is $(p - 1)$ and $M_i$ is any relatively compact subdomain of an open Riemann surface $M$, then there exists a neighbourhood $U$ of $t_0$ and a holomorphic map $h : \mathcal{F}(M_i \times U)$ into $M$ which maps each fibre biholomorphically into $M$.

Now, assumig $A_{p-1}$ we prove $A_p$.

Let $M_0$ be a relatively compact subdomain of $M$ such that $M_1 \subset M_0$. Let $W$ be a sufficiently small Stein neighbourhood of $t_0$ in $\mathcal{C}$, with $W$ compact, $W \subset U_0$. Then $\mathcal{F}(M_0 \times W)$ is a Stein manifold. If $\mathcal{F}$ denotes the holomorphic tangent bundle along the fibres, then by Theorem B on Stein manifolds $H'(\mathcal{F}(M_0 \times W), \mathcal{F}) = 0$. From the exact sequence

$$
H^0(\mathcal{F}(M_0 \times W), \Pi) \to H^0(W, T) \to H^1(\mathcal{F}(M_0 \times W), \mathcal{F})
$$

($\Pi$ denotes the sheaf of germs of holomorphic vector fields which are projectable, $T$ denotes the sheaf of germs of holomorphic vector fields on $W$), we see that the vector field $\frac{\partial}{\partial t}$ can be lifted into a holomorphic vector field $X$ of $\mathcal{F}(M_0 \times W)$.

We may suppose $t_0 = 0$. Let $f(x) = \exp(-\pi^p(x) X)$, $x \in M_i \times W'$ where $\pi^p(x)$ is defined as follows: if $x = (z, v', \ldots, v^p)$, $\pi^p(x) = v^p$. By the complex analytic analogue of Proposition 5.1 in [2], for a neighbourhood $W' \subset W$, $f$ maps $\mathcal{F}(M_i \times W')$ holomorphically into $\mathcal{F}(M_0 \times (v', \ldots, v^{p-1}, 0))$ mapping the fibre at $(v', \ldots, v^p)$ biholomorphically into $\mathcal{F}((v', \ldots, v^{p-1}, 0), M_0)$. $M_0$ being a relatively compact subdomain of $M$, there exists, by the inductive hypothesis, a holomorphic map $g$:

$$
M_0 \times (v', \ldots, v^{p-1}, 0) \to M
$$

which maps each fibre biholomorphically into $M$. Taking the
composite \( h = g \circ f \) we get a holomorphic map of \( \mathcal{J}(U_\ast \times M_\ast) \)
where \( U_\ast \) is a neighbourhood of \( t_\ast \) in \( \mathbb{C}^p \), mapping each fibre
biholomorphically into \( M \). This proves \( A_p \).

Once we have proved the assertion \( A_m \), consider the map
\[ \Phi: \mathcal{J}(U \times M_\ast) \to U \times M \]
defined by \((t, z) \mapsto (t, h(t, z))\). \( \Phi \) is holomorphic and one to
one. By a known theorem on holomorphic functions \( \Phi \) maps
\( \mathcal{J}(U \times M_\ast) \) biholomorphically onto an open subset \( \Omega \) of
\( U \times M \) and we have the commutative diagram
\[
\begin{array}{ccc}
S(U \times M_\ast) & \xrightarrow{\Phi} & \Omega \\
\downarrow{\pi_1} & & \downarrow{\pi} \\
U & \xrightarrow{\text{identity}} & U
\end{array}
\]
This completes the proof of Theorem 2.


Proof of Theorem 3.

Let \( \mathcal{J}(t, M) \) be a differentiable variation of complex struc-
tures on an open Riemann surface \( M, \ t \in U_0 \subset \mathbb{R}^n \). Let \( J_t \)
be the almost complex structure tensor corresponding to the
structure \( \mathcal{J}(t, M) \). On \( M \times U_0 \), let \( J \) denote the tensor along
the fibres composed of \( \{ J_t \} \). If \( X \) is a \textit{projectable} vector field
on \( M \times U_0 \) (with respect to the projection \( M \times U_0 \to U_0 \)) we
remark that the Lie derivative of \( J \) with respect to the vec-
tor field \( X \), denoted by \( [X, J] \), is defined as a tensor along
the fibres.

Let \( X \) be a projectable vector field on \( M \times U_0 \) satisfying
the condition \([X, J] = 0\). Let \( M' \) (resp. \( U'_0 \)) be a relatively
compact subdomain of \( M \) (resp. \( U'_0 \)). If \( \exp (sX) \) denotes the
one parameter family of transformations associated with \( X \),
\( \exp (sX) \) is a diffeomorphism of \( M' \times U'_0 \) into \( M \times U_0 \) which
maps \( \mathcal{J}(t, M') \), \( t \in U'_0 \) biholomorphically into \( \mathcal{J} \circ \exp (sv)(t), M \),
where \( \nu \) denotes the projection of \( X \) on \( U_0 \). Now referring
to the proof of Theorem 2, we see that to prove Theorem 3
it is sufficient to prove
Proposition 4. — Let $\mathcal{S}(t)$ be a differentiable family of complex structures on an open Riemann surface $M$. Let $M_1$ be a relatively compact subdomain of $M$. Then there exists a neighbourhood $U_1$ of $t_0$ in $\mathbb{R}^n$ such that every differentiable vector field (real) on $U_1$ can be lifted into a differentiable vector field $X$ on $M_1 \times U_1$ satisfying the condition $[X, J] = 0$, $J$ denoting the tensor along the fibres composed on the almost complex structure tensors along the fibres.

Proof of Proposition 4. — Let $M_0$ be a relatively compact subdomain (of $M$) with analytic boundary, with $M_1 \subset M_0$.

Let $\Theta_t$ denote the holomorphic tangent bundle of $\mathcal{S}(t, M)$. Let $\mathcal{F} = U_0\Theta_t$ be the bundle on $M \times U_0$ composed of the holomorphic tangent bundles along the fibres. If $U_2$ is a spherical neighbourhood of $t_0$ with $U_2 \subset U_0$ then $\mathcal{F}|M \times U_2$ is differentiably equivalent to the bundle $U_2 \times \Theta_{t_0}$ (Homotopy theorem). It follows that there exist isomorphisms

$$\psi_1(t) : H_{1, \alpha}(M_0, \Theta_t) \to H_{1, \alpha}(M_0, \Theta_{t_0}),$$
$$\psi_2(t) : H_{\alpha}(M_0, \Theta_t) \to H_{\alpha}(M_0, \Theta_{t_0}),$$

depending differentiably on $t$ such that $\psi_1(t_0) = \text{identity}$, $\psi_2(t_0) = \text{identity}$. Let

$$T_t = \psi_2(t) d\zeta(t) \psi_1(t)^{-1} : H_{1, \alpha}(M_0, \Theta_t) \to H_{\alpha}(M_0, \Theta_{t_0})$$

where $d\zeta(t)$ denotes the $d\zeta$ operator with respect to the structure $\mathcal{S}(t)$. $T_t$ depends differentiably on $t$. Since $T_{t_0} = d\zeta(t_0)$ admits of a right inverse by Lemma 5, there exist a neighbourhood $U_3$ of $t_0$ and operators

$$S_t : H_{\alpha}(M_0, \Theta_{t_0}) \to H_{1, \alpha}(M_0, \Theta_{t_0}), \quad t \in U_3$$

depending differentiably on $t \in U_3$ and such that $S_t$ is a right inverse of $T_t$ (Lemma 2).

Let $M_2$ be a relatively compact subdomain of $\overline{M}$ with $M_0 \subset M_2$.

Let $U_4$ be a neighbourhood of $t_0$ such that there exist a finite open covering $O_1$, $\ldots$, $O_k$ of $M_2$ and diffeomorphisms $g_i$ of $O_i \times U_4$ into $\mathbb{C} \times U_4$ which maps $\mathcal{S}(t_0)$ into $\mathbb{C} \times O_i$ biholomorphically into $\mathbb{C} \times (t)$. [Such a neighbourhood $U_4$ exists. This follows from the definition of differentiable variation of complex structures in the sense of Kodaira-Spencer. With our
definition this follows from the differentiable analogue of Proposition 1]. We denote the coordinate function in \( O_i \times U_4 \) by \((z^i, t)\).

Let \( U_1 \) be a relatively compact neighbourhood of \( t_0 \) in \( \mathbb{R}^n \) such that \( U_1 \subset U_2 \cap U_3 \cap U_4 \). Let \( \nu = (\nu_1(t), \ldots, \nu_m(t)) \) be a differentiable vector field in \( U_1 \). In \( O_i \times U_4 \) consider the vector field \( \pi_i \) defined by \((O, \nu_1(t), \ldots, \nu_m(t))\) with respect to the coordinate system \((z^i, t)\). We have \([\pi_i, J] = 0\). Put \( \theta_{ij} = \pi_i - \pi_j \) in \((O_i \times U_4) \cap (O_j \times U_4)\). Let \( \theta_{ij} = \theta_{ij} - iJ\theta_{ij}. \)

Then \( \theta_{ij} \) are sections of \( \mathcal{F} \) over \((O_i \times U_4) \cap (O_j \times U_4)\) whose restriction to each fibre is holomorphic. Evidently there exist differentiable sections \( f_i(z^i, t) \) of \( \mathcal{F} \) over \( O_i \times U_4 \) such that \( f_i' - f_j' = \theta_{ij}, \) in \((O_i \times U_4) \cap (O_j \times U_4)\). If we define \( \varphi(t) = d_z f_i(z^i, t), \) \( \varphi(t) \) is a \((0, 1)\) form on \( \mathcal{F}(t, M_0) \) with values in \( \Theta_t \) which depends differentiably on \( t \). Let \( \gamma(t) = \psi_z(t) \varphi(t) |_{M_0} \)

\([\psi_z(t) \varphi(t)] \) is the isomorphism defined earlier. Then \( \gamma(t) \) is a differentiable function with values in \( H^2(M_0, \Theta_t) \). For \( t \in U_1 \), let \( h_1(t) = S_1 \gamma(t) \). Then \( h_1(t) \) depends differentiably on \( t \). Let \( h_2(t) = \{\psi_z(t)\}^{-1}(h_1(t)) \). Then \( h_2(t) \) depends differentiably on \( t \) and satisfies \( d_z h_2(t) = \varphi(t) \). [It follows easily from differentiability theorem for elliptic differential equations that \( h_2(z, t) \) is a differentiable vector field on \( M_1 \times U_1 \). See proposition 1 in [3]]. Let

\[
h(z, t) = \frac{1}{2} \{ h_2(z, t) + \bar{h}_2(z, t) \}
\]

and

\[
f_i(z, t) = \frac{1}{2} \{ f'_i(z, t) + \bar{f}'_i(z, t) \}.
\]

Define

\( X = \pi_i + h - f_i \) in \((O_i \cap M_1) \times U_1\).

Then \( X \) is globally defined on \( M_1 \times U_1 \), projects into \( \nu \) and satisfies the equation \([X, J] = 0\). This completes the proof of Proposition 4.

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Tata Institute of Fundamental Research, Bombay
and
Centre National de la Recherche Scientifique, Paris.