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THEORY OF BESSEL POTENTIALS. PART I. (1, 2)

by **N. ARONSZAJN** and **K. T. SMITH**

INTRODUCTION

The present paper is the second in a series, the purpose of which is to give a basis for a treatment of differential eigenvalue and boundary value problems.

In the first paper [1], a general theory of functional spaces and functional completion was developed. Now, this general theory is applied to special spaces which are most important for the study of differential problems, especially of elliptic type.

Many results of this paper were announced several years ago and the paper was then referred to as « Theory of Potentials ». It was decided that the original title was misleading since we treat only potentials corresponding to special types of kernels and not those corresponding to more or less arbitrary kernels as has been done for instance in [7 *a*], [13 *a*], and [13 *c*].

Originally, the authors used the Riesz potentials of order α, i.e. potentials corresponding to kernels

$$(1) \qquad R_\alpha(x) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{n/2} \Gamma(\alpha/2)} |x|^{\alpha-n}, \qquad 0 < \alpha < n,$$

in n-dimensional Euclidean space R^n.

Despite the fact that many elegant and important results

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(2) Part II to appear in the next volume of this journal.

were obtained for these potentials by Riesz, Frostman, Cartan, and others, their application to differential problems was sometimes awkward. The reason for this was the limitation on the order, $\alpha < n$, whereas for differential problems we need potentials of arbitrarily high order. We were thus led to consider potentials based on the kernels

$$(2) \qquad G_\alpha(x) = \frac{1}{2^{\frac{n+\alpha-2}{2}} \pi^{n/2} \Gamma\left(\frac{\alpha}{2}\right)} K_{\frac{n-\alpha}{2}}(|x|)|x|^{\frac{\alpha-n}{2}}$$

where $K_{\frac{n-\alpha}{2}}$ is the modified Bessel function of third kind. It therefore seems appropriate to call the corresponding potentials « Bessel potentials of order α ».

The kernels (2) which are defined for all $\alpha > 0$ have the same basic properties as Riesz kernels, i.e. positiveness, composition theorem, etc., and in addition they converge to zero exponentially at infinity. This makes for much greater ease in the development of the theory. For $\alpha < n$, R_α represents the principal part of G_α at the origin with the result that the corresponding Riesz and Bessel potentials form the same classes of functions in every bounded portion of the space. The classes of potentials P^α which form the main object of research in this paper are exactly the Bessel potentials of order α of L^2 functions. The potentials of L^p functions would be of interest in themselves (in the case of Riesz potentials they were considered by B. Fuglede [11 *a*]), but they do not enter into the framework of our applications to differential problems which are based on Hilbert space or « quadratic » methods [3].

The first part of the paper (Chapters I and II presented here) gives the theory of potentials of order α in the whole space R^n. The second part will deal with these classes in subdomains of R^n and also on differentiable and Riemannian manifolds.

The contents and main results of Chapters I and II can be summarized as follows.

[3] This means they are based on the use of quadratic norms in functional spaces, or more generally, vector spaces; by some authors they are referred to as « L^2 methods ».

In Chapter I we recall the main results of the theory of functional spaces and functional completion [1] and add a few results not given before.

In the first section of Chapter II we consider functions $u \in C_0^\infty$ and define the Dirichlet integral of order α, $d_\alpha(u)$ for arbitrary $\alpha \geqq 0$. This is done at first by using Fourier transforms (as in [8] and [1]), after which a direct form for $d_\alpha(u)$ is given in terms of derivatives of u of orders $\leqq \alpha$. In [1] we showed that for $\alpha < \frac{n}{2}$, C_0^∞, with norm $\sqrt{d_\alpha(u)}$, has a perfect functional completion which coincides with the Riesz potentials of order α of L^2 functions. We show now that for $\alpha \geqq \frac{n}{2}$, C_0^∞ with this norm has no functional completion. We then consider the norm $|u|_\alpha^2 = ||u||_{L^2}^2 + d_\alpha(u)$, and the norm $||u||_\alpha$ (equivalent to $|u|_\alpha$) which is first expressed by Fourier transforms and then directly in terms of derivatives of u.

In Section 2 we show that C_0^∞, with the norm $|u|_\alpha$, has a functional completion for all α relative to the class $\mathfrak{A}_0$ of exceptional sets of Lebesgue measure 0. We study the basic properties of all (imperfect) functional completions of this space relative to a class of exceptional sets contained in $\mathfrak{A}_0$. This space has a perfect functional completion (shown in § 5 to be P^α).

In order to study the properties of the perfect completion P^α, it was found convenient to replace $|u|_\alpha$ in C_0^∞ by the equivalent norm $||u||_\alpha$, and this last norm is maintained in the remainder of the chapter.

In Sections 3 and 4 the basic properties of the Bessel functions K_ν are collected, and the resulting properties of the kernel G_α are given.

In Section 5, as was mentioned above, we prove that P^α is the perfect functional completion of C_0^∞ with norm $||u||_\alpha$, and the basic properties of P^α and of its class of exceptional sets $\mathfrak{A}_{2\alpha}$ (4) are given in so far as they are obtainable from the general theory of functional completion. For $\alpha > n/2$, P^α is a proper functional Hilbert space (its reproducing kernel is $G_{2\alpha}(x-y)$).

In Section 6 we define and investigate the capacities of

(4) This is in accordance with established notation: for $\alpha < n/2$, the sets in $\mathfrak{A}_{2\alpha}$ are the sets of outer capacity 0 of order 2α in the sense of Frostman.

order 2α in a manner similar to that used by Frostman [11] and Cartan [6] in their study of Riesz potentials. The outer capacity $\gamma_{2\alpha}$, of order 2α, thus defined, coincides with the capacity $c_2 = c_1^2$ as defined for P^α by the general theory of functional completion. It is worthwhile noticing that the logarithmic capacity γ_n, which for Riesz potentials requires special treatment with definitions and proofs somewhat changed, does not present any exceptional character for our potentials.

Sections 7 and 8 contain the most important results from the point of view of applications to differential problems. In Section 7, Theorem I gives differentiability and continuity properties of functions $u \in P^\alpha$ which allow $||u||_\alpha^2$ and $d_\alpha(u)$ to be defined by the same direct formulas which were used in Section 1 for functions in C_0^∞. In conjunction with Theorem I, Theorem II gives necessary and sufficient conditions for a function u to belong to P^α. Remarks 2 and 3 which follow Theorem II weaken these conditions quite considerably.

The theorems in Section 8 concern the restriction of a function $u \in P^\alpha$ in R^n to a subspace $R^k \subset R^n$. Theorems 1 *a* and 1 *b* show, essentially, that for a function u' defined on R^k to be a restriction of a function $u \in P^\alpha(R^n)$, it is necessary and sufficient that $u' \in P^{\alpha - \frac{n-k}{2}}(R^k)$. Theorem 1 *c* gives a basis for what we call the compensation method which is very useful in the study of elliptic differential problems. In Chapter IV (which is to appear shortly in Part II of this paper) these theorems, are extended to restrictions to submanifolds of R^n.

Section 9 treats functions u defined in an open set $D \subset R^n$ which are locally in P^α; the class of these functions is denoted by $P^\alpha_{\text{loc}}(D)$. These classes form a first step to the introduction of the classes P^α on a Riemannian manifold.

In Section 10, we study the relations between L^q and P^α classes. We depart from our general restriction and consider Bessel potentials of L^p functions (the proofs do not differ from those in the case $p = 2$). We obtain the following theorem: if $q \geq p \geq 1$ and one of the two conditions holds:

1° $\dfrac{1}{q} \geq \dfrac{1}{p} - \dfrac{\alpha}{n}$ with $p > 1$ and $\dfrac{1}{p} - \dfrac{\alpha}{n} \neq 0$;

2° $\dfrac{1}{q} > \dfrac{1}{p} - \dfrac{\alpha}{n}$ with $p = 1$ or $\dfrac{1}{p} - \dfrac{\alpha}{n} = 0$, then $f \in L^p$ implies $G_\alpha f \in L^q$. For Riesz potentials, by Soboleff's theorem we can consider only the case when condition 1 is satis fied and then we have $R_\alpha f \in L^q$ in general only for $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

In Section 11 — the last section — we compare our classes P^α with the corresponding classes of Riesz potentials, B-L classes, W^m and H^m classes. These classes, introduced by different authors, have similarities either in definition or purpose to our P^α classes.

Before finishing the Introduction we mention the recent papers of L. Slobodetzky [14 *a*] [14 *b*] where expressions similar to our direct formulars for $d_\alpha(u)$ are introduced and applied.

We should also mention that many of the results of the present paper and of its second part were referred to and applied in several papers by the authors, in particular in [0], [15 *a*], and [2]. In these references, however, we were considering the corresponding results for Riesz potentials.

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CHAPTER I

SUMMARY OF THE GENERAL THEORY OF FUNCTIONAL COMPLETION

This Chapter contains a short summary of the definitions and results from the general theory of functional completion which are needed in the rest of the paper. It is taken from [1] but it also contains a few minor observations which are not included in [1]. For simplicity, only complex spaces are considered. The changes which must be made in the real case are quite trivial.

§ 1. — Functional spaces and functional completion.

An exceptional class on a set $\mathcal{E}$ is a hereditary and σ-additive class of subsets of $\mathcal{E}$, that is, a class $\mathfrak{A}$ of subsets of $\mathcal{E}$ with the two properties: (*a*) If $A \subset B$ and $B \in \mathfrak{A}$, then $A \in \mathfrak{A}$; (*b*) if each member of a sequence of sets belongs to $\mathfrak{A}$, then the union belongs to $\mathfrak{A}$. Henceforth $\mathfrak{A}$ denotes an exceptional class on a set $\mathcal{E}$.

A property of points of $\mathcal{E}$ is said to hold except $\mathfrak{A}$ (to be written exc. $\mathfrak{A}$) if the set where it fails to hold belongs to $\mathfrak{A}$. If u and v are complex valued functions defined on $\mathcal{E} - A$ and $\mathcal{E} - B$, respectively, and α is a complex number, then $u + v$ denotes the function defined on $\mathcal{E} - (A \cup B)$ by pointwise addition and αu denotes the function defined on $\mathcal{E} - A$ by pointwise multiplication. It is obvious that if u and v are defined exc. $\mathfrak{A}$, then $u + v$ and αu are defined exc. $\mathfrak{A}$.

A linear functional class relative to $\mathfrak{A}$ (rel. $\mathfrak{A}$) is a class $\mathcal{F}$ of complex valued functions defined on $\mathcal{E}$ exc. $\mathfrak{A}$ such that if u and v belong to $\mathcal{F}$ and α is a complex number, then $u + v$ and αu belong to $\mathcal{F}$. $\mathfrak{A}$ is the exceptional class for $\mathcal{F}$. The saturated extension of $\mathcal{F}$ is the class of all functions defined on $\mathcal{E}$ exc. $\mathfrak{A}$ which are equal exc. $\mathfrak{A}$ to some function in $\mathcal{F}$, and $\mathcal{F}$ is saturated if it is identical with its saturated extension.

A normed functional class $\mathcal{F}$ rel $\mathfrak{A}$ is a linear functional class $\mathcal{F}$ rel $\mathfrak{A}$ on which there is defined a norm $\|u\| \geqq 0$ with the properties: 1° $\|u\| = 0$ if and only if $u(x) = 0$ exc. $\mathfrak{A}$; 2° $\|\alpha u\| = |\alpha| \|u\|$ for any complex α; 3° $\|u\| \leqq \|u - v\| + \|v\|$ (5). From 3° and 1° it follows: 4° if $u(x) = v(x)$ exc. $\mathfrak{A}$ then $\|u\| = \|v\|$. The saturated extension of a normed functional class rel $\mathfrak{A}$ is also a normed functional class rel. $\mathfrak{A}$ (when the norm is extended in the obvious way).

If in a functional class $\mathcal{F}$ we introduce the equivalence relation $f \sim f' \Longleftrightarrow f(x) = f'(x)$ exc. $\mathfrak{A}$, the set of equivalence classes obviously forms a vector space V. If $\mathcal{F}$ is normed, V becomes a normed space (since $f \sim f'$ implies $\|f\| = \|f'\|$). In this case we transfer without further explanation all the notions usual in a normed vector space to the class $\mathcal{F}$. For instance: $f_n \to f$ (f_n converges to f in norm); $\{f_n\}$ is a Cauchy sequence; a subset of $\mathcal{F}$ is dense in $\mathcal{F}$; $\mathcal{F}$ is complete or separable, etc.

A functional space rel $\mathfrak{A}$ is a normed functional class rel $\mathfrak{A}$ in which there is the following relation between the norm and the values of the functions:

1. 1. The functional space property. — *Every sequence which converges (in norm) to 0 contains a subsequence which converges to 0 pointwise exc.* $\mathfrak{A}$.

The saturated extension of a functional space rel. $\mathfrak{A}$ is also a functional space rel. $\mathfrak{A}$.

A functional completion of a normed functional class $\mathcal{F}$ rel. $\mathfrak{A}$ is a functional space $\tilde{\mathcal{F}}$ rel. $\tilde{\mathfrak{A}}$ such that:

(*a*) $\tilde{\mathfrak{A}} \supset \mathfrak{A}$.

(*b*) Each function $u \in \mathcal{F}$ belongs to $\tilde{\mathcal{F}}$ and has the same norm in both classes.

(*c*) $\mathcal{F}$ is dense (in norm) in $\tilde{\mathcal{F}}$.

(*d*) $\tilde{\mathcal{F}}$ is complete.

(5) **The usual form of Minkowski's inequality, $\|u + v\| \leqq \|u\| + \|v\|$, is not adequate here since in general F is not a vector space: $(u + v) - v$ is not identical with u (u may have a smaller exceptional set than $(u + v) - v$).**

We say that $\tilde{\mathfrak{F}}$ is a functional completion of $\mathfrak{F}$ rel. $\tilde{\mathfrak{A}}$. The saturated extension of a functional completion of a normed functional class is also a functional completion of the given normed functional class. Since it is technically convenient to work with saturated completions, and it involves no loss in generality ,it will be assumed that all functional completions are saturated.

A functional completion is perfect if its exceptional class is contained in the exceptional class of every functional completion.

The main problems in the theory of functional spaces and functional completion are: (i) to determine when a normed functional class is a functional space; (ii) to determine when a normed functional class has a functional completion; (iii) to determine when a normed functional class has a perfect functional completion [6]; (iv) to describe the exceptional class for the perfect completion.

It is easy to see that if a normed functional class has a functional completion relative to one exceptional class, then usually it also has a functional completion relative to infinitely many others. In this connection, however, the following result holds, and is easily proved.

1) *Relative to a given exceptional class there is at most one (saturated) functional space which is a functional completion of a given normed functional class. In particular, the perfect completion, when there is one, is uniquely determined.*

It is clear that the properties of a normed functional class which have been defined so far remain the same if the norm on the class is replaced by an equivalent norm. In particular, if there exists a functional completion rel. $\tilde{\mathfrak{A}}$ with respect to one of two equivalent norms, then there exists a functional completion rel. $\tilde{\mathfrak{A}}$ with respect to the other, and the two completions are composed of the same functions. A converse of this also holds.

2) *If a linear functional class* $\mathfrak{F}$ rel. $\mathfrak{A}$ *is a complete functional space with respect to two norms, then the two norms are equivalent.* More generally.

(6) It is not known whether the existence of some functional completion implies the existence of a perfect functional completion.

3) *If $\mathfrak{F}$ is a complete functional space rel. $\mathfrak{A}$, and if $\mathfrak{F}' \subset \mathfrak{F}$ is a complete functional space rel. $\mathfrak{A}$ with respect to a norm, $||u||'$, then there is a constant c such that for all $u \in \mathfrak{F}'$, $||u|| \leqq c||u||'$.*

PROFF. — The identity mapping from $\mathfrak{F}'$ into $\mathfrak{F}$ is a closed mapping. In fact, if $u_n \to u$ in $\mathfrak{F}'$ and $u_n \to v$ in $\mathfrak{F}$ then by the functional space property some subsequence $\{u_{n_k}\}$ converges pointwise exc. $\mathfrak{A}$ to both u and v. Hence $u = v$ exc. $\mathfrak{A}$. By using the closed graph theorem we obtain the statement.

§ 2. — The set functions δ and $\tilde{\delta}$ and capacities.

In this section we describe certain functions and classes of sets which lead toward solutions, partial or complete, to the problems listed in section 1. The classes provide explicit bounds for the exceptional class of a perfect completion; in every example where a perfect completion has been found, its exceptional class coincides with the bounds given. Throughout the section, $\mathfrak{A}$ is a fixed exceptional class and $\mathfrak{F}$ is a fixed normed functional class rel. $\mathfrak{A}$; $\tilde{\mathfrak{A}}$ is an exceptional class containing $\mathfrak{A}$.

The class $\mathfrak{L}$ is the class of all sets $B \subset \mathcal{E}$ for which there is a function $u \in \mathfrak{F}$ satisfying $|u(x)| \geqq 1$ on B exc. $\mathfrak{A}$. For each $B \in \mathfrak{L}$, $\delta(B)$ is the infimum of $||u||$ over all such u.

The class $\tilde{\mathfrak{L}}$ is the class of all sets $B \subset \mathcal{E}$ for which there is a Cauchy sequence $\{u_n\}$ in $\mathfrak{F}$ satisfying $\liminf |u_n(x)| \geqq 1$ on B exc. $\mathfrak{A}$. For each $B \in \tilde{\mathfrak{L}}$, $\tilde{\delta}(B)$ is the infimum of $\lim ||u_n||$ over all such Cauchy sequences.

REMARK. — If $\mathfrak{F}$ is a complete functional space, then clearly $\tilde{\delta} = \delta$.

If $\mathfrak{L}^0$ and $\tilde{\mathfrak{L}}^0$ are the classes of null sets of δ and $\tilde{\delta}$, respectively, then obviously $\mathfrak{A} \subset \mathfrak{L}^0 \subset \tilde{\mathfrak{L}}^0$. Conversely, there is the following result.

1) *If $\mathfrak{F}$ is a functional space rel. $\tilde{\mathfrak{A}}$, then $\tilde{\mathfrak{A}} \supset \mathfrak{L}^0_\sigma$. If $\mathfrak{F}$ has a functional completion rel. $\tilde{\mathfrak{A}}$, then $\tilde{\mathfrak{A}} \supset \tilde{\mathfrak{L}}^0_\sigma$* [7].

Upper bounds for the exceptional class of a perfect completion are provided by additional set functions called capacities.

[7] $\mathfrak{L}^0_\sigma$ is the class of countable unions of sets in $\mathfrak{L}^0$.

An admissible capacity for a normed functional class $\mathfrak{F}$ is a set function c on the hereditary σ-ring $\mathfrak{L}_\sigma$ with the following properties.

(*a*) c is an outer measure on $\mathfrak{L}_\sigma$ [8].

(*b*) For each $B \in \mathfrak{L}$, $c(B)$ is finite.

(*c*) To each $\varepsilon > 0$ corresponds an $\eta < 0$ such that if $\delta(B) \leqq \eta$, then $c(B) \leqq \varepsilon$.

Each real valued, non-decreasing function $\varphi(t)$, defined for $t \geqq 0$ and satisfying $\varphi(0) = \lim_{t \to 0} \varphi(t) = 0$ and $\varphi(t) > 0$ for $t > 0$, determines an admissible capacity c_φ as follows: For each $B \in \mathfrak{L}_\sigma$

$$c_\varphi(B) = \inf \sum_{n=1}^{\infty} \varphi[\delta(B_n)]$$

where the infimum is taken over all sequences $\{B_n\}$ in $\mathfrak{L}$ such that $B \subset \bigcup_{n=1}^{\infty} B_n$. The most important of the admissible capacities are the capacities c_α determined by the functions

$$\varphi(t) = t^\alpha,\ \alpha > 0.$$

We use especially c_1 and c_2.

In the following propositions c is an admissible capacity for $\mathfrak{F}$ and $\mathfrak{A}_c$ is its class of null sets.

2) *Every Cauchy sequence in $\mathfrak{F}$ contains a subsequence which, for each $\varepsilon > 0$, converges uniformly outside some set of capacity $< \varepsilon$.*

3) $\tilde{\mathfrak{L}}_\sigma^0 \subset \mathfrak{A}_c$.

4) *$\mathfrak{F}$ is a functional space rel. $\mathfrak{A}_c$ if and only if $\|u\| = 0$ whenever $u(x) = 0$ exc. $\mathfrak{A}_c$. $\mathfrak{F}$ has a functional completion rel. $\mathfrak{A}_c$ if and only if $\|u_n\| \to 0$ whenever $\{u_n\}$ is a Cauchy sequence which converges pointwise to 0 exc. $\mathfrak{A}_c$.*

5) *If $\mathfrak{F}$ is a functional space rel. $\tilde{\mathfrak{A}}$ or if $\mathfrak{F}$ has a functional completion rel. $\tilde{\mathfrak{A}}$, then the same is true rel. $\tilde{\mathfrak{A}} \cap \mathfrak{A}_c$.*

6) *If c_φ and $\tilde{c}_\varphi$ are the φ capacities formed for $\mathfrak{F}$ and $\tilde{\mathfrak{F}}$ where $\tilde{\mathfrak{F}}$ is a functional completion of $\mathfrak{F}$ rel. $\tilde{\mathfrak{A}} \subset \mathfrak{A}_{c_\varphi}$, then $\tilde{c}_\varphi = c_\varphi$.*

Propositions 1) and 5) give.

[8] Measurability with respect to c plays no role in this theory.

7) *If $\mathfrak{F}$ has a perfect functional completion, then its exceptional class $\tilde{\mathfrak{A}}$ satisfies $\tilde{\mathfrak{L}}^0_\sigma \subset \tilde{\mathfrak{A}} \subset \mathfrak{A}_c$.*

In every example where a perfect functional completion has been found, it has turned out that in fact $\tilde{\mathfrak{L}}^0_\sigma = \mathfrak{A}_{c_1}$. Conversely, if $\tilde{\mathfrak{L}}^0_\sigma = \mathfrak{A}_{c_1}$ holds, then if there is any functional completion, there is a perfect functional completion, and its exceptional class is $\tilde{\mathfrak{L}}^0_\sigma = \mathfrak{A}_{c_1}$.

§ 3. Majoration properties.

The object of the section is to describe three majoration properties and a few of the results that can be derived for normed functional classes that possess them. All of the functional spaces which are commonly used in differential problems do possess at least the weakest of the three. The majoration properties are as follows.

POSITIVE MAJORATION PROPERTY. — *The set $\mathcal{E}$ can be written as $\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$ and constants M_n can be chosen in such a way that for every function $u \in \mathfrak{F}$ and every n there exists a function $u'_n \in \mathfrak{F}$ satisfying*

$$||u'_n|| \leqq M_n ||u|| \text{ and } \operatorname{Re} u'_n(x) \geqq |u(x)| \text{ for } x \in \mathcal{E}_n \text{ exc. } \mathfrak{A}.$$

GLOBAL MAJORATION PROPERTY. — *There is a constant M such that for every function $u \in \mathfrak{F}$ there exists a function $u' \in \mathfrak{F}$ satisfying*

$$||u'|| \leqq M\,||u|| \qquad \text{and} \qquad \operatorname{Re} u'(x) \geqq |u(x)| \text{ exc. } \mathfrak{A}.$$

STRONG MAJORATION PROPERTY. — *For every function $u \in \mathfrak{F}$ there exists a function $u' \in \mathfrak{F}$ satisfying*

$$||u'|| \leqq ||u|| \qquad \text{and} \qquad \operatorname{Re} u'(x) \geqq |u(x)| \text{ exc. } \mathfrak{A}.$$

In so far as the general theory of functional completion is concerned, the main interest in the majoration properties lies in the next proposition.

1) *Let $\mathfrak{F}$ have the positive majoration property. Then $\tilde{\mathfrak{L}}^0_\sigma = \mathfrak{A}_{C_1}$, and the following statements are equivalent.*

(*a*) $\mathcal{F}$ *has a functional completion.*

(*b*) $\mathcal{F}$ *has a perfect functional completion, and the exceptional class for the perfect completion is* $\tilde{\mathfrak{L}}^0_\sigma = \mathfrak{A}_{C_1}$.

(*c*) $||u_n|| \to 0$ *whenever* $\{u_n\}$ *is a Cauchy sequence which converges pointwise to* 0 *exc.* $\mathfrak{A}_{C_1}$.

It is not difficult to see that if $\mathcal{F}$ has the global majoration property, then $B \in \tilde{\mathfrak{L}}$ if and only if $c_1(B) < \infty$, and if $B \in \tilde{\mathfrak{L}}$ then $\tilde{\delta}(B) \geqq c_1(B) \geqq \frac{1}{M}\tilde{\delta}(B)$.

2) *If* $\mathcal{F}$ *has the strong majoration property, then* $\tilde{\delta} = c_1$; *if* $\mathcal{F}$ *is also complete, then* $\delta = \tilde{\delta} = c_1$.

3) *If* $\mathcal{F}$ *has the strong majoration property and is reflexive, then the infimum in the definition of* δ *is attained. Moreover, if* B *is the union of an increasing sequence* $\{B_n\}$, *then*

$$c_1(B) = \lim c_1(B_n).$$

Proof. — Let Γ_B denote the closed convex set of all $u \in \mathcal{F}$ satisfying $\operatorname{Re} u(x) \geqq 1$ on B exc. $\mathfrak{A}$. From the strong majoration property it follows immediately that $\delta(B)$ is the distance from the origin to Γ_B, and in a reflexibe space this distance is attained.

Since a reflexive space is complete, it follows from 2) that $\delta = c_1$. Therefore, in order to prove the second part of the proposition, it is sufficient to show that if $\lim \delta(B_n) < \infty$ then $\delta(B) \leqq \lim \delta(B_n)$. For each n, let $u_n \in \Gamma_{B_n}$ be such that $||u_n|| = \delta(B_n)$. Then, since $\lim ||u_n|| < \infty$, there is a subsequence $\{u_{n_k}\}$ which converges weakly to some $u \in \mathcal{F}$. For every $i \geqq k$, $u_{n_i} \in \Gamma_{B_{n_k}}$. Therefore, since $\Gamma_{B_{n_k}}$ is closed and convex, $u \in \Gamma_{B_{n_k}}$, and, since this holds for every k, $u \in \Gamma_B = \bigcap_{k=1}^{\infty} \Gamma_{B_{n_k}}$. Hence

$$\delta(B) \leqq ||u|| \leqq \lim ||u_{n_k}|| = \lim \delta(B_n).$$

The main interest in the strong majoration property, however, comes from its application in another connection, namely, in the theory of pseudo-reproducing kernels and in the theory of balayage and classical type capacities. It has been shown in [2] that if $\mathcal{F}$ is a real functional Hilbert space with a pseudo-reproducing kernel, then the kernel is non-negative if and only if $\mathcal{F}$ has the strong majoration property. It will not be necessary to make use of this result in the later chapters, since the kernels are given by explicit formulas from which their properties can be derived. Nevertheless, the result underlies many of the developments. In particular, it is the need for the strong majoration property and positive pseudo-reproducing kernels which is responsible for the choice of the norm $||u||_\alpha$ in Chapter II.

In many questions a change from one norm, $||u||$, to an equivalent one, $||u||'$, is immaterial. The classes $\mathfrak{L}$, $\tilde{\mathfrak{L}}$, $\mathfrak{L}^0$, and $\tilde{\mathfrak{L}}^0$ are unchanged; δ and $\tilde{\delta}$ are replaced by δ' and $\tilde{\delta}'$, c_1 and c_2 by c_1' and c_2', where

$$\frac{\delta'}{\delta}, \quad \frac{\tilde{\delta}'}{\tilde{\delta}}, \quad \frac{c_1'}{c_1}, \quad \frac{c_2'}{c_2},$$

all lie between two positive constants. Admissible capacities remain admissible capacities. The validity of the positive and global majoration properties is unchanged. However, the validity of the strong majoration property is dependent on the particular norm used, and the need for this property can impose a particular norm, even one which is more complicated than some equivalent norms.

§ 4. — Propre functional spaces.

A proper normed functional class is a normed functional class rel. $\{0\}$; a proper functional space is a functional space rel. $\{0\}$.

The complete functional spaces occuring in analysis arise most often as functional completions of proper normed functional classes. This does not mean, however, that the complete spaces are proper functional spaces, for in the process of completion it usually happens that some sets become exceptional. This cannot happen if the original proper normed functional class is a proper functional space.

1) *A proper normed functional class $\mathcal{F}$ is a proper functional space if and only if for each $x \in \mathcal{E}$ there is a constant M_x such that for every function $u \in \mathcal{F}$,* $|u(x)| \leq \mathrm{M}_x\, ||u||$.

It can be shown that if $\mathcal{F}$ is a proper functional space, then $c_1(\{x\}) \neq 0$, provided there is at least one function in $\mathcal{F}$ which does not vanish at x. More precisely, if μ_x denotes the set-function which takes the value 1 on every set containing x and the value 0 on every set not containing x, then for each set $B \in \mathfrak{K}_\sigma$, $\mu_x(B) \leq M_x c_1(B)$. The main result on functional completion of proper functional spaces is obtained easily from this fact and the results of the precedings section. Also it can be obtained directly.

2) *A proper functional space $\mathcal{F}$ has a functional completion if and only if $\|u_n\| \to 0$ whenever $\{u_n\}$ is a Cauchy sequence which converges to 0 at each point. If a proper functional space has a functional completion, then it has a proper functional completion.*

§ 5. — Restrictions to a subset of $\mathcal{E}$.

Let $\mathcal{F}$ be a normed functional class rel. $\mathfrak{A}$, and let D be a subset of $\mathcal{E}$ which does not belong to $\mathfrak{A}$. Let $\mathfrak{A}(D)$ denote the class of all subsets of D which belong to $\mathfrak{A}$. If $u \in \mathcal{F}$, let u' denote the restriction of u to D. Each function u' is then defined on D exc. $\mathfrak{A}(D)$, and the class $\mathcal{F}(D)$ of all u' is a linear functional class on D rel. $\mathfrak{A}(D)$. There is a natural norm on $\mathcal{F}(D)$ given by

$$\|u'\|_D = \inf \|v\|,$$

the infimum being taken over all $v \in \mathcal{F}$ for which $v' = u'$ exc. $\mathfrak{A}(D)$. In general, $\bar{\mathcal{F}}(D)$ is not a normed functional class rel. $\mathfrak{A}(D)$. However,

1) *If $\mathcal{F}$ is a functional space rel. $\mathfrak{A}$ then $\mathcal{F}(D)$ is a functional space rel. $\mathfrak{A}(D)$. Moreover, the set functions δ', $\tilde{\delta}'$, and c'_φ corresponding to $\mathcal{F}(D)$ are the restrictions to D of δ, $\tilde{\delta}$, and c_φ. If c is any admissible capacity for $\mathcal{F}$ then the restriction of c to D is an admissible capacity for $\mathcal{F}(D)$. If $\mathcal{F}$ is complete, so is $\mathcal{F}(D)$.*

Proof. — If $u' = 0$ exc. $\mathfrak{A}(D)$, then $u' = 0'$ exc. $\mathfrak{A}(D)$, so that $\|u'\|_D \leq \|0\| = 0$. On the other hand, if $\|u'\|_D = 0$. then there is a sequence $\{v_n\}$ in $\mathcal{F}$ such that $v'_n = u'$ exc.

$\mathfrak{A}(\mathrm{D})$ and such that $||\varphi_n|| \to 0$. By choosing a subsequence if necessary, it can be assumed that $\varphi_n \to 0$ pointwise exc. $\mathfrak{A}$. This obviously requires that $u' = 0$ exc. $\mathfrak{A}(\mathrm{D})$. Hence $\mathcal{F}(\mathrm{D})$ is a normed functional class rel. $\mathfrak{A}(\mathrm{D})$. A similar argument shows that $\mathcal{F}(\mathrm{D})$ is a functional space rel. $\mathfrak{A}(\mathrm{D})$.

It is evident that δ' is the restriction of δ to D, and also that if $\mathrm{A} \subset \mathrm{D}$ then $\tilde{\delta}'(\mathrm{A}) \leqq \tilde{\delta}(\mathrm{A})$. On the other hand, if $\mathrm{A} \subset \mathrm{D}$ and $\tilde{\delta}'(\mathrm{A}) < d$, then there exists a Cauchy sequence $\{u'_n\}$ in $\mathcal{F}(\mathrm{D})$ satisfying

$$\liminf |u'_n(x)| \geqq 1 \textit{ on } \mathrm{A} \text{ exc. } \mathfrak{A}(\mathrm{D}) \textit{ and } \lim ||u'_n||_{\mathrm{D}} < d.$$

By picking a subsequence if necessary, it can be assumed that

$$||u'_1||_{\mathrm{D}} + \sum_{n=1}^{\infty} ||u'_{n+1} - u'_n||_{\mathrm{D}} < d.$$

Functions φ_n in $\mathcal{F}$ exist such that $\varphi'_1 = u'_1$ exc. $\mathfrak{A}(\mathrm{D})$ and $\varphi'_{n+1} = u'_{n+1} - u'_n$ exc. $\mathfrak{A}(\mathrm{D})$ and $\sum_{n=1}^{\infty} ||\varphi_n|| < d$. If $w_n = \sum_{k=1}^{n} \varphi_k$, then clearly $w'_n = u'_n$ exc. $\mathfrak{A}(\mathrm{D})$. Therefore,

$$\liminf |w_n(x)| \geqq 1 \text{ on A exc. } \mathfrak{A}$$

and in addition $\{w_n\}$ is a Cauchy sequence such that

$$\lim ||w_n|| \leqq \sum_{\infty}^{n=1} ||\varphi_n|| < d.$$

Hence $\tilde{\delta}(\mathrm{A}) < d$.

The assertions about capacities are immediate consequences of the fact that δ' is the restriction of δ to D.

Finally, if $\mathcal{F}$ is complete, the argument above shows that $\mathcal{F}(\mathrm{D})$ is complete. Indeed, the sequence $\{w_n\}$ converges to some $w \in \mathcal{F}$, and, therefore, the sequence $\{u'_n\} = \{w'_n\}$ converges to $w' \in \mathcal{F}(\mathrm{D})$.

CHAPTER II

SPACES OF POTENTIALS

§ 1. — Definition and elementary properties of the Dirichlet integral.

In this section the Dirichlet integral over the Euclidean space R^n, $d_\alpha(u) = d_{\alpha, R^n}(u)$, of arbitrary order $\alpha \geq 0$, is defined and expressed in terms of the function u and its derivatives. The following notation is used: if $i = (i_1, \ldots, i_m)$ where i_k is an integer between 1 and n, then $|i| = m$, $\xi^i = \prod_{k=1}^{m} \xi_{i_k}$ when $\xi = (\xi_1, \ldots, \xi_n)$, and

$$D_i u = \frac{\partial^m u}{\partial x_{i_1} \ldots \partial x_{i_m}} = \frac{\partial^m u}{(\partial x)^i}.$$

If α is an integer, the Dirichlet integral of order α is commonly defined by the formula

$$(1, 1) \qquad d_\alpha(u) = d_{\alpha, R^n}(u) = \sum_{|i|=\alpha} \int |D_i u|^2 \, dx.$$

If $\hat{u}$ is the Fourier transform of u, that is. if

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int e^{-i(x, \xi)} u(x) \, dx,$$

then $d_\alpha(u)$ is expressed in terms of $\hat{u}$ by the formula

$$(1, 2) \qquad d_\alpha(u) = \int |\xi|^{2\alpha} |\hat{u}(\xi)|^2 \, d\xi.$$

Formula (1, 2) can be used to define $d_\alpha(u)$ for arbitrary

$\alpha \geqq 0$. However, it is convenient to have an expression for $d_\alpha(u)$ which, like (1, 1) involves u and its derivatives, but not the Fourier transform. This will make it possible (in Chapter III) to define the Dirichlet integral of order α not only for functions on the whole space R^n, but also for functions on an open set $D \subset R^n$, and it will simplify a number of proofs.

If $0 < \alpha < 1$, then by Parseval's formula we have

$$\iint \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy = \iint \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}}\,dx\,dz$$
$$= \int |\hat{u}(\xi)|^2 \int \frac{|e^{i(z,\xi)}-1|^2}{|z|^{n+2\alpha}}\,dz\,d\xi = \int F(\xi)|\hat{u}|^2 d\xi.$$

It is easy to see that $F(\xi)$ is homogeneous of degree 2α and is invariant under orthogonal transformations. Hence $F(\xi) = C(n, \alpha)|\xi|^{2\alpha}$, where if $z = (z', z_n)$,

$$C(n, \alpha) = \int \frac{|e^{iz_n}-1|^2}{\left(|z'|^2+z_n^2\right)^{\frac{n+2\alpha}{2}}}\,dz'\,dz_n$$
$$= \int_{-\infty}^{\infty} \frac{|e^{iz_n}-1|^2}{|z_n|^{2\alpha+1}}\,dz_n \int_{R^{n-1}} \frac{dw'}{\left(1+|w'|^2\right)^{\frac{n+2\alpha}{2}}}$$
$$= 2^{3-2\alpha}\omega_{n-1} \int_0^{\infty} \frac{\sin^2 r}{r^{2\alpha+1}}\,dr \int_0^{\infty} \frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n+2\alpha}{2}}}\,d\rho.$$

The last two integrals and ω_{n-1} (the area of the unit sphere in R^{n-1}) can be evaluated in terms of the Gamma function to give

$$(1, 3) \qquad C(n, \alpha) = \frac{2^{-2\alpha+1}\pi^{\frac{n+2}{2}}}{\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{n}{2}\right)\sin \pi\alpha}.$$

Thus

$$(1, 4) \quad d_\alpha(u) = \frac{1}{C(n, \alpha)} \iint \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy \quad \textit{if} \quad 0<\alpha<1.$$

It is important to notice that for $\alpha \searrow 0$ or $\alpha \nearrow 1$, $\frac{1}{C(n, \alpha)}$ converges to 0 like α or $1-\alpha$ respectively. In general, if for arbitrary $\alpha > 0$, α^* denotes the largest integer strictly less than α, then [9]

$$d_\alpha(u) = \sum_{|i|=\alpha} \int |D_i u|^2 \, dx \qquad \text{if } \alpha \text{ is an integer;}$$

$$(1, 5) \quad d_\alpha(u) = \frac{1}{C(n, \alpha - \alpha^*)} \sum_{|i|=\alpha^*} \iint \frac{|D_i u(x) - D_i u(y)|^2}{|x-y|^{n+2\alpha-2\alpha^*}} \, dx \, dy$$

otherwise.

It is obvious from the expression (1, 2) of $d_\alpha(u)$ by Fourier transforms that for every function u such that u and $|\xi|^{\alpha_0}\hat{u}$ are square integrable, $d_\alpha(u)$ is continuous in α in the range $0 \leq \alpha \leq \alpha_0$ [10]. If U is an orthogonal transformation on R^n and $\varphi(x) = u(Ux)$, then clearly $\hat{\varphi}(\xi) = \hat{u}(U\xi)$, so that by (1, 2) $d_\alpha(\varphi) = d_\alpha(u)$. In other words $d_\alpha(u)$ is independent of the (orthogonal) coordinates which are used in R^n.

1) *The Dirichlet integral $d_\alpha(u)$ is continuous in α and independent of the (orthogonal) coordinates which are used in* R^n.

In the classical potential theory of Frostman and Riesz, which is valid for $0 < \alpha < \frac{n}{2}$, it is shown that the functions on R^n which are representable as potentials

$$(1, 6) \qquad u(x) = \int |x-y|^{\alpha-n} g(y) \, dy,$$

where g is square integrable, form with the norm $\sqrt{d_\alpha}$ a complete functional space relative to the exceptional class composed of the sets of (outer) capacity 0 of order 2α. (See [1].) This complete space is the perfect functional completion of the space $C_0^\infty(R^n)$. It is a space with a positive pseudo-reproducing kernel, namely the Riesz kernel

$$(1, 7) \qquad R_{2\alpha}(x-y) = \frac{\Gamma\left(\frac{n}{2} - \alpha\right)}{\pi^{\frac{n}{2}} 2^{2\alpha} \Gamma(\alpha)} |x-y|^{2\alpha-n}.$$

Spaces with positive pseudo-reproducing kernels are the natural setting for the classical type theory of capacity. With

(9) In formula (1, 5) the notation α^* is needed only when α is not an integer; it will be needed later, however, for all α.

(10) It is not obvious from (1, 5) that $d_\alpha(u)$ is continuous at α = integer; the corresponding result for domains $\neq R^n$, which will be considered in chapter III, is deeper.

this particular space there are associated two kinds of capacities, the classical capacity of order 2α and the capacities defined in Chapter I for arbitrary functional spaces. It is proved in [1] that the classical capacity of order 2α is identical with the functional space capacity c_2. For $\alpha \geqq \frac{2}{n}$ the situation is quite different: the potentials in (1, 6) cannot be formed for all square integrable g, since $|x|^{\alpha-n}$ is not square integrable at ∞; the pseudo-reproducing kernel in (1, 7) is not usable; and, as we shall now show, the space $C_0^\infty(R^n)$ normed by $\sqrt{d_\alpha}$ is not a functional space.

2) *If* $\alpha \geqq \frac{n}{2}$ *the space* $C_0^\infty(R^n)$ *normed by* $\sqrt{d_\alpha}$ *is not a functional space relative to any exceptional class.*

PROOF. — Let u be a function in $C_0^\infty(R^n)$ which is identically 1 on a neighborhood of 0. If $u_{(\rho)}(x) = u\left(\frac{x}{\rho}\right)$, then

$$\hat{u}_{(\rho)}(\xi) = \rho^n \hat{u}(\rho\xi) \text{ and } d_\alpha(u_{(\rho)}) = \rho^{n-2\alpha} d_\alpha(u).$$

Thus, if $\alpha > \frac{n}{2}$, then as $\rho \to \infty$, $d_\alpha(u_{(\rho)}) \to 0$, while, for each x, $u_{(\rho)}(x) \to 1$. This shows that the space in question cannot be a functional space (for the whole R^n would have to be an exceptional set). If $\alpha = \frac{n}{2}$, choose ε so that $0 < \varepsilon < \alpha$ and let $\varphi \in C_0^\infty(R^n)$. Then, if $d_\alpha(u, \varphi)$ is the bilinear form corresponding to the quadratic form $d_\alpha(u)$ we have

$$d_\alpha(u_{(\rho)}, \varphi) = \int |\xi|^{2\alpha} \hat{u}_{(\rho)}(\xi)\overline{\hat{\varphi}(\xi)}\, d\xi = \int |\xi|^{\alpha+\varepsilon}\hat{u}_{(\rho)}(\xi)\, |\xi|^{\alpha-\varepsilon}\overline{\hat{\varphi}(\xi)}\, d\xi$$
$$\leqq d_{\alpha+\varepsilon}(u_{(\rho)})^{1/2} d_{\alpha-\varepsilon}(\varphi)^{1/2},$$

so, by what has been proved, $d_\alpha(u_{(\rho)}, \varphi) \to 0$. Since this holds for each $\varphi \in C_0^\infty(R^n)$ and since $d_\alpha(u_{(\rho)})$ is bounded, it follows that $u_{(\rho)} \to 0$ weakly in the Hilbert space which is the abstract completion of $C_0^\infty(R^n)$ with the norm $\sqrt{d_\alpha}$. Therefore, by a well known theorem there is a sequence $\rho_k \to \infty$ such that the arithmetic means of the sequence $\{u_{(\rho_k)}\}$ converge strongly to 0. The sequence of arithmetic means converges pointwise to 1 everywhere, so, as before, the space cannot be a functional space.

This suggests the problem of defining an α-norm with the following properties : (a) On the subspace of $C_0^\infty(R^n)$ of functions vanishing outside any fixed compact set the α-norm is equivalent to $\sqrt{d_\alpha}$. (b) The space $C_0^\infty(R^n)$ normed by the α-norm is a functional space which has a perfect functional completion. (c) The completion has a positive pseudo-reproducing kernel. (d) The classical type capacity for this kernel coincides with the functional space capacity c_2.

One of the simplest norms on $C_0^\infty(R^n)$ for which (a) and (b) are true is the norm

$$(1, 8) \qquad |u|_\alpha^2 = d_0(u) + d_\alpha(u) = \int (1 + |\xi|^{2\alpha})|\hat{u}(\xi)|^2 d\xi.$$

In the next section we prove the existence of a functional completion relative to the class of sets of Lebesgue measure 0 and derive some properties of the completion relative to any smaller exceptional class. Later we shall replace $|u|_\alpha$ by the equivalent norm

$$(1, 9) \qquad ||u||_\alpha^2 = \int (1 + |\xi|^2)^\alpha |\hat{u}(\xi)|^2 d\xi$$

for which all the properties (a) — (d) hold.

There is a direct expression for $||u||_\alpha$ in terms of the function u and its derivatives, similar to the expression for d_α.

Suppose first that $0 < \alpha < 1$, and consider

$$I = \int_{-\infty}^{\infty}\int_{R^n}\int_{R^n} \frac{\left|e^{\frac{1}{2}iz_0}u(x) - e^{-\frac{1}{2}iz_0}u(y)\right|^2}{[|x-y|^2 + z_0^2]^{\frac{n+1+2\alpha}{2}}} dx\, dy\, dz_0.$$

If we put $x - y = z$, and write $\tilde{z}$ for the point $(z_0, z) \in R^{n+1}$ and $\tilde{\xi}$ for $(1, \xi)$, then

$$\begin{aligned} I &= \int_{R^{n+1}}\int_{R^n} \frac{\left|e^{\frac{1}{2}iz_0}e^{i(z,\xi)} - e^{-\frac{1}{2}iz_0}\right|^2}{|\tilde{z}|^{n+1+2\alpha}} |\hat{u}(\xi)|^2 d\xi\, d\tilde{z} \\ &= \int_{R^n}\int_{R^{n+1}} \frac{|e^{iz_0}e^{i(z,\xi)} - 1|^2}{|\tilde{z}|^{n+1+2\alpha}} |\hat{u}(\xi)|^2 d\tilde{z}\, d\xi. \\ &= \int_{R^n}\int_{R^{n+1}} \frac{|e^{i(\tilde{z},\tilde{\xi})} - 1|^2}{|\tilde{z}|^{n+1+2\alpha}} |\hat{u}(\xi)|^2 d\tilde{z}\, d\xi = C(n+1, \alpha)\int (1 + |\xi|^2)^\alpha |\hat{u}(\xi)^2| d\xi, \end{aligned}$$

by the formulas which were used to express d_α. Therefore we can write

$$(1,\ 10) \qquad \|u\|_0^2 = \int |u|^2\, dx;$$

for $0 < \alpha < 1$,

$$\|u\|_\alpha^2 = \frac{1}{C(n+1,\ \alpha)} \int_{-\infty}^{\infty} \int_{R^n} \int_{R^n} \frac{\left| e^{\frac{1}{2} i z_0} u(x) - e^{-\frac{1}{2} i z_0} u(y) \right|^2}{[\,|x-y|^2 + z_0^2]^{\frac{n+1+2\alpha}{2}}}\, dx\, dy\, dz_0;$$

if m *is the greatest integer* $\leqq \alpha$,

$$\|u\|_\alpha^2 = \sum_{k=0}^{m} \binom{m}{k} \sum_{|i|=k} \|D_i u\|_{\alpha-m}^2.$$

The last formula in (1, 10) is obtained from the expressions of the various norms by Fourier transforms.

Another formula will be given later (formula (4, 9)) which does not use an extra integration.

§ 2. — Functional completion with respect to $|u|_\alpha$.

In this section we investigate the normed functional class $\mathfrak{F}_\alpha$ obtained by giving the class $C_0^\infty(R^n)$ the α-norm $|u|_\alpha$. Using the results quoted in Chapter I we show that $\mathfrak{F}_\alpha$ has a functional completion relative to the class of exceptional sets of Lebesgue measure 0, and we establish some properties of the completion relative to any smaller exceptional class.

It is obvious that the class $\mathfrak{L}$ of sets B on which some function in $\mathfrak{F}_\alpha$ is $\geqq 1$ is the class of all bounded subsets of R^n, and, therefore, that the class $\mathfrak{L}_\sigma$ is the class of all subsets of R^n. Consequently, an admissible capacity for $\mathfrak{F}_\alpha$ is an outer measure c on R^n such that

$$(2.\ 1) \quad \begin{cases} (a) \text{ Each bounded set has finite (outer) measure.} \\ (b) \text{ To each } \varepsilon > 0 \text{ corresponds an } \eta > 0 \text{ such that if} \\ \qquad B \in \mathfrak{L} \text{ and } \delta(B) \leqq \eta, \text{ then } c(B) \leqq \varepsilon. \end{cases}$$

If c denotes the (outer) Lebesgue measure on R^n, then obviously (a) is satisfied, and since $|u|_\alpha \geqq \|u\|_{L^2}$ it follows that for each bounded set B, $c(B) \leqq \delta(B)^2$. Hence (b) is also satisfied, and the Lebesgue measure is an admissible capacity for $\mathfrak{F}_\alpha$.

1) *$\mathcal{F}_\alpha$ has a functional completion relative to the exceptional class of sets of Lebesgue measure* 0 (11).

PROOF. — It has been shown that the Lebesgue measure is an admissible capacity for $\mathcal{F}_\alpha$. Therefore, by virtue of proposition 4, § 2, Chapter I, it is sufficient to prove that if $\{u_n\}$ is a Cauchy sequence in $\mathcal{F}_\alpha$ which converges pointwise to 0 almost everywhere, then $|u_n|_\alpha \to 0$. Since $|u|_\alpha \geq \|u\|_{L^2}$, the sequence $\{u_n\}$ is a Cauchy sequence in L^2, and so by the usual Lebesgue theory, $\|u_n\|_{L^2} \to 0$. Therefore the sequence $\{\hat{u}_n\}$ of Fourier transforms is Cauchy in the L^2 space formed with the measure $(1 + |\xi|^{2\alpha})d\xi$, and converges to 0 in the ordinary L^2 space. It follows from the usual Lebesgue theory that $\{\hat{u}_n\}$ converges to 0 in the L^2 space formed with the measure $(1 + |\xi|^{2\alpha})d\xi$, that is, that $|u_n|_\alpha \to 0$.

In the rest of the section $\tilde{\mathcal{F}}_\alpha$ will denote any saturated functional completion of $\mathcal{F}_\alpha$ relative to an exceptional class $\tilde{\mathfrak{A}}_{2\alpha}$, which is contained in the class of sets of Lebesgue measure 0 (12).

2) *If $u \in \tilde{\mathcal{F}}_\alpha$, then $u \in L^2$, and* $|u|_\alpha^2 = \int (1 + |\xi|^{2\alpha})|\hat{u}(\xi)|^2\, d\xi$.

PROOF. — If $u \in \tilde{\mathcal{F}}_\alpha$, then there is a Cauchy sequence $\{u_n\}$ in $\mathcal{F}_\alpha$ which converges pointwise to u exc. $\tilde{\mathfrak{A}}_{2\alpha}$, hence almost everywhere. It follows that $u \in L^2$. The sequence $\{\hat{u}_n\}$ of Fourier transforms is Cauchy in the L^2 space formed with the measure $(1 + |\xi|^{2\alpha})d\xi$ and converges to $\hat{u}$ in the ordinary L^2 space. It follows that $\{\hat{u}_n\}$ converges to $\hat{u}$ in the L^2 space formed with the measure $(1 + |\xi|^{2\alpha})d\xi$. Thus,

$$|u_n|_\alpha^2 \to \int (1 + |\xi|^{2\alpha})\hat{u}(\xi)^2\, d\xi.$$

At the same time, by definition $|u|_\alpha = \lim |u_n|_\alpha$.

COROLLARY. — *If two functions in $\tilde{\mathcal{F}}_\alpha$ are equal almost everywhere, then they are equal exc.* $\tilde{\mathfrak{A}}_{2\alpha}$.

PROOF. — By 2) their difference has norm 0.

(11) It is easy to see that if $\alpha = 0$ the class of sets of Lebesgue measure 0 is the exceptional class for the perfect completion of $\mathcal{F}_\alpha$; the perfect functional completion of $\mathcal{F}_0$ is simply L^2.

(12) The notation $\tilde{\mathfrak{A}}_{2\alpha}$ is chosen to agree with the notation which will be used later.

3) *If* B *is a set of finite measure, then on the subspace of* $\tilde{\mathfrak{F}}_\alpha$ *of functions which vanish outside* B *exc.* $\tilde{\mathfrak{A}}_{2\alpha}$ *the norms* $|u|_\alpha$ *and* $\sqrt{d_\alpha}$ *are equivalent. In fact, there is a constant c such that if u vanishes outside* B *exc.* $\tilde{\mathfrak{A}}_{2\alpha}$ *then*

$$d_0(u) \leqq c|\mathrm{B}|^{2\alpha/n} d_\alpha(u).$$

Proof. — If u vanishes outside B, then for every ξ

$$|\hat{u}(\xi)|^2 \leqq (2\pi)^{-n}|\mathrm{B}|d_0(u). \tag{2, 2}$$

Hence, for every $r > 0$,

$$\begin{aligned} d_0(u) &\leqq \int_{|\xi|\leqq r} (2\pi)^{-n}|\mathrm{B}|d_0(u)\, d\xi + \int_{|\xi|\geqq r} \frac{|\xi|^{2\alpha}}{r^{2\alpha}} |\hat{u}(\xi)|^2 d\xi \\ &\leqq \frac{\omega_n}{n}(2\pi)^{-n}|\mathrm{B}|r^n d_0(u) + \frac{1}{r^{2\alpha}} d_\alpha(u), \end{aligned}$$

and so, for $r < 2\pi\left(\dfrac{n}{\omega_n|\mathrm{B}|}\right)^{1/n}$,

$$d_0(u) \leqq \frac{n(2\pi)^n}{r^{2\alpha}[n(2\pi)^n - \omega_n|\mathrm{B}|r^n]} d_\alpha(u).$$

The inequality in the proposition is obtained by minimizing the coefficient of $d_\alpha(u)$.

4) *If* $\beta < \alpha$ *and* B *is a set of finite measure, then* $|u|_\beta$ *is completely continuous on the subspace of* $\tilde{\mathfrak{F}}_\alpha$ *of functions which vanish outside* B *exc.* $\tilde{\mathfrak{A}}_{2\alpha}$.

Proof. — (The idea of this proof is due to Gårding.) It will be shown that if u_n vanishes outside B and the sequence $\{u_n\}$ converges weakly to 0 in $\tilde{\mathfrak{F}}_\alpha$, then $|u_n|_\beta \to 0$. For every $r > 1$

$$|u_n|_\beta^2 \leqq \int_{|\xi|\leqq r} (1 + |\xi|^{2\beta})\, |\hat{u}_n(\xi)|^2\, d\xi + \frac{2}{r^{2\alpha-2\beta}} d_\alpha(u_n). \tag{2, 3}$$

Since a weakly convergent sequence is necessarily bounded, for every positive number ε a positive number r can be chosen large enough so that the second term on the right side of (2, 3) is less than ε for all n. For fixed r the first term on the right side of (2, 3) converges to 0, for the functions $\hat{u}_n(\xi)$

converge pointwise to 0 [13] and by (2, 2) they are uniformly bounded. Thus if u_n vanishes outside B and $\{u_n\}$ converges weakly to 0 in $\tilde{\mathcal{F}}_\alpha$, then $|u_n|_\beta \to 0$.

A function is said to be of class $C^{(m,1)}$ on an open set if the function is of class C^m on the open set and every derivative of order $\leqq m$ is Lipschitzian [14]. If u is a function of class $C^{(\alpha^*,1)}$ on R^n, then the integrals in (1, 5) are defined, and they are obviously finite if u has compact support [15].

In the proof of the next two propositions and in several later proofs we will need the well known process of regularization. A family of regularizing functions is a family

$$e_\rho(x) = \rho^{-n} e\left(\frac{x}{\rho}\right) \qquad for \qquad 0 < \rho \leqq 1,$$

where e is a non-negative function in $C_0^\infty(R^n)$ satisfying

$$e(x) = 0 \qquad for \qquad |x| \geqq 1 \qquad and \qquad \int e(x)\,dx = 1.$$

If u is a locally integrable function, the functions

$$u_\rho(x) = u * e_\rho(x) = \int u(y)e(x-y)\,dy$$

are called the functions obtained from u by regularization, or, more briefly the regularizations of u. A few of the standard properties of the family $\{u_\rho\}$ are as follows:

(*a*) Each u_ρ is of class C^∞.

(*b*) $u_\rho(x) \to u(x)$ almost everywhere.

(*c*) If u belongs to L^p (or locally to L^p) then so do the u_ρ, and $u_\rho \to u$ in L^p (or locally in L^p).

(*d*) If $u \in L^p$, $1 \leqq p \leqq 2$, then $u_\rho \in L^p$ and $\hat{u}_\rho = (2\pi)^{n/2}\hat{u}\hat{e}_\rho$.

Other properties of u_ρ will be stated when they are needed.

5) *If u is square integrable and $|u|_\alpha$ is finite* [16], *then u is*

[13] In fact, $\hat{u}_n(\xi)$ is the inner product in L^2 of u_n with $(2\pi)^{-n/2}e^{-i(x,\xi)}$ times the characteristic function of B, and weak convergence in $\tilde{\mathcal{F}}_\alpha$ implies weak convergence in L^2 (which has a smaller norm).

[14] By this it is meant that there is a constant M such that if $|i| \leqq m$ then

$$|D_i u(x) - D_i u(y)| \leqq M|x-y|$$

for all x and y in the open set.

[15] If α is not an integer this is self-evident; if α is an integer it follows from the classical theorem that the partial derivatives of a Lipschitz function exist a.e. and are bounded.

[16] If it is known only that u is square integrable then the expression of $|u|_\alpha$ by Fourier transforms must be used here, but if it is known that the necessary derivatives of u exist, then either expression of $|u|_\alpha$ can be used.

equal almost everywhere to a function in $\tilde{\mathcal{F}}_\alpha$; *if, in addition, u is continuous, then* $u \in \tilde{\mathcal{F}}_\alpha$.

6) *There is a constant c (depending only on* α^* *and n) such that if* φ *is of class* $C^{(\alpha^*, 1)}$ *on* R^n *and satisfies* $|D_i\varphi(x)| \leqq M$ a.e. *for* $|i| \leqq \alpha^* + 1$ *and if* $u \in \tilde{\mathcal{F}}_\alpha$ *then*

$$(2, 4) \qquad \varphi u \in \tilde{\mathcal{F}}_\alpha \qquad \textit{and} \qquad |\varphi u|_\alpha \leqq cM|u|_\alpha.$$

PROOFS. — First we show that if φ is of class $C^{(\alpha^*, 1)}$ on R^n and if u is of class C^∞ and $|u|_\alpha$ is finite, then the inequality (2, 4) holds. For this the direct expression (1, 5) of $|u|_\alpha$ is used.

Since each derivative $D_i(\varphi u)$ is a sum of products $D_j\varphi D_k u$ with $|j| + |k| = |i|$, if α is an integer, then, by (1, 5), $d_\alpha(\varphi u)$ is majorated by a constant (depending only on α) times a sum of terms of the form

$$\int |D_j\varphi D_k u|^2 dx \leqq M^2 d_{|k|}(u) \qquad \text{where} \qquad |j| + |k| = \alpha.$$

It is obvious from (1, 8) that if $\beta \leqq \alpha$ then $d_\beta(u) \leqq |u|_\alpha^2$. Hence (2, 4) is proved if α is an integer. If α is not an integer, then by (1, 5) $d_\alpha(\varphi u)$ is majorated by a constant (depending only on α^*) times a sum of terms of the form

$$\frac{1}{C(n, \alpha - \alpha^*)} \iint \frac{|D_j\varphi(x)D_k u(x) - D_j\varphi(y)D_k u(y)|^2}{|x - y|^{n+2\alpha-2\alpha^*}} dx\, dy$$
$$\leqq 2M^2 d_{|k|+\alpha-\alpha^*}(u) + \frac{2}{C(n, \alpha-\alpha^*)} \iint \frac{|D_k u(y)|^2 |D_j\varphi(x) - D_j\varphi(y)|^2}{|x - y|^{n+2\alpha-2\alpha^*}} dx dy,$$

where $|j| + |k| = \alpha^*$. The first term on the right has already been considered, and for the inner integral in the second we have

$$\int \frac{|D_j\varphi(x) - D_j\varphi(y)|^2}{|x - y|^{n+2\alpha-2\alpha^*}} dx = \int \frac{|D_j\varphi(y + z) - D_j\varphi(y)|^2}{|z|^{n+2\alpha-2\alpha^*}} dz$$
$$\leqq M^2 \int_{|z| \leqq 1} |z|^{-n-2\alpha+2\alpha^*+2} dz + 4M^2 \int_{|z| \geqq 1} |z|^{-n-2\alpha+2\alpha^*} dz \leqq \frac{c}{\alpha - \alpha^*} M^2.$$

Hence the second term on the right above is at most $cM^2 d_{|k|}(u)$, and the inequality (2, 4) is established for φ of class $C^{(\alpha^*, 1)}$ and u of class C^∞ with $|u|_\alpha$ finite.

Next we show that if u is of class C^∞ and $|u|_\alpha$ is finite, then $u \in \tilde{\mathcal{F}}_\alpha$. Let φ be a function in $C_0^\infty(R^n)$ which is 1 on a neighborhood of 0, and for $\rho > 0$ put $\varphi_{(\rho)}(x) = \varphi(\rho x)$. If $|D_i\varphi(x)| \leq M$ for $|i| \leq \alpha^* + 1$, then $|D_i\varphi_{(\rho)}(x)| \leq \rho^{|i|}M$ for $|i| \leq \alpha^* + 1$, so by what has been proved, as $\rho \to 0$, $|\varphi_{(\rho)}u|_\alpha$ is bounded. Consequently there is a sequence $\rho_k \to 0$ such that the arithmetic means of the sequence $\{\varphi_{(\rho_k)}u\}$ converge in $\tilde{\mathfrak{F}}_\alpha$, and since the arithmetic means obviously converge pointwise everywhere to u, it follows that $u \in \tilde{\mathfrak{F}}_\alpha$.

Now we prove 5) by regularization. We note first that for regularizing functions e_ρ, $\hat{e}_\rho(\xi) = \hat{e}(\rho\xi)$, and since

$$|\hat{e}(\xi)| \leq (2\pi)^{-n/2} \qquad \textit{and} \qquad \hat{e}(0) = (2\pi)^{-n/2},$$

it follows that as $\rho \to 0$ the functions

$$\hat{u}_\rho(\xi) = (2\pi)^{n/2}\hat{e}_\rho(\xi)\,\hat{u}(\xi)$$

are uniformly bounded by $|\hat{u}(\xi)|$ and converge pointwise to $\hat{u}\ (\xi)$.

Therefore, since $|u|_\alpha$ is finite, $|u_\rho|_\alpha$ is finite, and, as $\rho \to 0$, $|u - u_\rho|_\alpha \to 0$. Since u_ρ is of class C^∞, it follows from what has been shown that $u_\rho \in \tilde{\mathfrak{F}}_\alpha$. Thus, as $\rho \to 0$ $\{u_\rho\}$ is Cauchy in $\tilde{\mathfrak{F}}_\alpha$, so there exists a function $v \in \tilde{\mathfrak{F}}_\alpha$ and a sequence $\rho_k \to 0$ such that $u_{\rho k} \to v$ pointwise almost everywhere. Therefore $u = v$ almost everywhere. Moreover, if u is continuous, then $u_{\rho k} \to u$ everywhere, so that $u = v$ exc. $\tilde{\mathfrak{A}}_{2\alpha}$ and hence $u \in \tilde{\mathfrak{F}}_\alpha$. This completes the proof of 5).

The proof of 6) is completed as follows. Proposition 5) shows that each function of class $C^{(\alpha^*,1)}$ with compact support belongs to $\tilde{\mathfrak{F}}_\alpha$. Therefore, the transformation $Tu = \varphi u$ transforms $\mathfrak{F}_\alpha$ into $\tilde{\mathfrak{F}}_\alpha$, and by the first paragraph in the proof this transformation is continuous. If $\tilde{T}$ denotes its extension to $\tilde{\mathfrak{F}}_\alpha$ by continuity, then for each $u \in \tilde{\mathfrak{F}}_\alpha$ there is a sequence $\{u_n\}$ in $\mathfrak{F}_\alpha$ such that $u_n \to u$ both in $\tilde{\mathfrak{F}}_\alpha$ and pointwise exc. $\tilde{\mathfrak{A}}_{2\alpha}$. and such that $Tu_n \to \tilde{T}u$ both in $\tilde{\mathfrak{F}}_\alpha$ and pointwise exc. $\tilde{\mathfrak{A}}_{2\alpha}$. Since $Tu_n = \varphi u_n$ and since $\varphi u_n \to \varphi u$ pointwise wherever u is defined, *a fortiori* exc. $\tilde{\mathfrak{A}}_{2\alpha}$, it follows that $\tilde{T}u = \varphi u$. This completes the proof of 6).

Corollary 1. — *If u is locally in $\tilde{\mathfrak{F}}_\alpha$ and $|u|_\alpha$ is finite, then $u \in \tilde{\mathfrak{F}}_\alpha$.*

PROOF. — To say that u is locally in $\tilde{\mathfrak{F}}_\alpha$ means that each point in R^n has a neighborhood on which u coincides with a function in $\tilde{\mathfrak{F}}_\alpha$. If this is the case, then by using 6) we can choose for each point a function φ in $C_0^\infty(R^n)$ such that $\varphi = 1$ on a neighborhood of the point and such that $\varphi u \in \tilde{\mathfrak{F}}_\alpha$. Now, by 5) u is equal almost everywhere to some function $v \in \tilde{\mathfrak{F}}_\alpha$. Hence $\varphi u = \varphi v$ almost everywhere, and by the corollary to proposition 2, $\varphi u = \varphi v$ exc. $\tilde{\mathfrak{A}}_{2\alpha}$. It follows that $u = v$ exc. $\tilde{\mathfrak{A}}_{2\alpha}$, so that $u \in \tilde{\mathfrak{F}}_\alpha$.

COROLLARY 2. — *If φ is bounded and of class $C^{(\alpha^*, 1)}$ on R^n and $\varphi = 1$ on a neighborhood of 0, and if $\varphi_{(\rho)}(x) = \varphi(\rho x)$, then, as $\rho \to 0$, $\varphi_{(\rho)}u \to u$ in $\tilde{\mathfrak{F}}_\alpha$ for every $u \in \tilde{\mathfrak{F}}_\alpha$.*

PROOF. — By (6) the transformations $T_\rho u = \varphi_{(\rho)}u$ of $\tilde{\mathfrak{F}}_\alpha$ into itself are uniformly bounded. Therefore, it is sufficient to show that $T_\rho u \to u$ for all u in some dense subset of $\tilde{\mathfrak{F}}_\alpha$. It is obvious that this is the case for $u \in \mathfrak{F}_\alpha$.

7) *Let u be a square integrable function and let m be an integer $\leqq \alpha$. Then u is equal almost everywhere to a function in $\tilde{\mathfrak{F}}_\alpha$ if and only if for each j with $|j| = m$ there is a function $v_j \in \tilde{\mathfrak{F}}_{\alpha-m}$ such that $D_j u = v_j$ in the sense of distributions.*

PROOF. — To say that $D_j u = v_j$ in the sense of distributions means that for every function $\varphi \in C_0^\infty(R^n)$

$$\int u\overline{D_j\varphi}dx = (-1)^{|j|}\int v_j\bar{\varphi}dx.$$

Suppose first that $u \in \tilde{\mathfrak{F}}_\alpha$, and let $\{u_n\}$ be a sequence in $C_0^\infty(R^n)$ which converges to u in $\tilde{\mathfrak{F}}_\alpha$. From the obvious relation

$$d_\alpha(v) = \sum_{|j|=m} d_{\alpha-m}(D_j v)$$

it follows that the sequence $\{D_j u_n\}$ is Cauchy in $\tilde{\mathfrak{F}}_{\alpha-m}$, and hence that there exists $v_j \in \tilde{\mathfrak{F}}_{\alpha-m}$ such that $D_j u_n \to v_j$ in $\tilde{\mathfrak{F}}_{\alpha-m}$. If $\varphi \in C_0^\infty(R^n)$, then

$$(u, D_j\varphi)_{L^2} = \lim_{n\to\infty}(u_n, D_j\varphi)_{L^2} = \lim_{n\to\infty}(-1)^m(D_j u_n, \varphi) = (-1)^m (v_j, \varphi)_{L^2}$$

and the first part of the statement is proved.

Following a general theorem about temperate distributions and their Fourier transforms [14, vol. 2], if $D_j u = v_j$ and both u and v_j are square integrable, then $i^m \xi_{j_1} \ldots \xi_{j_m} \hat{u} = \hat{v}_j$ (17). Hence, if $v_j \in \tilde{\mathcal{F}}_{\alpha-m}$, then

$$\int |\xi|^{2\alpha} |\hat{u}|^2 d\xi = \Sigma \int |\xi|^{2\alpha-2m} |\hat{v}_j|^2 d\xi < \infty.$$

By 5), u is equal a.e. to a function in $\tilde{\mathcal{F}}_\alpha$.

Propositions 1)-7) describe most of the properties of the functional completion of $\mathcal{F}_\alpha$ which subsist for an arbitrary completion $\tilde{\mathcal{F}}_\alpha$ whose exceptional class is contained in the class of sets of Lebesgue measure 0. The finer properties which are developed in later sections are properties of the perfect completion. Therefore, we shall close this section with a few remarks of introduction for the next.

We have stated at the beginning of this chapter that $\mathcal{F}_\alpha$ has a perfect functional completion and that if the norm $|u|_\alpha$ is replaced by the equivalent norm

$$\|u\|_\alpha^2 = \int (1 + |\xi|^2)^\alpha |\hat{u}(\xi)|^2 \, d\xi$$

then the perfect completion is a space with a positive pseudo-reproducing kernel; this kernel will be denoted by $G_{2\alpha}$. It will be shown that every function u in the perfect completion can be represented as a potential

$$u(x) = G_\alpha g(x) \doteq \int G_\alpha(x-y)\, g(y) dy \tag{2, 5}$$

of a square integrable function g, and that the following relation holds:

$$\|u\|_\alpha = \|g\|_{L^2} = \int |g|^2 \, dx. \tag{2, 6}$$

If (2, 5) and (2, 6) are accepted for the present, we can deduce an expression for the kernel G_α.

Since the potential $G_\alpha g$ defined in (2, 5) is a product of composition, if follows (provided G_α is integrable) that

$$(G_\alpha g)^\wedge(\xi) = (2\pi)^{n/2} \, \hat{G}_\alpha(\xi) \hat{g}(\xi).$$

If (2, 6) is to hold, then

$$\int (1 + |\xi|^2)^\alpha |(2\pi)^{n/2} \hat{G}_\alpha(\xi)|^2 |\hat{g}(\xi)|^2 \, d\xi = \int |\hat{g}(\xi)|^2 \, d\xi$$

(17) This simple case of the general theorem can be proved directly without much trouble.

for every square integrable function g. Our condition is therefore satisfied by

$$(2, 7) \qquad \hat{G}_\alpha(\xi) = (2\pi)^{-n/2}(1 + |\xi|^2)^{-\alpha/2}.$$

This gives the expression for the Fourier transform of G_α. However, if $\alpha > n$, $(1 + |\xi|^2)^{-\alpha/2}$ is integrable, and (2, 7) can be inverted to give

$$(2, 8) \qquad G_\alpha(x) = (2\pi)^{-n}\int \frac{e^{i(x,\xi)}}{(1 + |\xi|^2)^{\alpha/2}} d\xi.$$

There is a well known formula expressing the Fourier transform of a function $f(\xi)$ which is a function of $|\xi|$ alone as a single integral involving the Bessel function J_ν [18]:

$$(2, 9) \qquad \hat{f}(x) = |x|^{\frac{2-n}{2}} \int_0^\infty f(\rho)\rho^{n/2} J_{\frac{n-2}{2}}(\rho|x|)\, d\rho.$$

Formulas (2, 8) and (2, 9) give

$$G_\alpha(x) = (2\pi)^{-n/2}|x|^{\frac{2-n}{2}} \int_0^\infty \frac{\rho^{n/2}}{(1 + \rho^2)^{\alpha/2}} J_{\frac{n-2}{2}}(\rho|x|)d\rho$$

From this we obtain for $\alpha > n$ [19].

$$(2, 10) \quad G_\alpha(x) = \frac{1}{2^{\frac{n+\alpha-2}{2}} \pi^{n/2}\Gamma\left(\frac{\alpha}{2}\right)} K_{\frac{n-\alpha}{2}}(|x|)|x|^{\frac{\alpha-n}{2}}, \ [20]$$

where $K_\nu(z)$ is the modified Bessel function of the third kind. It will be shown that if $G_\alpha(x)$ is defined by (2, 10) for all $\alpha > 0$, then G_α is integrable and its Fourier transform is given by (2, 7), and, in fact, that G_α has all the properties that have been attributed to it. In the next section some pertinent formulas and properties of the Bessel function K_ν are listed.

§ 3. — Formulas and properties of K_ν.

Most of the results listed in this section can be found both in [10] and in [16], and all can be found in one or the other.

(18) See, for example, S. Bochner [3].
(19) Formula 20, p. 24 of [9].
(20) L. Schwartz [14] introduced functions $L_\alpha(|x|)$ related to $G_\alpha(x)$ by the equation $G_\alpha(x) = L_\alpha\left(\frac{|x|}{2\pi}\right)(2\pi)^{-n}$.

The modified Bessel function of the third kind, K_ν, is defined in terms of the more common Bessel functions by

$$(3,1) \qquad K_\nu = \frac{\pi}{2}\,\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}$$

where
$$I_\nu(z) = e^{-i\frac{1}{2}\nu\pi} J_\nu(iz),$$

and J_ν is the Bessel function of the first kind of order ν. The above formula for K_ν for ν an integer should be understood as a limit $K_\nu = \lim_{\epsilon \to 0} K_{\nu+\epsilon}$.

The functions K_ν and I_ν are defined for complex values of ν and z, but we shall be concerned only with real values of ν and positive real values of z. The function I_ν (the modified Bessel function of the first kind) has the series expansion

$$(3,2) \qquad I_\nu(z) = \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}z\right)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)}$$

from which it follows that $K_\nu(z)$ is an analytic function of z except at $z = 0$, and for $z \neq 0$, $K_\nu(z)$ is an entire function of ν. Obviously

$$(3,3) \qquad K_{-\nu}(z) = K_\nu(z).$$

From (3, 2) it follows immediately that

$$(3,4) \qquad \begin{aligned} &K_\nu(z) \sim 2^{\nu-1}\Gamma(\nu)z^{-\nu}\ {}^{(21)} && \text{as } z \to 0, \quad \text{for } \nu > 0,\\ &K_0(z) \sim \log 1/z && \text{as } z \to 0. \end{aligned}$$

It is known that

$$(3,5) \qquad K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \qquad \text{as } z \to \infty \textit{ for all } \nu.$$

The following integral formula holds.

$$(3,6) \qquad K_\nu(z) = \frac{\left(\frac{1}{2}\pi\right)^{1/2} z^\nu e^{-z}}{\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^\infty e^{-zt} t^{\nu-\frac{1}{2}} \left(1 + \frac{1}{2}t\right)^{\nu-\frac{1}{2}} dt$$

for $z > 0$, $\nu > -\frac{1}{2}$.

[21] As usual, we write $f(z) \sim g(z)$ as $z \to a$ if $\lim_{z\to a} \frac{f(z)}{g(z)} = 1$.

We shall also need the differentiation formula

$$(3,\,7)\qquad \left(\frac{1}{z}\frac{d}{dz}\right)^m [z^{-\nu}K_\nu(z)] = (-1)^m z^{-\nu-m} K_{\nu+m}(z).$$

We mention also the differential equation of second order satisfied by K_ν (which can be deduced easily from (3, 7) and (3, 3))

$$(3,\,8)\qquad \frac{d^2}{dz^2}(z^{-\nu}K_\nu(z)) + \frac{(2\nu+1)}{z}\frac{d}{dz}(z^{-\nu}K_\nu(z)) - z^{-\nu}K_\nu(z) = 0.$$

§ 4. — Formulas and properties of G_α.

The kernel G_α is defined for $\alpha > 0$ by

$$(4,\,1)\qquad G_\alpha(x) = \frac{1}{2^{\frac{n+\alpha-2}{2}}\pi^{n/2}\Gamma\left(\frac{\alpha}{2}\right)} K_{\frac{n-\alpha}{2}}(|x|)\,|x|^{\frac{\alpha-n}{2}}.$$

Most of the necessary formulas and properties of G_α are almost immediate consequences of the corresponding formulas and properties of $K_{\frac{n-\alpha}{2}}$. They are listed in this section.

The kernel $G_\alpha(x)$ is an analytic function of $|x|$ except at $x = 0$, and for $x \neq 0$, $G_\alpha(x)$ is an entire function of α. From (3, 4) we obtain

$$(4,2)\qquad As\ x \to 0 \qquad \begin{aligned} &G_\alpha(x) \sim \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{n/2}\Gamma\left(\frac{\alpha}{2}\right)}|x|^{\alpha-n} \quad \textit{if} \quad \alpha < n \\ &G_n(x) \sim \frac{1}{2^{n-1}\pi^{n/2}\Gamma\left(\frac{n}{2}\right)}\log\frac{1}{|x|} \\ &G_\alpha(x) \sim \frac{\Gamma\left(\frac{\alpha-n}{2}\right)}{2^n\pi^{n/2}\Gamma\left(\frac{\alpha}{2}\right)} \quad \textit{if} \quad \alpha > n, \end{aligned}$$

and from (3, 5) we obtain

$$(4, 3) \quad as\, |x| \to \infty, \quad G_\alpha(x) \sim \frac{1}{2^{\frac{n+\alpha-1}{2}} \pi^{\frac{n-1}{2}} \Gamma\left(\frac{\alpha}{2}\right)} |x|^{\frac{\alpha-n-1}{2}} e^{-|x|}.$$

Clearly G_α is a function of $|x|$ alone; it will sometimes be convenient to write $G_\alpha(r)$ for $G_\alpha(x)$ with $|x| = r$. With this notation (3, 7) gives

$$(4, 4) \quad \frac{d}{dr} G_\alpha(r) = \frac{-1}{2^{\frac{n+\alpha-2}{2}} \pi^{n/2} \Gamma\left(\frac{\alpha}{2}\right)} r^{\frac{\alpha-n}{2}} K_{\frac{n-\alpha+2}{2}}(r).$$

Hence for $\alpha > 1$, there is a constant c such that

$$(4, 5) \qquad \left|\frac{\partial G_\alpha(x)}{\partial x_i}\right| \leqq c[G_\alpha(x) + G_{\alpha-1}(x)].$$

Formula (3, 6) shows that all K_ν, and therefore all G_α, are everywhere positive; (4, 4) then shows that G_α is a decreasing function of $|x|$. Formulas (4, 2) and (4, 3) show that G_α is integrable.

Since G_α is integrable, the Fourier transform $G_\alpha(\xi)$ exists for each ξ; as a function of α it is analytic for $\alpha > 0$. Therefore, from (2, 7) we obtain by analytic continuation

$$(4, 6) \quad \hat{G}_\alpha(\xi) = (2\pi)^{-n/2} (1 + |\xi|^2)^{-\alpha/2} \qquad for \qquad \alpha > 0.$$

A simple consequence is

$$(4, 6') \qquad \int G_\alpha(x)\, dx = (2\pi)^{n/2} \hat{G}_\alpha(0) = 1.$$

From (4, 6) it is evident that the following composition formula holds.

$$(4, 7) \quad G_{\alpha+\beta}(x) = G_\alpha * G_\beta(x) = \int G_\alpha(y) G_\beta(x-y)\, dy.$$

We give now a mean value theorem similar to the Frostman mean value theorem for the kernel $|x|^{\alpha-n}$.

1) *For each $r_0 > 0$ there is a constant c (depending only*

on r_0, α, and n) such that for every point z, every sphere $S(x, r)$ *with $r \leqq r_0$, and every function $g \geqq 0$,*

$$(a) \qquad \frac{1}{|S(x, r)|}\int_{S(x,r)} G_\alpha(z-y)\,dy \leqq cG_\alpha(z-x).$$

$$(b) \qquad \frac{1}{|S(x, r)|}\int_{S(x,r)} G_\alpha g(y)\,dy \leqq cG_\alpha g(x),$$

$$(c) \qquad \lim_{r \to 0}\frac{1}{|S(x, r)|}\int_{S(x,r)} G_\alpha g(y)\,dy = G_\alpha g(x).$$

We use here the notation $G_\alpha g$ introduced in (2, 5). When $g \geqq 0$, $G_\alpha g(x)$ will be considered as defined everywhere possibly with the value $+\infty$.

Proof. — Parts (b) and (c) are obvious consequences of part (a), and part (a) is an obvious consequence of the special case in which $x = 0$. This special case can be formulated as follows: if f_r denotes $|S(0, r)|^{-1}$ times the characteristic function of $S(0, r)$, then for every x

$$(4, 8) \qquad f_r * G_\alpha(x) \leqq cG_\alpha(x) \qquad \textit{for} \qquad r \leqq r_0.$$

From the composition formula (4, 7) it is seen that (4, 8) has only to be proved for small values of α, in particular, for $\alpha < n$. To simplify the notation we shall suppose $r_0 = 1$.

By using (4, 2) and the fact that $|x|^{\alpha-n}$ and $G_\alpha(x)$ are both positive we obtain the existence of positive constants c_1 and c_2 such that

$$c_1|x|^{\alpha-n} \leqq G_\alpha(x) \leqq c_2|x|^{\alpha-n} \qquad \textit{for} \qquad |x| < 3.$$

Hence (4, 8) holds for $|x| < 2$ (with $r_0 = 1$) if and only if it holds when $G_\alpha(x)$ is replaced by $|x|^{\alpha-n}$. That it does hold in this case is the assertion of the Frostman mean value theorem [22]. For $|x| \geqq 2$ we have, since $G_\alpha(x)$ is a decreasing function of $|x|$,

$$\frac{f_r * G_\alpha(x)}{G_\alpha(x)} \leqq \sup_{\rho \geqq 2}\frac{G_\alpha(\rho - 1)}{G_\alpha(\rho)}.$$

[22] O. Frostman [11]. In the Frostman theorem the constant c is independent of r_0. From the exponential decrease of $G_\alpha(\rho)$ as $\rho \to \infty$, it is easy to see that such is not the case here.

The supremum on the right side is finite, since G_α is continuous and positive and since, by (4, 3),

$$\lim_{\rho \to \infty} \frac{G_\alpha(\rho - 1)}{G_\alpha(\rho)} = e.$$

Corollary. — *If $\{e_\rho\}$ is a family of regularizing functions, there is a constant c such that for every point x and every function* $g \geqq 0$

$$\begin{array}{ll} (a) & G_\alpha * e_\rho(x) \leqq cG_\alpha(x) \\ (b) & (G_\alpha g) * e_\rho(x) \leqq cG_\alpha g(x) \\ (c) & \lim_{\rho \to 0} (G_\alpha g) * e_\rho(x) = G_\alpha g(x). \end{array}$$

Proof. — Part (a) follows from (4, 8), since

$$e_\rho(x) = \rho^{-n} e\left(\frac{x}{\rho}\right) \leqq cf_\rho(x).$$

The other parts follow from (a).

With the aid of the kernels G_α we can give the other direct expression of $\|u\|_\alpha$ for $0 < \alpha < 1$ promised earlier. From (1, 10) we obtain

$$\|u\|_\alpha^2 = \frac{1}{C(n+1, \alpha)} \int_{-\infty}^{\infty} \int_{R^n} \int_{R^n} \frac{\left| i \sin \frac{1}{2} z_0 [u(x) + u(y)] + \cos \frac{1}{2} z_0 [u(x) - u(y)] \right|^2}{[|x-y|^2 + z_0^2]^{\frac{n+1+2\alpha}{2}}} dx\, dy\, dz_0,$$

$$= \frac{1}{C(n+1, \alpha)} \int_{-\infty}^{\infty} \int_{R^n} \int_{R^n} \frac{|u(x) + u(y)|^2 \sin^2 \frac{1}{2} z_0}{[|x-y|^2 + z_0^2]^{\frac{n+1+2\alpha}{2}}} dx\, dy\, dz_0,$$

$$+ \frac{1}{C(n+1, \alpha)} \int_{-\infty}^{\infty} \int_{R^n} \int_{R^n} \frac{|u(x) - u(y)|^2 \cos^2 \frac{1}{2} z_0}{[|x-y|^2 + z_0^2]^{\frac{n+1+2\alpha}{2}}} dx\, dy\, dz_0.$$

Integration with respect to z_0 yields the kernel $G_{n+1+2\alpha}$ for the space R^1. By using (4, 1) this kernel can be transformed into the kernel $G_{2n+2\alpha}$ for the space R^n. Making these transformations and using the formula (1, 3) for $C(n+1, \alpha)$ we get

$$(4, 9)\quad \|u\|_\alpha^2 = 2^{n+2\alpha-2}\Gamma(n+\alpha)\Gamma(1+\alpha)$$
$$\left[\frac{\sin \pi\alpha}{\pi}\iint |u(x)+u(y)|^2 \frac{G_{2n+2\alpha}(0)-G_{2n+2\alpha}(x-y)}{|x-y|^{n+2\alpha}}\,dx\,dy\right.$$
$$\left.+\frac{\sin \pi\alpha}{\pi}\iint |u(x)-u(y)|^2 \frac{G_{2n+2\alpha}(0)+G_{2n+2\alpha}(x-y)}{|x-y|^{n+2\alpha}}\,dx\,dy\right].$$

REMARK. — It is easily seen that the first term in the square brackets is equivalent to the L^2 norm of u, the second to $d_\alpha(u)$.

Another interesting formula for $\|u\|_\alpha$, $0<\alpha<1$, is the following

$$(4, 10)\quad \|u\|_\alpha^2 = \frac{2^n\pi^{n/2}\Gamma(n+\alpha)}{\Gamma\left(\frac{n}{2}+\alpha\right)}\int d_\alpha(G_{n+\alpha}(x-y)u(x))\,dy.$$

The Dirichlet integral d_α is taken with respect to x. Beside the formulas used above, we apply here the composition formula $G_{2n+2\alpha} = G_{n+\alpha} * G_{n+\alpha}$.

We give now an important formula connecting $G_\alpha(x-y)$ with the Laplace operator Δ. It will be convenient in this connection to extend the definition of G_α to all real α, by formula (4, 1). This gives for all even integers $\alpha \leqq 0$ a function G_α identically 0.

(4, 11) *For fixed y and $x \neq y$,*

$$(1-\Delta)G_\alpha(x-y) = G_{\alpha-2}(x-y).$$

To prove it, we use y as origin and apply the elementary formula for Δf where f depends only on $r=|x-y|$:

$$\Delta f(r) = \frac{d^2}{dr^2}f(r) + \frac{(n-1)}{r}\frac{d}{dr}f(r).$$

We get therefore

$$(1-\Delta)G_\alpha(x-y) = G_\alpha(r) - \frac{(n-1)}{r}\frac{d}{dr}G_\alpha(r) - \frac{d^2}{dr^2}G_\alpha(r).$$

Comparing with (4, 1) and (3, 8) we transform the right-hand side into $\dfrac{2-\alpha}{r}\dfrac{d}{dr}G_\alpha(r)$ and by (3, 7) and (4, 1) we see that it is $= G_{\alpha-2}(x-y)$.

As corollary we have for all positive integers m

$$(4, 11') \qquad (1-\Delta)^m G_\alpha(x-y) = G_{\alpha-2m}(x-y).$$

The function $G_{2m}(x-y)$ is a fundamental solution for the operator $(1-\Delta)^m$ (more will be said about this in section 7).

§ 5. — The perfect functionnal completion of $\mathcal{F}_\alpha$.

It will be shown that the normed functional class $\mathcal{F}_\alpha$ has a perfect functional completion. For $\alpha > 0$, the exceptional class for the perfect completion is the class of sets on which a potential $G_\alpha g$ of a square integrable function g can be undefined; the functions in the saturated perfect completion are those which are equal except on an exceptional set to such a potential.

For $\alpha > 0$, let $\mathfrak{A}_{2\alpha}$ denote the class of all sets A such that for some square integrable function $g \geqq 0$,

$$(5, 1) \qquad A \subset \mathop{E}_{x}\,[G_\alpha g(x) = +\infty],$$

and let P^α denote the class of all functions u, defined exc. $\mathfrak{A}_{2\alpha}$, such that for some square integrable function g,

$$(5, 2) \qquad u(x) = G_\alpha g(x) \text{ exc. } \mathfrak{A}_{2\alpha}.$$

Since the kernel G_α is integrable, it follows from a standard theorem on products of composition that for every square integrable function g, $G_\alpha g$ is defined and finite almost everywhere and is square integrable. In particular, every set in $\mathfrak{A}_{2\alpha}$ has Lebesgue measure 0. Furthermore, the Fourier transform of $G_\alpha g$ is

$$(5, 3) \quad (G_\alpha g)\hat{}\,(\xi) = (2\pi)^{n/2}\,\hat{G}_\alpha(\xi)\hat{g}(\xi) = (1+|\xi|^2)^{-\alpha/2}\hat{g}(\xi),$$

which shows that [23].

$$(5, 4) \qquad \|G_\alpha g\|_\alpha = \|g\|_{L^2} \qquad \textit{if} \qquad g \in L^2.$$

Formula (5, 4) shows that if $g \in L^2$ then the following conditions are equivalent: (*a*) $g = 0$ almost everywhere; (*b*) $G_\alpha g$ is identically 0; (*c*) $G_\alpha g = 0$ exc. $\mathfrak{A}_{2\alpha}$; (*d*) $G_\alpha g = 0$ almost everywhere; (*e*) $\|G_\alpha g\|_\alpha = 0$. Indeed, it is obvious that each

[23] We use here the expression (1, 9) for $\|u\|_\alpha$.

condition implies the one following, and by (5, 4), (*e*) implies (*a*). Henceforth, P^α will denote the normed class in which the norm is $||u||_\alpha$.

1) $\mathfrak{A}_{2\alpha}$ *is an exceptional class.* P^α *is a complete functional space relative to* $\mathfrak{A}_{2\alpha}$.

Proof. — In order to prove that $\mathfrak{A}_{2\alpha}$ is an exceptional class it must be proved that $\mathfrak{A}_{2\alpha}$ is hereditary (that is, that if $A \in \mathfrak{A}_{2\alpha}$ and $B \subset A$, then $B \in \mathfrak{A}_{2\alpha}$) and σ-additive. The first is obvious. To see the second, if $A_n \in \mathfrak{A}_{2\alpha}$, let g_n be a function $\geqq 0$ in L^2 such that

$$A_n \subset \mathop{E}_{x} [G_\alpha g_n(x) = +\infty] \qquad \textit{and} \qquad ||g_n||_{L^2} \leqq 2^{-n}.$$

Then, if

$$A = \bigcup_{n=1}^{\infty} A_n \qquad \textit{and} \qquad g = \sum_{n=1}^{\infty} g_n,$$

clearly g is a function $\geqq 0$ in L^2 such that (5, 1) holds, so $A \in \mathfrak{A}_{2\alpha}$.

From the equivalence of (*a*) — (*e*) above it follows that P^α is a normed functional class rel. $\mathfrak{A}_{2\alpha}$, i.e. that the conditions $u = 0$ exc. $\mathfrak{A}_{2\alpha}$ and $||u||_\alpha = 0$ are equivalent. From (5, 4) it is obvious that P^α is complete. By definition P^α is saturated. All that remains is to prove the functional space property.

From any sequence converging to 0 we can choose a subsequence $\{u_n\}$ such that

$$\sum_{n=1}^{\infty} ||u_n||_\alpha < \infty.$$

If $u_n = G_\alpha g_n$ except on the set $A_n \in \mathfrak{A}_{2\alpha}$, let

$$g(x) = \sum_{n=1}^{\infty} |g_n(x)|.$$

Then $g \in L^2$, and by the Lebesgue convergence theorem, $G_\alpha g_n(x) \to 0$ for every $x \notin A_0 = \mathop{E}_{x} [G_\alpha g(x) = +\infty]$. Hence $u_n(x) \to 0$ for every x not in the set $\bigcup_{n=0}^{\infty} A_n$, which belongs to $\mathfrak{A}_{2\alpha}$. This proves the functional space property, and the proof of 1) is complete.

As was mentioned before, the norm $||u||_\alpha$ is obviously equivalent to $|u|_\alpha$. Henceforth in this chapter we will consider $\mathcal{F}_\alpha$ as provided with the norm $||u||_\alpha$. The set functions δ, c_1, c_2, etc., are formed with this norm.

2) P^α *is the perfect functional completion of* $\mathcal{F}_\alpha$.

Proof. — In order to show that $\mathcal{F}_\alpha \subset \mathrm{P}^\alpha$, let $u \in \mathcal{F}_\alpha$, put $\hat{g} = (1 + |\xi|^2)^{\alpha/2}\hat{u}$, and let g be the inverse Fourier transform of $\hat{g}$. Since u is in $\mathrm{C}_0^\infty(\mathrm{R}^n)$, $\hat{g}$ is obviously both integrable and square integrable, so that g is continuous, bounded and square integrable. Therefore, $\mathrm{G}_\alpha g$ is continuous and in P^α. Since $\hat{u} = (\mathrm{G}_\alpha g)^\wedge$, it follows that $u = \mathrm{G}_\alpha g$ almost everywhere, but since both functions are continuous, $u = \mathrm{G}_\alpha g$ everywhere. Thus $u \in \mathrm{P}^\alpha$, and so $\mathcal{F}_\alpha \subset \mathrm{P}^\alpha$.

Let $\tilde{\mathcal{F}}_\alpha$ denote the closure of $\mathcal{F}_\alpha$ in P^α. $\tilde{\mathcal{F}}_\alpha$ is a functional completion of $\mathcal{F}_\alpha$ of the type considered in section 2, and it must be shown that $\tilde{\mathcal{F}}_\alpha = \mathrm{P}^\alpha$, and that this completion is perfect. Since for each $u \in \mathrm{P}^\alpha$, $||u||_\alpha$ is finite, it follows from 5) section 2, that each $u \in \mathrm{P}^\alpha$ is equal almost everywhere to some $v \in \tilde{\mathcal{F}}_\alpha$. Both u and v are in P^α, however, and so from $u = v$ a. e. follows $||u - v||_\alpha = 0$, and hence $u = v$ exc. $\mathfrak{A}_{2\alpha}$. Thus $\tilde{\mathcal{F}}_\alpha = \mathrm{P}^\alpha$. All of the results of section 2 are now applicable to P^α.

Finally, we show that if $\mathrm{A} \in \mathfrak{A}_{2\alpha}$ and if S is any sphere, then there exists a Cauchy sequence in $\mathcal{F}_\alpha$ which converges pointwise to $+\infty$ everywhere on $\mathrm{A} \cap \mathrm{S}$. This will show that $\mathrm{A} \cap \mathrm{S}$, and hence A itself, must be an exceptional set for any functional completion of $\mathcal{F}_\alpha$, and this will complete the proof of 2). By 6), section 2 (take $\varphi \in \mathrm{C}_0^\infty$ and $= 1$ on S), it is sufficient to show that there is a Cauchy sequence $\{u_n\}$ in P^α such that u_n is of class C^∞ and such that $u_n(x) \to +\infty$ at every point of A. The existence of such a sequence is given by the following proposition.

3) *Let* $\{e_\rho\}$ *be a family of regularizing functions. If* $u \in \mathrm{L}^2$, *then* $u_\rho = u * e_\rho$ *is of class* C^∞ *and belongs to* P^α *for all* α. *If* $u \in \mathrm{P}^\alpha$, *then* $||u_\rho||_\alpha \leq ||u||_\alpha$ *and* $u_\rho \to u$ *both in* P^α *and pointwise exc.* $\mathfrak{A}_{2\alpha}$. *Moreover, if* $u = \mathrm{G}_\alpha g$ *where* $g \geq 0$, *then* $u_\rho \to u$ *pointwise everywhere.*

Proof. — It is clear that u_ρ is of class C^∞. For the Fourier transform of u_ρ we have

$$\hat{u}_\rho(\xi) = (2\pi)^{n/2}\hat{u}(\xi)\hat{e}_\rho(\xi) = (2\pi)^{n/2}\hat{u}(\xi)\hat{e}(\rho\xi).$$

Since e is of class C_0^∞, it follows that the product of $\hat{e}$ with any polynomial is bounded, and hence $||u_\rho||_\alpha < \infty$ so that $u_\rho \in P^\alpha$ for all α.

If $u \in P^\alpha$, then

$$||u_\rho||_\alpha^2 = \int (1 + |\xi|^2)^\alpha |(2\pi)^{n/2}\hat{e}(\rho\xi)|^2 |\hat{u}(\xi)|^2 d\xi \leqq ||u||_\alpha^2$$

since $|\hat{e}(\xi)| \leqq (2\pi)^{-n/2}$.

Finally, if $u = G_\alpha g$ with $g \in L^2$, then $u_\rho = G_\alpha(g * e_\rho)$. It is well known that when $g \in L^2$, $g * e_\rho \in L^2$ and $g * e_\rho$ converges in L^2 to g. Hence $\{u_\rho\}$ converges in P^α to u. If $g \geqq 0$, then by the corollary to the mean value theorem in the last section $u_\rho \to u$ pointwise everywhere. It follows that whether $g \geqq 0$ or not, $u_\rho(x) \to u(x)$ at every x such that $u(x) = G_\alpha g(x)$ is defined, i.e. pointwise exc. $\mathfrak{A}_{2\alpha}$.

4) P^α *is a proper functional space if and only if* $\alpha > \frac{n}{2}\cdot$ *If* $\alpha > \frac{n}{2}$ *and* m *is an integer* $< \alpha - \frac{n}{2}$, *then every function in* P^α *is of class* C^m.

PROOF. — If $\alpha \leqq \frac{n}{2}$, then, by (4, 2) $G_\alpha(x-y)$ as a function of y is not square integrable, and there exists a square integrable function $g \geqq 0$ such that

$$\int G_\alpha(x-y)\, g(y) dy = +\infty,$$

which shows that the set $\{x\}$ belongs to $\mathfrak{A}_{2\alpha}$. On the other hand, if $\alpha > \frac{n}{2}$, then, by (4, 2) and (4, 3), G_α is square integrable, so for every square integrable function g, $G_\alpha g$ is defined everywhere and is a bounded continuous function. The second statement is proved by making use of (4, 4) and a similar argument. It can also be proved easily by Fourier transforms.

It can be shown that the exceptional class $\mathfrak{A}_{2\alpha}$ is precisely the class of sets of capacity 0 for the capacity c_1 associated with the space P^α ([24]). This result depends on the strong majoration property defined in section 3, of chapter I.

5) P^α *has the strong majoration property.*

([24]) From now on, $\mathfrak{L}$ and δ are the class of sets and set function associated with P^α (not as earlier, those associated with $\mathcal{F}_\alpha$). The capacity c_1 associated with P^α is the same as that associated with $\mathcal{F}_\alpha$.

Indeed, if $u = G_\alpha g$ and if $u' = G_\alpha|g|$, then $u'(x) \geqq |u(x)|$ exc. $\mathfrak{A}_{2\alpha}$ and $\|u'\|_\alpha = \|u\|_\alpha$.

6) *If* $B \in \mathfrak{L}$, *then* $c_1(B) < \infty$. *If* $c_1(B) < \infty$, *then* $B \in \mathfrak{L}$ *and there is a function* $g \geqq 0$ *in* L^2 *such that* $u = G_\alpha g$ *satisfies*

$$(5, 5) \quad u \geqq 1 \text{ on } B \text{ exc. } \mathfrak{A}_{2\alpha} \quad \text{and} \quad \|u\|_\alpha = \delta(B) = c_1(B).$$

All the assertions follow from the general theory in chapter I (propositions 2 and 3, section 3) except the fact that the minimizing function u is equal to $G_\alpha g$ with $g \geqq 0$. But if $u = G_\alpha f$, take $g = |f|$.

It is evident from 6) that

7) $\mathfrak{A}_{2\alpha}$ *is the class of sets of capacity* 0 *for the capacity* c_1.

Before beginning a detailed study of capacities we record one additional consequence of the general theory (chapter I, § 3, 3)).

8) *If* B *is the union of the increasing sequence* $\{B_n\}$, *then* $c_1(B) = \lim c_1(B_n)$.

Remark. — It was mentioned in the footnote to proposition 1, § 2, that the perfect functional completion of $\mathcal{F}_0$ is L^2. In order to maintain a systematic notation we shall sometimes use P^0 to designate L^2 and $\mathfrak{A}_0$ to designate the class of sets of Lebesgue measure 0. It is not difficult to prove (see [1]) that the set functions associated with P^0 satisfy

$$\delta(A)^2 = c_1(A)^2 = c_2(A) = |A|.$$

§ 6. — Capacities.

A theory of capacity of the classical type rests ultimately on the use of positive pseudo-reproducing kernels. In the classical theory of Riesz and Frostman of capacity of order 2α the kernel is the Riesz kernel (1, 7), which is the pseudo-reproducing kernel of the completion of $C_0^\infty(R^n)$ with respect to the norm $\sqrt{d_\alpha}$. In the present theory the kernel is $G_{2\alpha}$, which is the pseudo-reproducing kernel for P^α ([25]). In the first part of the section we assume $\alpha > 0$.

([25]) It is not necessary that the reader be acquainted with the theory of pseudo-reproducing kernels. The necessary details will be given fully. Some additional results on this subject can be found in [2].

The capacity of order 2α of a compact set C is defined to be the number $\gamma_{2\alpha}(C)$ determined by

$$(6,\ 1) \qquad \frac{1}{\gamma_{2\alpha}(C)} = \inf \iint G_{2\alpha}(x-y)\, d\mu(x)\, d\mu(y),$$

where the infimum is taken over all positive Borel measures on C of total mass 1. The inner capacity $\gamma^i_{2\alpha}(A)$ of an arbitrary set A is the upper bound of the capacities of the compact sets $C \subset A$. The outer capacity $\gamma^0_{2\alpha}(A)$ of an arbitrary set A is the lower bound of the inner capacities of the open sets $G \supset A$.

Remark 1. — The standard capacity of order $2\alpha < n$ is obtained in the same way by simply replacing the kernel $G_{2\alpha}$ by the Riesz kernel (1, 7). The standard capacity $\tilde{\gamma}_{2\alpha}$ is easily seen to be invariant under translations and rotations and to have the following property with respect to a homothetic transformation with ratio t: $\tilde{\gamma}_{2\alpha}(tC) = t^{n-2\alpha}\tilde{\gamma}_{2\alpha}(C)$. Our present capacity obviously retains the invariance under translations and rotations, but it does not have as simple a behavior with respect to homothetic transformations. It is easy to show by means of (4, 2) that the following relation holds between our present capacity and the standard capacity:

$$\lim_{t \to 0} t^{2\alpha-n}\, \gamma_{2\alpha}(tC) = \tilde{\gamma}_{2\alpha}(C),$$

for every compact set C. Corresponding statements hold for the inner and outer capacities of arbitrary bounded sets, and in each case the limit is uniform when the diameter of the set remains bounded.

A capacity of order n has been studied under the name of logarithmic capacity, studied rather extensively in the case $n = 2$ and rather sketchily for larger n. The logarithmic capacity is obtained as above by replacing the kernel G_n by the kernel $\log \frac{r}{|x|}$, where r is any number larger than the diameter of the set C. The resulting set function $\tilde{\gamma}_n(r, C)$ is then defined for all compact sets C of diameter $\leqq r$. Sometimes the set function

$$\tilde{\gamma}_n(C) = r \exp\left(-\frac{1}{\tilde{\gamma}_n(r, C)}\right),$$

which is independent of r, is used in place of the logarithmic capacity.

It is clear from (4, 2) that for each r there is a constant c such that

$$\frac{1}{c}\tilde{\gamma}_n(r, C) \leqq \gamma_n(C) \leqq c\tilde{\gamma}_n(r, C)$$

holds for every compact set C of diameter $\leqq r$. The exact limiting relation corresponding to the one given above for $\gamma_{2\alpha}$ and $\tilde{\gamma}_{2\alpha}$ is a little more complicated in this case.

If γ is Euler's constant, we have

$$\lim_{t\to 0}\left\{\frac{2^{n-1}\pi^{n/2}\Gamma(n/2)}{\gamma_n(tC)} - \log\frac{1}{t}\right\} + \log\frac{r}{2} + \gamma = \frac{1}{\tilde{\gamma}_n(r; C)}$$

for every compact set C of diameter $\leqq r$. Equivalently

$$\lim_{t\to 0} 2e^{-\gamma}\frac{1}{t}\exp\left\{-\frac{2^{n-1}\pi^{n/2}\Gamma(n/2)}{\gamma_n(tC)}\right\} = \tilde{\gamma}_n(C)$$

for every compact set C. To establish these relations, an improvement of (4, 2) is needed, namely

$$as \quad x \to 0, \quad G_n(x) = \frac{1}{2^{n-1}\pi^{n/2}\Gamma(n/2)}\log\frac{2e^{-\gamma}}{|x|} + O(|x|^{2-\epsilon})$$ [26].

Since $G_\alpha(r)/R_\alpha(r)$ (with $0 < \alpha < n$) is decreasing for $0 < r < \infty$ from 1 to 0 (using (3, 7), (3, 3), (3, 4) and (3, 5)), we obtain for sets A of diameter $\leqq r$

$$\tilde{\gamma}^0_\alpha(A) \leqq \gamma^0_\alpha(A) \leqq \frac{R_\alpha(r)}{G_\alpha(r)}\tilde{\gamma}^0_\alpha(A);$$

hence the sets with $\gamma^0_\alpha(A) = 0$ are the same as those with $\tilde{\gamma}^0_\alpha(A) = 0$.

REMARK 2. — Comparison of our potentials with the Riesz potentials of the same order $\alpha < \frac{n}{2}$ shows that locally the

[26] The first part of (4, 2) can be improved to

$$as\ x \to 0, \quad G_\alpha(x) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha\pi^{n/2}\Gamma(\alpha/2)}|x|^{\alpha-n} + O\left(\max\left(|x|^{\alpha-n+2}, 1\right)\right)$$

for $\alpha < n$, but this will not be needed.

potentials are the same, but globally (because of the exponential decrease of G_α) the Riesz potentials form a larger class (see Remark in § 9). Consequently, there are many statements which are true for the functions in P^α and untrue for the Riesz potentials. For instance, every function in P^α is square integrable, along with all derivatives of order $\leqq \alpha$. Also, the product of a function in P^α with a bounded function of class $C^{(\alpha^*, 1)}$ is in P^α (see 6) § 2). In addition, the proofs of many common theorems are simpler for P^α. On the other hand, several of the formulas become more complicated due to the fact that $G_{2\alpha}$ is not homogeneous — for instance, the formulas in Remark 1 and those in proposition 20) below.

The potential of a measure with respect to the kernel $G_{2\alpha}$ is defined as follows ([27])

$$G_{2\alpha}\mu(x) = \int G_{2\alpha}(x-y)d\mu(y).$$

$G_{2\alpha}\mu$ is defined everywhere, provided $+\infty$ is admitted as a value, and is lower semi-continuous. Of primary interest are the measures μ for which the 2α-energy

$$(6,2) \quad \|\mu\|_{2\alpha}^2 = \iint G_{2\alpha}(x-y)d\mu(x)d\mu(y) = \int G_{2\alpha}\mu d\mu$$

is finite. However, we shall begin by proving a few results about arbitrary measures.

If μ is a measure with finite total mass, then, by (4, 6′),

$$(6,3) \quad \int G_{2\alpha}\mu(x)\,dx = \iint G_{2\alpha}(x-y)d\mu(y)dx = |\mu|,$$

and in particular, $G_{2\alpha}\mu$ is finite almost everywhere and is an integrable function. By using this and (4, 3) the following result is easily proved.

1) *If μ is a measure such that*

$$(6,4) \qquad \int (1+|x|)^{\frac{2\alpha-n-1}{2}} e^{-|x|}d\mu(x) < \infty$$

then $G_{2\alpha}\mu$ is finite almost everywhere and is integrable over every bounded set. If (6, 4) holds and $2\alpha > n$, $G_{2\alpha}\mu$ is finite and

([27]) Henceforth the term measure will refer to a positive Borel measure unless otherwise stated. If μ is such a measure, $|\mu| = \mu(R^n)$ is its total mass (possibly $+\infty$). A set E is said to be a support of μ if $\mu(R^n - E) = 0$; a support need not be closed.

continuous everywhere. If (6, 4) *does not hold, then* $G_{2\alpha}\mu$ *is identically* $+\infty$.

The following result does not require proof.

2) *The mean value theorem (proposition* 1, § 4*) holds for potentials* $G_{2\alpha}\mu$ *as well as for potentials* $G_\alpha g$.

The next two propositions were proved by Frostman [11] for the Riesz kernel. The proofs given here are quite similar.

3) *Let* μ *be a measure with a compact support* C. *There is a constant c (depending on the diameter of* C, α, *and n, but not on* μ*) such that if* $G_{2\alpha}\mu(x) \leqq 1$ a. e. (μ) [28] *then* $G_{2\alpha}\mu(x) \leqq c$ *everywhere.*

PROOF. — Since $G_{2\alpha}\mu$ is lower semi-continuous, the subset F of C on which $G_{2\alpha}\mu \leqq 1$ is closed, and by assumption F contains a support of μ. If for an arbitrary point x, $\bar{x}$ denotes a point in F closest to x, then for any point $y \in F$, $|x-y| \geqq \frac{1}{2}|\bar{x}-y|$. Hence, $G_{2\alpha}(x-y) \leqq G_{2\alpha}\left(\frac{1}{2}(\bar{x}-y)\right)$. By (4, 2), there is a constant c (depending only on the diameter of C, α, and n) such that for every $\rho \leqq$ the diameter of C, $G_{2\alpha}(\rho/2) \leqq cG_{2\alpha}(\rho)$. Thus, we have $G_{2\alpha}\mu(x) \leqq cG_{2\alpha}\mu(\bar{x}) \leqq c$.

4) *If* $G_{2\alpha}\mu$ *is continuous on a closed support of* μ, *then* $G_{2\alpha}\mu$ *is continuous on* R^n.

PROOF. — In view of proposition 1) we may assume $2\alpha \leqq n$. This implies that if $\mu(\{x_0\}) > 0$, $G_{2\alpha}\mu(x_0) = +\infty$, hence, by our hypothesis, μ cannot have point-masses. Let F be a closed support of μ on which $G_{2\alpha}\mu$ is continuous and let x_0 be an arbitrary point. For each n, write $\mu = \mu_n' + \mu_n''$, where μ_n'' is the restriction of μ to the sphere $S(x_0, 1/n)$. Since $\mu(\{x_0\}) = 0$, it follows that $G_{2\alpha}\mu(x) = \lim G_{2\alpha}\mu_n'(x)$ for each x. Therefore, if for arbitrary $\varepsilon > 0$,

$$O_n = F \cap \mathop{E}_{x}[G_{2\alpha}\mu_n'(x) > G_{2\alpha}\mu(x) - \epsilon],$$

then $F = \bigcup_{n=1}^{\infty} O_n$. Moreover, O_n is relatively open in F, for $G_{2\alpha}\mu$ is continuous on F and $G_{2\alpha}\mu_n'$ is lower semi-continuous. Consequently for sufficiently large n, O_n contains $F \cap S(x_0, 1)$,

[28] a.e. (μ) means « except on a set of μ-measure 0 ».

and, therefore, for sufficiently large n, $G_{2\alpha}\mu_n'' \leqq \varepsilon$ on $F \cap S(x_0, 1)$, and by proposition 3, $G_{2\alpha}\mu_n'' \leqq c\,\varepsilon$ everywhere. Thus

$$G_{2\alpha}\mu_n' \to G_{2\alpha}\mu$$

uniformly, and since each $G_{2\alpha}\mu_n'$ is continuous at x_0, so is $G_{2\alpha}\mu$.

We turn now to measures for which the 2α-energy defined in (6, 2) is finite. The class of such measures will be called $\Omega_{2\alpha}$. Note that (6, 3) shows that $\Omega_{2\alpha}$ contains the restriction of the Lebesgue measure to any bounded set.

5) *The following conditions on a measure μ are equivalent.*

(*a*) $\mu \in \Omega_{2\alpha}$.

(*b*) $G_\alpha\mu$ *is square integrable.*

(*c*) $G_{2\alpha}\mu \in P^\alpha$.

(*d*) *Every function in* P^α *is* μ*-integrable.*

(*e*) *Every function in* P^α *is* μ*-integrable, and the integral is a continuous linear functional on* P^α.

Proof. — The composition formula (4, 7) gives

$$G_{2\alpha}\mu = G_\alpha G_\alpha\mu \qquad \textit{and} \qquad \|\mu\|_{2\alpha} = \|G_\alpha\mu\|_{L^2} \qquad \textit{and}$$

$$(6, 5) \quad \int G_\alpha g\,d\mu = \int G_\alpha\mu(x)g(x)dx \quad \textit{for any} \quad g \geqq 0.$$

It follows that $\mu \in \Omega_{2\alpha}$ if and only if $G_\alpha\mu \in L^2$, and that $G_\alpha\mu \in L^2$ if and only if every function in P^α is μ-integrable. Therefore, (*a*), (*b*), (*d*) and (*e*) are equivalent and imply (*c*).

If (*c*) holds, then for some $g \in L^2$, $G_\alpha g = G_{2\alpha}\mu$, and for every $f \geqq 0$ in L^2 we have,

$$\int G_\alpha f\,g dx = \int G_\alpha g\,f dx = \int G_\alpha G_\alpha\mu\,f dx = \int G_\alpha f\,G_\alpha\,\mu dx.$$

Hence, for every $f \in L^2$,

$$\int G_\alpha f\,g dx = \int G_\alpha f\,G_\alpha\mu dx,$$

and since f can be chosen so that $G_\alpha f$ is an arbitrary function in $C_0^\infty(R^n)$, it follows that $G_\alpha\mu = g$ a.e. Therefore, $G_\alpha\mu \in L^2$.

The third formula in (6, 5) gives the equation which expresses the reproducing property of $G_{2\alpha}$.

$$(6, 6) \quad \textit{For every } u \in P^\alpha \textit{ and } \mu \in \Omega_{2\alpha}, \qquad \int u\,d\mu = (u, G_{2\alpha}\mu)_\alpha \text{ [29]}.$$

[29] $(u, v)_\alpha$ denotes the inner product in the Hilbert space P^α.

6) *If* μ *and* ν *belong to* $\Omega_{2\alpha}$ *and if* $G_{2\alpha}\mu = G_{2\alpha}\nu$ *almost everywhere, then* $\mu = \nu$ [30].

PROOF. — For every $f \in L^2$ (first for $f \geqq 0$, then for all $f \in L^2$)

$$\int G_{2\alpha} f d\mu = \int G_{2\alpha}\mu f dx = \int G_{2\alpha}\nu f dx = \int G_{2\alpha} f d\nu,$$

and f can be chosen so that $G_{2\alpha} f$ is an arbitrary function in $C_0^\infty(R^n)$.

Proposition 6) (and the second equality in (6, 5)) show that $\|\mu\|_{2\alpha}^2$ is a positive definite quadratic form on $\Omega_{2\alpha}$. If $(\mu, \nu)_{2\alpha}$ denotes the corresponding bilinear form, then

$$\text{(6, 7)} \quad (\mu, \nu)_{2\alpha} = \int G_{2\alpha}(x-y) d\mu(y) d\nu(x) = \int G_{2\alpha}\mu(x) d\nu(x)$$
$$= \int G_{2\alpha}\nu(y) d\mu(y) = (G_{2\alpha}\mu, G_{2\alpha}\nu)_\alpha.$$

By virtue of 6) and (6, 7) the correspondence between the measures of finite 2α-energy and their potentials is 1-1, linear, and inner product preserving. In order to simplify the notation, we shall use the same symbol $\Omega_{2\alpha}$ to denote both the class of measures of finite 2α-energy and the class of their potentials.

7) $\Omega_{2\alpha}$ *is a closed convex cone in* P^α. *The subspace generated by* $\Omega_{2\alpha}$ *is dense in* P^α [31].

PROOF. — It is obvious that $\Omega_{2\alpha}$ is a convex cone. If $u_n \to u$ in P^α, where $u_n = G_{2\alpha}\mu_n$, then for each $\varphi \in C_0^\infty(R^n)$

$$(\varphi, u)_\alpha = \lim_{n\to\infty} (\varphi, u_n)_\alpha = \lim_{n\to\infty} \int \varphi(x) d\mu_n(x).$$

Hence, if $\varphi \geqq 0$, then $(\varphi, u)_\alpha \geqq 0$, and by the well known theorem of Riesz on the representation of non-negative linear functionals there exists a measure μ such that

$$\text{(6, 8)} \qquad (\varphi, u)_\alpha = \int \varphi(x)\, d\mu(x) \quad \textit{for} \quad \varphi \in C_0^\infty(R^n).$$

In general, if a sequence $\{\mu_n\}$ of measures is such that

$$\int \varphi(x)\, d\mu(x) = \lim_{n\to\infty} \int \varphi(x)\, d\mu_n(x) \quad \textit{for} \quad \varphi \in C_0^\infty(R^n),$$

[30] The same statement can be proved (by a different argument) when μ and ν are only supposed to satisfy (6, 4).

[31] For RIESZ potentials this result is due to H. CARTAN [5, 6] and J. DENY [8].

then for every non-negative lower semi-continuous function φ

$$\int \varphi(x)\, d\mu(x) \leqq \liminf \int \varphi(x)\, d\mu_n(x).$$

Applying this remark first to $\varphi = G_{2\alpha}$ and then to $\varphi = G_{2\alpha}\mu$, we get

$$\|\mu\|_{2\alpha} \leqq \liminf \|\mu_n\|_{2\alpha} < \infty,$$

so $\mu \in \Omega_{2\alpha}$. By (6, 6) and (6, 8), $(\nu, u)_\alpha = (\nu, G_{2\alpha}\mu)_\alpha$ for every $\nu \in C_0^\infty(R^n)$, and since such ν are dense in P^α, $u = G_{2\alpha}\mu \in \Omega_{2\alpha}$.

Finally, to see that the subspace generated by $\Omega_{2\alpha}$ is dense in P^α, we observe that if a function $u \in P^\alpha$ is such that

$$\int u\, d\mu = 0 \qquad \textit{for all} \qquad \mu \in \Omega_{2\alpha},$$

then, since the restriction of the Lebesgue measure to any bounded set belongs to $\Omega_{2\alpha}$, the mean value of u over every sphere is 0, and by the mean value theorem (proposition 1, § 5) $u = 0$ exc. $\mathfrak{A}_{2\alpha}$.

For each set A in R^n, $\Omega_{2\alpha}(A)$ will denote the class of measures $\mu \in \Omega_{2\alpha}$ which are supported by A (i. e., $\mu(R^n - A) = 0$), as well as the corresponding class of potentials. It results from 7) that if A is a closed set then $\Omega_{2\alpha}(A)$ is a closed convex cone, and if A is any set $\overline{\Omega_{2\alpha}(A)} \subset \Omega_{2\alpha}(\overline{A})$. It is obvious from the definition that every restriction of a measure in $\Omega_{2\alpha}(A)$ belongs also to $\Omega_{2\alpha}(A)$. A similar statement is needed for $\overline{\Omega_{2\alpha}(A)}$.

8) *If f is a bounded non-negative Borel function and $\mu \in \overline{\Omega_{2\alpha}(A)}$ then the measure μ_f defined by*

$$\mu_f(E) = \int_E f(x)\, d\mu(x)$$

belongs to $\overline{\Omega_{2\alpha}(A)}$. In particular, the restriction of μ to any Borel set belongs to $\overline{\Omega_{2\alpha}(A)}$.

Proof. — We prove first that 8) holds when $f \in C_0^\infty(R^n)$. Since $\mu \in \overline{\Omega_{2\alpha}(A)}$, there is a sequence $\{\mu_n\}$ in $\Omega_{2\alpha}(A)$ converging to μ. It is clear that

$$\|(\mu_n)_f\|_{2\alpha} \leqq (\sup f)\, \|\mu_n\|_{2\alpha}$$

and for every $\varphi \in C_0^\infty(R^n)$

$$(\varphi, G_{2\alpha}(\mu_n)_f)_\alpha = \int \varphi d(\mu_n)_f = \int f\varphi\, d\mu_n = (f\varphi, G_{2\alpha}\mu_n)_\alpha$$
$$\rightarrow (f\varphi, G_{2\alpha}\mu)_\alpha = (\varphi, G_{2\alpha}\mu_f)_\alpha.$$

Therefore, $G_{2\alpha}(\mu_n)_f \rightarrow G_{2\alpha}\mu_f$ weakly, and as $\overline{\Omega_{2\alpha}(A)}$ is closed and convex (hence weakly closed) $G_{2\alpha}\mu_f \in \overline{\Omega_{2\alpha}(A)}$, which proves 8) if $f \in C_0^\infty(R)$.

Now, it is well known that there exists a uniformly bounded sequence $\{f_n\}$ in $C_0^\infty(R^n)$ such that $f_n \rightarrow f$ a.e. (μ). Then for every $\varphi \in P^\alpha$

$$(\varphi, G_{2\alpha}\mu_{f_n})_\alpha = \int \varphi f_n\, d\mu \rightarrow \int \varphi f\, d\mu = (\varphi, G_{2\alpha}\mu_f)_\alpha.$$

Therefore, $G_{2\alpha}\mu_{f_n} \rightarrow G_{2\alpha}\mu_f$ weakly, and as before $G_{2\alpha}\mu_f \in \overline{\Omega_{2\alpha}(A)}$.

Next we describe the process of sweeping or balayage. If A is any set in R^n and if $\varphi \in P^\alpha$, then, since $\overline{\Omega_{2\alpha}(A)}$ is closed and convex, there is a unique $u = G_{2\alpha}\mu \in \overline{\Omega_{2\alpha}(A)}$ which realizes the minimum distance form φ to functions of $\overline{\Omega_{2\alpha}(A)}$. This u, or the corresponding measure μ, is called the result of sweeping φ onto A. Simple and familiar arguments show that if μ is the result of sweeping φ onto A, then $(G_{2\alpha}\mu - \varphi, G_{2\alpha}\nu)_\alpha \geqq 0$ for every $\nu \in \overline{\Omega_{2\alpha}(A)}$, and $(G_{2\alpha}\mu - \varphi, G_{2\alpha}\mu)_\alpha = 0$. The inequality, combined with proposition 8), shows that $G_{2\alpha}\mu \geqq \varphi$ a. e. (ν) for each $\nu \in \overline{\Omega_{2\alpha}(A)}$; and then the equality shows that $G_{2\alpha}\mu = \varphi$ a. e. (μ).

9) *Let* μ *be the result of sweeping* φ *onto* A. *Then*

(*a*) $G_{2\alpha}\mu \geqq \varphi$ *a. e.* (ν) *for each* $\nu \in \overline{\Omega_{2\alpha}(A)}$.

(*b*) $G_{2\alpha}\mu = \varphi$ *a. e.* (μ).

(*c*) *If* A *is either open or closed,* $G_{2\alpha}\mu \geqq \varphi$ *on* A *exc.* $\mathfrak{A}_{2\alpha}$.

(*d*) *If* A *is open and* φ *is continuous on* A, $G_{2\alpha}\mu \geqq \varphi$ *everywhere on* A.

Proof. — Parts (*a*) and (*b*) have been proved already. Part (*d*) and the half of (*c*) which is concerned with open sets follow from the mean value theorem.

To prove the remaining half of (*c*), let $A = \bigcap_{k=1}^{\infty} D_k$, where the D_k are open and $\overline{D}_k \subset D_{k-1}$. Then $\Omega_{2\alpha}(A) = \bigcap_{k=1}^{\infty} \overline{\Omega_{2\alpha}(D_k)}$, from which if follows easily that if μ_k is the result of sweeping v onto D_k, then $G_{2\alpha}\mu_k \to G_{2\alpha}\mu$. By what has been shown, $G_{2\alpha}\mu_k \geqq v$ on D_k exc. $\mathfrak{A}_{2\alpha}$. Hence $G_{2\alpha}\mu_k \geqq v$ on A exc. $\mathfrak{A}_{2\alpha}$, and by the functional space property, $G_{2\alpha}\mu \geqq v$ on A exc. $\mathfrak{A}_{2\alpha}$.

If A is a bounded set there exist functions in P^α which are equal to 1 everywhere on $\bar{A}$. The measure μ_A that results from sweeping any such function onto A is called the capacitary distribution of A. The corresponding potential $u_A = G_{2\alpha}\mu_A$ is called the capacitary potential of A [32].

10) (*a*) $u_A \geqq 1$ a.e. (ν) *for each* $\nu \in \Omega_{2\alpha}(\bar{A})$; *in particular,* $u_A \geqq 1$ *on* A a.e. (ν) *for each* $\nu \in \Omega_{2\alpha}$.

(*b*) $u_A = 1$ a.e. (μ_A).

(*c*) *There is a constant c depending only on r, α, and n such that if the diameter of* A *is* $\leqq r$, *then* $u_A \leqq c$ *everywhere.*

(*d*) $|\mu_A| = ||\mu_A||^2_{2\alpha} = ||u_A||^2_\alpha$.

(*e*) *If* A *is open,* $u_A \geqq 1$ *everywhere on* A.

(*f*) *If* A *is closed,* $u_A \geqq 1$ *on* A *exc.* $\mathfrak{A}_{2\alpha}$.

PROOF. — Parts (*a*), (*b*), (*e*), and (*f*) are direct consequences of 9). Part (*c*) follows from part (*b*) and proposition 3). Part (*d*) follows from part (*b*) and (6, 6) and (6, 7).

The next proposition, which shows that u_B can be taken for the function u of proposition 6), § 5 and that the normalized capacitary distribution realizes the minimum in (6, 1), is the first step in showing that the relations $(c_1)^2 = c_2 = \gamma^0_{2\alpha}$ hold.

11) *If* C *is a compact set, then* u_C *minimizes the expression* $||v||^2_\alpha$ *among all* $v \in P^\alpha$ *such that* $v \geqq 1$ *on* C a.e. (ν) *for each* $\nu \in \Omega_{2\alpha}$. *Moreover,* $\frac{\mu_C}{|\mu_C|}$ *realizes the minimum in* (6, 1), *and*

$$||u_C||^2_\alpha = ||\mu_C||^2_{2\alpha} = |\mu_C| = \delta(C)^2 = \gamma_{2\alpha}(C).$$

PROOF. — If $v \geqq 1$ on C a.e. (ν) for each $\nu \in \Omega_{2\alpha}$, then in particular $v \geqq 1$ a.e. (μ_C). and hence

$$||u_C||^2_\alpha = |\mu_C| \leqq \int v \, d\mu_C = (v, u_C)_\alpha \leqq ||v||_\alpha ||u_C||_\alpha.$$

Therefore, $||u_C||_\alpha \leqq ||v||_\alpha$, and the first part of 11) is proved.

[32] Is is easily seen that μ_A and u_A are independent of the particular function which is swept onto A.

From this and 10)-(f) it follows that u_C realizes the minimum in the definition of $\delta(C)$ ([33]) which gives

$$||u_C||_\alpha^2 = ||\mu_C||_{2\alpha}^2 = |\mu_C| = \delta(C)^2.$$

If $\nu \in \Omega_{2\alpha}(C)$ and $|\nu| = 1$, then by 10)-(a)

$$1 = |\nu| \leqq \int u_C \, d\nu = (\mu_C, \nu)_{2\alpha} \leqq ||\mu_C||_{2\alpha} ||\nu||_{2\alpha}.$$

Thus,

$$||\nu||_{2\alpha}^2 \geqq \frac{1}{||\mu_C||_{2\alpha}^2}$$

while if $\nu = \dfrac{\mu_C}{|\mu_C|}$,

$$||\nu||_{2\alpha}^2 = \frac{||\mu_C||_{2\alpha}^2}{|\mu_C|^2} = \frac{1}{|\mu_C|} = \frac{1}{||\mu_C||_{2\alpha}^2}$$

This shows that $\dfrac{\mu_C}{|\mu_C|}$ realizes the minimum in (6, 1) and that the value of the minimum is as stated.

12) *If* A *is a set* F_σ, *then* $\gamma_{2\alpha}^i(A) = c_1(A)^2$.

Proof. — By 11) and (5, 5), proposition 12) holds for compact sets. Therefore, by the definition of $\gamma_{2\alpha}^i(A)$ and proposition 8), § 5, proposition 12) holds for sets F_σ.

13) *For every set* A, $c_1(A) = \inf c_1(D)$, *the infimum being taken over all open sets* $D \supset A$.

Proof. — It can be supposed that $c_1(A) < \infty$, in which case, by 6) § 5, there exists, $g \geqq 0$ in L^2 such that $G_\alpha g \geqq 1$ on A exc. $\mathfrak{A}_{2\alpha}$ and $||G_\alpha g||_\alpha = c_1(A)$. For each $\rho < 1$ let

$$D_\rho = \underset{x}{E}[G_\alpha g(x) > \rho].$$

D_ρ is an open set, $D_\rho \supset A$ exc. $\mathfrak{A}_{2\alpha}$, and

$$c_1(D_\rho) = \delta(D_\rho) \leqq \frac{1}{\rho}\delta(A) = \frac{1}{\rho}c_1(A).$$

Let $g_\rho \in L^2$ be $\geqq 0$ and such that $||G_\alpha g_\rho||_\alpha = 1$ and such that $A - D_\rho \subset \underset{x}{E}[G_\alpha g_\rho(x) = +\infty]$. Let $D'_\rho = \underset{x}{E}\left[G_\alpha g_\rho(x) > \dfrac{1}{\varepsilon}\right]$.

([33]) We use the fact that, by definition of $\mathfrak{A}_{2\alpha}$ and by (6, 6), each set in $\mathfrak{A}_{2\alpha}$ is of measure 0 for each $\nu \in \Omega_{2\alpha}$.

Then D'_ρ is open and $c_1(D'_\rho) = \delta(D'_\rho) \leqq \epsilon$. Hence $A \subset D_\rho \cup D'_\rho$, $D_\rho \cup D'_\rho$ is open and

$$c_1(D_\rho \cup D'_\rho) \leqq c_1(D_\rho) + c_1(D'_\rho) \leqq \frac{1}{\rho} c_1(A) + \epsilon .$$

14) *For every set* A, $\gamma^0_{2\alpha}(A) = c_1(A)^2$.

Proof. — It is obvious that if A is open then $\gamma^0_{2\alpha}(A) = \gamma^i_{2\alpha}(A)$. Therefore, from 12) it follows that 14) holds if A is open. From this, 13), and the definition of $\gamma^0_{2\alpha}(A)$, it follows that 14) holds for arbitrary A.

15) *If* $A = \bigcup_{k=1}^{\infty} A_k$, *then* $\gamma^0_{2\alpha}(A) \leqq \sum_{k=1}^{\infty} \gamma^0_{2\alpha}(A_k)$.

Proof. — We first show that if C_1 and C_2 are compact and $C = C_1 \cup C_2$, then $\gamma_{2\alpha}(C) \leqq \gamma_{2\alpha}(C_1) + \gamma_{2\alpha}(C_2)$. If μ_1 and μ_2 denote the restrictions of μ_C to C_1 and C_2, then

$$\gamma_{2\alpha}(C) = |\mu_C| \leqq |\mu_1| + |\mu_2|.$$

On the other hand,

$$|\mu_1| \leqq \int u_{C_1}\, d\mu_1 \leqq ||u_{C_1}||_\alpha ||\mu_1||_{2\alpha}.$$

Since $G_{2\alpha}\mu_1 \leqq G_{2\alpha}\mu_C = 1$ a.e. (μ_C), it follows that

$$G_{2\alpha}\mu_1 \leqq 1 \text{ a.e. } (\mu_1)$$

so that

$$||\mu_1||^2_{2\alpha} = \int G_{2\alpha}\mu_1\, d\mu_1 \leqq |\mu_1|.$$

Combining these inequalities we get

$$|\mu_1| \leqq ||u_{C_1}||^2_\alpha = \gamma_{2\alpha}(C_1).$$

Similarly, $|\mu_2| \leqq \gamma_{2\alpha}(C_2)$.

Now let D_1 and D_2 be open and let $D = D_1 \cup D_2$. It is well known that an arbitrary compact subset C of D can be expressed as $C = C_1 \cup C_2$, where $C_1 \subset D_1$, $C_2 \subset D_2$, and both are compact. By what has been shown,

$$\gamma_{2\alpha}(C) \leqq \gamma_{2\alpha}(C_1) + \gamma_{2\alpha}(C_2) \leqq \gamma^i_{2\alpha}(D_1) + \gamma^i_{2\alpha}(D_2),$$

and, as C is arbitrary, $\gamma^i_{2\alpha}(D) \leqq \gamma^i_{2\alpha}(D_1) + \gamma^i_{2\alpha}(D_2)$. If

$D = \bigcup_{k=1}^{\infty} D_k$, D_k open, and if $C \subset D$ is compact, then for some m, $C \subset \bigcup_{k=1}^{m} D_k$ and

$$\gamma_{2\alpha}(C) \leqq \gamma_{2\alpha}^{i}\left(\bigcup_{k=1}^{m} D_k\right) \leqq \sum_{k=1}^{m} \gamma_{2\alpha}^{i}(D_k) \leqq \sum_{k=1}^{\infty} \gamma_{2\alpha}^{i}(D_k).$$

Therefore, since C is arbitrary

$$\gamma_{2\alpha}^{i}(D) \leqq \sum_{k=1}^{\infty} \gamma_{2\alpha}^{i}(D_k).$$

Since the inner and outer capacities are obviously the same for open sets, this gives 15) for open sets, and 15) for arbitrary sets follows immediately.

16) *For every set* A, $\gamma_{2\alpha}^{0}(A) = c_2(A)$.

Proof. — If $A \subset \bigcup_{k=1}^{\infty} A_k$ with $A_k \in \mathfrak{L}$, then

$$\gamma_{2\alpha}^{0}(A) \leqq \sum_{k=1}^{\infty} \gamma_{2\alpha}^{0}(A_k) = \sum_{k=1}^{\infty} c_1(A_k)^2 = \sum_{k=1}^{\infty} \delta(A_k)^2.$$

Hence $\gamma_{2\alpha}^{0}(A) \leqq c_2(A)$. On the other hand, if $\gamma_{2\alpha}^{0}(A) < \infty$, then $\gamma_{2\alpha}^{0}(A) = c_1(A)^2 = \delta(A)^2 \geqq c_2(A)$.

Because of the above results it is possible to make use of an important theorem of Choquet [7] on capacitability. Choquet's theorem can be stated as follows:

Let γ *be an increasing non-negative set function defined on all compact sets, and let* γ^i *and* γ^0 *be constructed from* γ *as in the paragraph after* (6, 1). *If* $\gamma^0(C) = \gamma(C)$ *for every compact set* C *and if* $\gamma^0(A) = \lim \gamma^0(A_n)$ *whenever* A *is the union of the increasing sequence* $\{A_n\}$, *then* $\gamma^i(A) = \gamma^0(A)$, *for every analytic set* A.

By 14) we have $\gamma_{2\alpha}^{0}(A) = c_1(A)^2$. Hence, if C is compact, then by 11), $\gamma_{2\alpha}^{0}(C) = c_1(C)^2 = \delta(C)^2 = \gamma_{2\alpha}(C)$. In addition if A is the union of the increasing sequence $\{A_n\}$, then by 8), § 5, $\gamma_{2\alpha}^{0}(A) = \lim \gamma_{2\alpha}^{0}(A_n)$. Thus, the second part of the theorem below follows from Choquet's theorem. The first part has been proved already.

THEOREM 1. — *For every set* A, $\gamma_{2\alpha}^{0}(A) = c_2(A) = c_1(A)^2$, *and if* $c_1(A) < \infty$, $c_1(A) = \delta(A)$. *For every analytic set* A, $\gamma_{2\alpha}^{i}(A) = \gamma_{2\alpha}^{0}(A)$.

The several notations for the capacity of order 2α will now be dropped. Henceforth $\gamma_{2\alpha}$ will denote the outer capacity of order 2α, i. e., $\gamma_{2\alpha} = \gamma_{2\alpha}^{0} = c_2 = c_1^2$. Also $\gamma_{2\alpha}(A)$ will be called the 2α-capacity of A. The sets in $\mathfrak{A}_{2\alpha}$ are the sets of 2α-capacity 0.

REMARK 3. — In accordance with the results which have been proved here for $\alpha > 0$ and which were mentioned in the remark at the end of the last section for $\alpha = 0$, γ_0 should denote the Lebesgue measure, and Ω_0 should denote the class of measures which are absolutely continuous with respect to γ_0 and have a square integrable derivative. The results in the rest of this section (except 18) which has no meaning for $\alpha = 0$) hold for $\alpha \geq 0$. Many are rather trivial for $\alpha = 0$, or are standard results from measure theory. When this is so we do not take account of the case $\alpha = 0$ in the proof.

Using the capacitability (i.e. equality of inner and outer capacities) of sets G_δ we can give another characterization of the sets in $\mathfrak{A}_{2\alpha}$.

17) $A \in \mathfrak{A}_{2\alpha}$ *if and only if* A *is a subset of a set* G_δ *which has* ν*-measure* 0 *for every* $\nu \in \Omega_{2\alpha}$.

PROOF. — If $A \in \mathfrak{A}_{2\alpha}$, then by definition there exists $g \geq 0$ in L^2 such that

$$A \subset \mathop{E}_{x} [G_{2\alpha} g(x) = +\infty].$$

The set on the right is a set G_δ which has ν-measure 0 for every $\nu \in \Omega_{2\alpha}$. If $A \subset B$ and B is a set G_δ which has ν-measure 0 for every $\nu \in \Omega_{2\alpha}$, then for every compact set $C \subset B$, $\gamma_{2\alpha}(C) = |\mu_C| = 0$. Therefore, B has inner capacity 0, and by Theorem 1, $c_1(B) = 0$, from which it follows that $B \in \mathfrak{A}_{2\alpha}$ and hence that $A \in \mathfrak{A}_{2\alpha}$.

The capacitability of G_δ's also gives an improvement of proposition 1 on the infinities of the potential of an arbitrary measure.

18) *If* μ *is a measure satisfying* (6, 4) *then* $G_{2\alpha}\mu$ *is finite exc.* $\mathfrak{A}_{2\alpha}$.

Proof. — The proof is easily reduced to the case in which $|\mu| < \infty$. If C is an arbitrary compact subset of

$$\underset{x}{\mathrm{E}}\,[G_{2\alpha}\mu(x) = +\infty],$$

then by 10)-(*d*) there is a constant c such that $u_C \leqq c$ everywhere. Hence

$$\int G_{2\alpha}\mu\, d\mu_C = \int u_C\, d\mu \leqq c|\mu| < \infty,$$

and since $G_{2\alpha}\mu = +\infty$ everywhere on C it follows that $\gamma_{2\alpha}(C) = |\mu_C| = 0$. Thus, $\underset{x}{\mathrm{E}}\,[G_{2\alpha}\mu(x) = +\infty]$ is a set G_δ with inner capacity 0.

We end this section with some results on the nature of the capacities $\gamma_{2\alpha}$.

19) *For fixed* A, $\gamma_{2\alpha}(A)$ *is an increasing function of* α. *If* A *is open*, $\gamma_{2\alpha}(A)$ *is continuous on the left; if* A *is compact* $\gamma_{2\alpha}(A)$ *is continuous on the right.*

Proof. — It is evident that if $\beta \leqq \alpha$, then $||u||_\beta \leqq ||u||_\alpha$. Therefore $P^\alpha \subset P^\beta$, $\mathfrak{A}_{2\alpha} \subset \mathfrak{A}_{2\beta}$, $\Omega_{2\alpha} \supset \Omega_{2\beta}$ [34] and $||\mu||_{2\alpha} \leqq ||\mu||_{2\beta}$. If $\gamma_{2\alpha}(A) < \infty$, then there is a function $u \in P^\alpha$ such that $u \geqq 1$ on A exc. $\mathfrak{A}_{2\alpha}$ and such that $||u||_\alpha^2 = \gamma_{2\alpha}(A)$. Since $u \geqq 1$ on A exc. $\mathfrak{A}_{2\beta}$ we have $\gamma_{2\beta}(A) \leqq ||u||_\beta^2 \leqq ||u||_\alpha^2 = \gamma_{2\alpha}(A)$, which proves the first statement in the proposition.

If A is compact and $\varepsilon > 0$ is given, let $D \supset A$ be open and bounded and such that $\gamma_{2\beta}(D) < \gamma_{2\beta}(A) + \varepsilon$, and let $u \in P^\beta$ be such that $u \geqq 1$ everywhere on D and such that $||u||_\beta^2 = \gamma_{2\beta}(D)$. ($u = u_D$ has these properties.) By using Fourier transforms we get that every regularization $u_\rho = u * e_\rho$ belongs to P^α for all α, and that if ρ is fixed and $\alpha \searrow \beta$, then $||u_\rho||_\alpha \searrow ||u_\rho||_\beta$. Let ρ_0 be small enough so that $u_{\rho_0} \geqq 1$ everywhere on A. Then, as $\alpha \searrow \beta$, by 3) § 5,

$$\gamma_{2\alpha}(A) \leqq ||u_{\rho_0}||_\alpha^2 \searrow ||u_{\rho_0}||_\beta^2 \leqq ||u||_\beta^2 = \gamma_{2\beta}(D) \leqq \gamma_{2\beta}(A) + \varepsilon.$$

This proves the continuity on the right when A is compact.

If A is open and $\varepsilon > 0$ is given, let $C \subset A$ be compact and such that $\gamma_{2\alpha}(A) < \gamma_{2\alpha}(C) + \varepsilon$, and let μ be a measure with support in C such that $|\mu| = 1$ and such that $\dfrac{1}{\gamma_{2\alpha}(C)} = ||\mu||_{2\alpha}^2$.

[34] In the present case $\Omega_{2\alpha}$ and $\Omega_{2\beta}$ are considered as sets of measures and compared as such.

Every regularization $\mu_\rho = \mu * e_\rho$ belongs to $\Omega_{2\beta}$ for all β and has total mass 1, and if ρ is fixed and $\beta \nearrow \alpha$, then $||\mu_\rho||_{2\beta} \searrow ||\mu_\rho||_{2\alpha}$ ([35]). Let ρ_0 be small enough so that a closed support of μ_{ρ_0} is contained in A. Since $||\nu||_{2\alpha} = ||G_{2\alpha}\nu||_\alpha$ and $G_{2\alpha}\mu_\rho = (G_{2\alpha}\mu)_\rho$, by 3) § 5, as $\beta \nearrow \alpha$,

$$\frac{1}{\gamma_{2\beta}(A)} \leqq ||\mu_{\rho_0}||^2_{2\beta} \searrow ||\mu_{\rho_0}||^2_{2\alpha} \leqq ||\mu||^2_{2\alpha} = \frac{1}{\gamma_{2\alpha}(C)} \leqq \frac{1}{\gamma_{2\alpha}(A) - \epsilon}.$$

The next result is an improvement of a proposition of Brelot [4] on the relation between the capacity of a set A and the capacity of T(A), when T is a Lipschitz transformation. Later we shall use the result in cases when the Lispchitz constant tends to 0 and in cases when it tends to ∞. It is well known that a Lipschitz transformation defined on an arbitrary subset of R^n with Lipschitz constant M can be extended to a Lipschitz transformation on R^n, with the same Lipschitz constant (see [13]) so there is no loss in generality in assuming from the beginning that the transformation is defined on R^n.

20) *Let* T *be a transformation of* R^n *into* R^n *satisfying* $|Tx - Ty| \leqq M|x-y|$. *If* A *and* B = T(A) *both have diameter* $\leqq r$, *then*

$$\gamma_{2\alpha}(B) \leqq M^{n-2\alpha}\gamma_{2\alpha}(A) \quad \textit{if} \quad 0 \leqq \alpha < \frac{n}{2} \quad \textit{and} \quad M \leqq 1.$$

$$\gamma_{2\alpha}(B) \leqq \frac{2^{\frac{n-2\alpha}{2}-1}\Gamma\left(\frac{n-2\alpha}{2}\right)}{r^{\frac{n-2\alpha}{2}}K_{\frac{n-2\alpha}{2}}(r)} M^{n-2\alpha}\gamma_{2\alpha}(A) \ \textit{if} \ 0 \leqq \alpha < \frac{n}{2}$$

and $M > 1$.

$$\gamma_n(B) \leqq \frac{\gamma_n(A)}{1 + \dfrac{rK_1(r)}{2^{n-1}\pi^{n/2}\Gamma\left(\dfrac{n}{2}\right)}\log\dfrac{1}{M}\,\gamma_n(A)} \quad \textit{if} \quad M \leqq 1.$$

$$\gamma_n(B) \leqq \left(1 + \frac{1}{K_0(r)}\log M\right)\gamma_n(A) \quad \textit{if} \quad M > 1.$$

([35]) μ_ϱ is the measure with density $h_\varrho(x) = \int e_\varrho(x-y)\,d\mu(y)$. Since μ has compact support, h_ϱ is of class C_0^∞, and $||\mu_\varrho||^2_{2\alpha} = \int (1+|\xi|^2)^{-\alpha}|h_\varrho|^2\,d\xi$.

PROOF. — Suppose first that A is compact, put $B = T(A)$, and let $\mathcal{C}(A)$ and $\mathcal{C}(B)$ denote the spaces of continuous functions on A and B normed by the upper bound. Each function $\varphi \in \mathcal{C}(B)$ defines a function $T^*\varphi \in \mathcal{C}(A)$ by the equation $T^*\varphi(x) = \varphi(Tx)$. Since $\|T^*\varphi\| = \|\varphi\|$, T^* is one to one and the range $R(T^*)$ is a closed subspace of $\mathcal{C}(A)$. If $\mu = \frac{\mu_B}{|\mu_B|}$ is the normalized capacitary distribution for B, then

$$l(T^*\varphi) = \int \varphi(z)\, d\mu(z)$$

is a positive linear functional on $R(T^*)$. By a well known theorem of Hahn-Banach type, l has a positive linear extension to $\mathcal{C}(A)$, and by the representation theorem of Riesz, this extension is given by a measure ν on A. Thus, ν is a positive measure on A such that for every continuous function φ on B, and hence for every non-negative lower semi-continuous function φ on B.

$$\int \varphi(z)\, d\mu(z) = \int \varphi(Tx)\, d\nu(x).$$

Taking $\varphi = G_{2\alpha}$, we have

$$(6, 9)\qquad \frac{1}{\gamma_{2\alpha}(B)} = \|\mu\|^2_{2\alpha} = \iint G_{2\alpha}(z - w)\, d\mu(z)\, d\mu(w)$$
$$= \iint G_{2\alpha}(Tx - Ty)\, d\nu(x)\, d\nu(y).$$

Put $|x - y| = \rho$ and $|Tx - Ty| = \rho_1$. By our assumption $\rho_1 \leqq M\rho$, $\rho \leqq r$, and $\rho_1 \leqq r$. Now, for $0 < \alpha < \frac{n}{2}$ we evaluate the quotient

$$\frac{G_{2\alpha}(Tx - Ty)}{G_{2\alpha}(x - y)} = \frac{\rho_1^{\frac{2\alpha - n}{2}} K_{\frac{n-2\alpha}{2}}(\rho_1)}{\rho^{\frac{2\alpha - n}{2}} K_{\frac{n-2\alpha}{2}}(\rho)} = \left(\frac{\rho_1}{\rho}\right)^{2\alpha - n} \frac{\rho_1^{\frac{n-2\alpha}{2}} K_{\frac{n-2\alpha}{2}}(\rho_1)}{\rho^{\frac{n-2\alpha}{2}} K_{\frac{n-2\alpha}{2}}(\rho)}$$

By (3, 3) and (3, 7) $z^\nu K_\nu(z)$ is a decreasing function of z. From this and (3, 4) we get

$$\frac{G_{2\alpha}(Tx - Ty)}{G_{2\alpha}(x - y)} \geqq \begin{cases} M^{2\alpha - n} & \textit{if}\quad M \leqq 1 \\ M^{2\alpha - n} \dfrac{r^{\frac{n-2\alpha}{2}} K_{\frac{n-2\alpha}{2}}(r)}{2^{\frac{n-2\alpha-2}{2}} \Gamma\left(\dfrac{n-2\alpha}{2}\right)} & \textit{if}\quad M > 1. \end{cases}$$

Combining this with (6, 9) and the fact that $|\nu| = 1$ we get

$$\frac{1}{\gamma_{2\alpha}(B)} \geqq M^{2\alpha-n} \|\nu\|^2_{2\alpha} \geqq \frac{M^{2\alpha-n}}{\gamma_{2\alpha}(A)} \qquad \textit{for} \quad M \leqq 1,$$

$$\frac{1}{\gamma_{2\alpha}(B)} \geqq \frac{r^{\frac{n-2\alpha}{2}} K_{\frac{n-2\alpha}{2}}(r)}{2^{\frac{n-2\alpha-2}{2}} \Gamma\left(\frac{n-2\alpha}{2}\right)} M^{2\alpha-n} \frac{1}{\gamma_{2\alpha}(A)} \quad \textit{for} \quad M > 1.$$

Next, consider $\alpha = \dfrac{n}{2}$ and $M \leqq 1$. Then $\rho_1 \leqq M\rho \leqq \rho$ and, by (4, 1) and (3, 7) we have

$$G_n(Tx - Ty) - G_n(x-y) = \frac{1}{2^{n-1}\pi^{n/2}\Gamma\left(\frac{n}{2}\right)} [K_0(\rho_1) - K_0(\rho)]$$

$$= \frac{1}{2^{n-1}\pi^{n/2}\Gamma\left(\frac{n}{2}\right)} \int_{\rho_1}^{\rho} K_1(t)\, dt.$$

By (3, 7) $tK_1(t)$ is decreasing, hence $K_1(t) \geqq \dfrac{1}{t} rK_1(r)$ for $t \leqq r$. Therefore, since $\rho_1 \leqq M\rho$

$$G_n(Tx - Ty) - G_n(x-y) \geqq \frac{rK_1(r)}{2^{n-1}\pi^{n/2}\Gamma\left(\frac{n}{2}\right)} \log \frac{1}{M},$$

and from (6, 9) and the fact that $|\nu| = 1$ we get

$$\frac{1}{\gamma_n(B)} \geqq \|\nu\|^2_{2\alpha} + \frac{rK_1(r)}{2^{n-1}\pi^{n/2}\Gamma\left(\frac{n}{2}\right)} \log \frac{1}{M}$$

$$\geqq \frac{1}{\gamma_n(A)} + \frac{rK_1(r)}{2^{n-1}\pi^{n/2}\Gamma\left(\frac{n}{2}\right)} \log \frac{1}{M}.$$

Finally, we show that for $M > 1$, we have

$$G_n(Tx - Ty) \geqq \frac{1}{1 + \dfrac{\log M}{K_0(r)}} G_n(x-y),$$

which, combined with (6, 9) will give the required result.

First if $\rho_1 \leqq \rho$, this inequality certainly holds, since G_n is a decreasing function. Suppose, therefore, that $\rho_1 > \rho$. Then

$$\frac{G_n(x-y)-G_n(Tx-Ty)}{G_n(Tx-Ty)}=\frac{K_0(\rho)-K_0(\rho_1)}{K_0(\rho_1)}=\frac{1}{K_0(\rho_1)}\int_\rho^{\rho_1} K_1(t)\,dt.$$

By (3, 7), $tK_1(t)$ is decreasing, so $tK_1(t) \leqq \lim_{t\to 0} tK_1(t) = 1$. Since K_0 is a decreasing function

$$\frac{G_n(x-y)-G_n(Tx-Ty)}{G_n(Tx-Ty)} \leqq \frac{1}{K_0(r)}\log\frac{\rho_1}{\rho} \leqq \frac{1}{K_0(r)}\log M$$

which gives the inequality at the beginning of the paragraph.

Now that we have proved the proposition when A is a compact set, we see from proposition 8, § 5, that it holds when A is an F_σ, and in particular when A is an open set; and having the proposition when A is an open set, we deduce immediately that it holds when A is arbitrary.

Corollary. — *If* A *has diameter* $\leqq r$, *then*

$$\gamma_n(A) \leqq \frac{2^{n-1}\pi^{n/2}\Gamma\left(\frac{n}{2}\right)}{rK_0(r)K_1(r)}.$$

Proof. — Apply the last two inequalities in the proposition to the homothetic transformations $y \to \rho y$ and $y \to \frac{1}{\rho} y$ with any $\rho < 1$.

The following result is used in the proof of a generalization of the Frostman mean value theorem.

21) *If* $\rho \leqq 1$ *we have: for* $\alpha < \frac{n}{2}$

$$\frac{2K_{\frac{n-2\alpha}{2}}(2)}{\Gamma\left(\frac{n-2\alpha}{2}\right)}\gamma_{2\alpha}(S(0,1)) \leqq \frac{\gamma_{2\alpha}(S(0,\rho))}{\rho^{n-2\alpha}} \leqq \gamma_{2\alpha}(S(0,1))$$

and for $\alpha = \frac{n}{2}$,

$$K_0(2)\gamma_n(S(0,1)) \leqq \gamma_n(S(0,\rho))\log\frac{e}{\rho} \leqq \frac{2^{n-2}\pi^{n/2}\Gamma\left(\frac{n}{2}\right)}{K_0(2)K_1(2)}.$$

PROOF. — Proposition 20) applied to the transformations $y \to \rho y$ of $S(0,1)$ onto $S(0, \rho)$ and $y \to \frac{y}{\rho}$ of $S(0, \rho)$ onto $S(0,1)$ gives the inequalities stated for $\alpha < \frac{n}{2}$ and gives

$$\frac{\gamma_n(S(0,1))\left(1 + \log\frac{1}{\rho}\right)}{1 + \frac{1}{K_0(2)}\log\frac{1}{\rho}} \leqq \gamma_n(S(0, \rho))\log\frac{e}{\rho}$$

$$\leqq \frac{\gamma_n(S(0,1))\left(1 + \log\frac{1}{\rho}\right)}{1 + \frac{K_1(2)}{2^{n-2}\pi^{n/2}\Gamma\left(\frac{n}{2}\right)}\log\frac{1}{\rho}\gamma_n(S(0,1))}.$$

For $0 < \rho < 1$ the left side varies monotonically between $\gamma_n(S(0,1))$ and $\gamma_n(S(0,1))\, K_0(2)$ and the latter is the lower bound since $e^2K_0(2) = .841...$ The right side varies monotonically between $\gamma_n(S(0,1))$ and $\frac{2^{n-2}\pi^{n/2}\Gamma\left(\frac{n}{2}\right)}{K_1(2)}$. By the preceding corollary and the fact that $K_0(2) < 1$, both of these are smaller than the constant given in the proposition.

22) *There is a constant* $c > 0$ *(depending only on* α, β, *and* n*) such that if* $0 \leqq \beta \leqq \alpha \leqq \frac{n}{2}$ *and* $A \subset S(x, \rho)$ *with* $\rho \leqq 1$, *then*

$$\frac{\gamma_{2\alpha}(A)}{\gamma_{2\alpha}(S(x, \rho))} \geqq c\frac{\gamma_{2\beta}(A)}{\gamma_{2\beta}(S(x, \rho))}.$$

PROOF. — Let $A \subset S(0, \rho)$ and let B denote the image of A under the transformation $y \to \frac{y}{\rho}$. Then by propositions 19) and 20) we have for $\alpha < \frac{n}{2}$

$$\gamma_{2\alpha}(A) \geqq c\rho^{n-2\alpha}\gamma_{2\alpha}(B) \geqq c\rho^{n-2\alpha}\gamma_{2\beta}(B) \geqq c\frac{\rho^{n-2\alpha}}{\rho^{n-2\beta}}\gamma_{2\beta}(A).$$

In view of 21) this is sufficient to prove our statement. For $\alpha = \frac{n}{2}$ the proof is similar with $\rho^{n-2\alpha}$ replaced by $\left(\log\frac{e}{\rho}\right)^{-1}$.

Next we consider subsets of a subspace $R^k \subset R^n$, $k < n$, and the relations between their capacities as subsets of R^k and their capacities as subsets of R^n. Quantities associated with R^k will be primed: thus x' denotes a point of R^k, $\gamma'_{2\beta}$ denotes the capacity of order 2β in the Euclidean space R^k, $\mathfrak{A}'_{2\alpha}$ is the class of subsets $A' \subset R^k$ with $\gamma'_{2\alpha}(A') = 0$.

23) *If* $2\alpha > n - k$, *then for every measure* μ *on* R^k *and every subset* A *of* R^k

$$(6, 10) \qquad \begin{aligned} \|\mu\|'^2_{2\alpha-(n-k)} &= \frac{2^{n-k}\pi^{\frac{n-k}{2}}\Gamma(\alpha)}{\Gamma\left(\alpha - \frac{n-k}{2}\right)}\|\mu\|^2_{2\alpha} \\ \gamma'_{2\alpha-(n-k)}(A) &= \frac{\Gamma\left(\alpha - \frac{n-k}{2}\right)}{2^{n-k}\pi^{\frac{n-k}{2}}\Gamma(\alpha)}\gamma_{2\alpha}(A). \end{aligned}$$

Proof. — It is easy to see from (4, 1) that

$$(6, 11) \qquad G'_{2\alpha-(n-k)}(x') = \frac{2^{n-k}\pi^{\frac{n-k}{2}}\Gamma(\alpha)}{\Gamma\left(\alpha - \frac{n-k}{2}\right)} G_{2\alpha}(x'),$$

and from this the first formula in (6, 10) is obvious. For compact sets A, the second formula in (6, 10) follows at once from the first. The validity of the second formula for compact sets implies the validity for sets F_σ, and then the validity for sets F_σ implies the validity for all sets.

24) *If* $2\alpha > n - k$, *then, as* $\rho \to 0$, $\gamma_{2\alpha}[S(0, \rho) \cap R^k]$ *is of order* $\rho^{n-2\alpha}$ *or of the order* $\frac{1}{\log 1/\rho}$, *according as* $\alpha < \frac{n}{2}$ *or* $\alpha = \frac{n}{2}$. *If* $0 \leq 2\alpha \leq n - k$, *then* $\gamma_{2\alpha}(R^k) = 0$.

Proof. — The first part of 24) is obtained by applying 21) to $\gamma'_{2\alpha-(n-k)}$ $(S(0, \rho) \cap R^k)$ and then using 23).

As for the second part of 24), if in the second equation in (6, 10) we take A to be any compact subset of R^k and let $2\alpha \searrow n - k$, we can conclude from 10) that $\gamma_{n-k}(A) = 0$, and hence that $\gamma_{n-k}(R^k) = 0$. Using 19) again, we see that if $2\alpha < n - k$, then $\gamma_{2\alpha}(R^k) = 0$.

The next proposition, which we state without proof, shows the relation between sets of capacity 0 and sets of Hausdorff measure 0. It is due to Frostman [11].

If $h(t)$ is a continuous non-decreasing function of $t \geqq 0$ with $h(0) = 0$ and $h(t) > 0$ for $t > 0$, the Hausdorff outer measure H corresponding to h is defined as follows:

$$\mathrm{H}(\mathrm{A}) = \lim_{\rho \to 0} \mathrm{H}_\rho(\mathrm{A}) \qquad \textit{where} \qquad \mathrm{H}_\rho(\mathrm{A}) = \inf \Sigma h(d(\mathrm{A}_k)),$$

where $d(\mathrm{A}_k)$ is the diameter of A_k and the infimum is taken over all sequences A_k satisfying $\mathrm{A} \subset \bigcup \mathrm{A}_k$ and $d(\mathrm{A}_k) < \rho$.

For $\alpha > 0$, the Hausdorff α-dimensional measure is the Hausdorff measure corresponding to $h(t) = c(\alpha) t^\alpha$, $c(\alpha)$ a suitable constant; the 0-dimensional, or logarithmic Hausdorff measure is the one corresponding to $h(t) = \dfrac{1}{\log \dfrac{1}{t}}$.

We say that $\tilde{\mathrm{H}}$ is weak relative to H if

$$\int_0^\varepsilon \frac{\tilde{h}(t)}{t\,h(t)}\,dt < \infty.$$

(Thus the β-dimensional measure is weak relative to the α-dimensional measure if $\beta > \alpha$.)

Frostman's theorem is as follows.

25) *If the* $(n\text{-}2\alpha)$*-dimensional measure of* A *is* 0, *then* $\gamma_{2\alpha}(\mathrm{A}) = 0$. *If* $\gamma_{2\alpha}(\mathrm{A}) = 0$ *then* $\mathrm{H}(\mathrm{A}) = 0$ *for every Hausdorff measure which is weak relative to the* $(n\text{-}2\alpha)$*-dimensional measure.*

Remark 4. — In the case $\alpha = n/2$ Frostman proves a slightly stronger statement: *if* $\gamma_n(\mathrm{A}) = 0$ *then* $\mathrm{H}(\mathrm{A}) = 0$ *for every Hausdorff measure with* $\int_0^\varepsilon t^{-1} h(t)\,dt < \infty$. The first part of 25) can be strengthened as follows: *if the* $(n - 2\alpha)$*-dimensional measure of A is finite then* $\gamma_{2\alpha}(\mathrm{A}) = 0$. This result for $\alpha = n/2$ and $n = 2$ is essentially due to P. Erdös and J. Gillis [10*a*] (a simpler proof was given more recently by L. Carleson [4*a*]). The proof was extended to arbitrary n and $\alpha \leqq n/2$ by W. F. Donoghue (as yet unpublished).

It is clear from proposition 20) that if A is a set of 2α-capacity 0, then the projection of A on any hyperplane has 2α-capacity 0. To conclude the section we give a partial converse of this which will be used in the next section.

26) *If* A *is a set whose projection on some hyperplane of dimension n-k*, $k \leqq \alpha$, *has* 2α*-capacity* 0, *then* $\gamma_{2\alpha-2k}(\mathrm{A}) = 0$ ([35a]).

PROOF. — It is sufficient to consider $k = 1$, in which case, by virtue of proposition 20), the assertion is equivalent to the following lemma.

LEMMA 1. — *If* $\mathrm{A} \in \mathfrak{A}_{2\alpha}$, $\alpha \geqq 1$, *then the union of all lines which meet* A *and are parallel to the* x_n*-axis belongs to* $\mathfrak{A}_{2\alpha-2}$.

PROOF. — First we consider $\alpha = 1$. In this case, by 20), the projection of A on the hyperplane $x_n = 0$ has 2-capacity 0, and by 23) it has 1-capacity 0 relative to the hyperplane. Therefore, it has (n-1)-dimensional Lebesgue measure 0. By a standard theorem in measure theory, the union of the lines which meet A and are parallel to the x_n-axis must have n-dimensional Lebesgue measure 0, that is, O-capacity 0.

Now assume that $\alpha > 1$ and that $\mathrm{A} \subset \underset{x}{\mathrm{E}}[\mathrm{G}_\alpha g(x) = +\infty]$, $0 \leqq g \in \mathrm{L}^2$, and put

$$\bar{g}(x) = \sup_h \frac{1}{h} \int_{x_n}^{x_n+h} g(x', t)\, dt.$$

According to an important inequality of Hardy and Littlewood [12], $\bar{g} \in \mathrm{L}^2$. It will be shown that if x is any point such that $\mathrm{G}_\alpha g(x)$, $\mathrm{G}_\alpha \bar{g}(x)$, and $\mathrm{G}_{\alpha-1}\bar{g}(x)$ are all finite, then the line through x parallel to the x_n-axis does not meet A. This will prove the proposition, because the set of x such that either $\mathrm{G}_\alpha g(x)$, $\mathrm{G}_\alpha \bar{g}(x)$ or $\mathrm{G}_{\alpha-1}\bar{g}(x)$ is $= +\infty$ has (2α-2)-capacity 0.

From (4, 5) it is clear that if $\mathrm{G}_\alpha \bar{g}(x) + \mathrm{G}_{\alpha-1}\bar{g}(x) < \infty$, then $\int \left| \frac{\partial \mathrm{G}_\alpha(y)}{\partial y_n} \right| \bar{g}(x-y)\, dy < \infty$. Thus, if $\mathrm{G}_\alpha g(x) < \infty$, $\mathrm{G}_\alpha \bar{g}(x) < \infty$

([35a]) *Added in proofs.* A stronger result holds due to M. Ohtsuka [13*d*]: the hypothesis $k \leqq \alpha$ is replaced by $k \leqq 2\alpha$ and the thesis $\gamma_{2\alpha-2k}(\mathrm{A}) = 0$ is replaced by $\gamma_{2\alpha-k}(\mathrm{A}) = 0$. However the weaker result of the text is sufficient for our purposes.

and $G_{\alpha-1}\bar{g}(x) < \infty$, we have for $h \neq 0$

$$\int \left|\frac{\partial G_\alpha(y)}{\partial y_n}\right| \bar{g}(x-y)\,dy \geq \left|\int \frac{\partial G_\alpha(y)}{\partial y_n}\left\{\frac{1}{h}\int_{x_n}^{x_n+h} g(x'-y', t-y_n)\,dt\right\}dy\right|$$

$$= \left|\frac{1}{h}\int_{x_n}^{x_n+h}\int \frac{\partial G_\alpha(y)}{\partial y_n}\, g(x'-y',\, t-y_n)\,dy\,dt\right|$$

$$= \left|\frac{1}{h}\int_{x_n}^{x_n+h}\int_{R^{n-1}}\int_{R^1} \frac{\partial G_\alpha(y',\, t-z_n)}{\partial y_n}\, g(x'-y',\, z_n)\,dy'\,dz_n\,dt\right|,$$

$$= \left|\int g(x'-y',\, z_n)\,\frac{1}{h}\int_{x_n}^{x_n+h}\frac{\partial G_\alpha(y',\, t-z_n)}{\partial y_n}\,dt\,dz_n\,dy'\right|,$$

$$= \left|\int g(x'-y',\, z_n)\left\{\frac{G_\alpha(y', x_n - z_n + h) - G_\alpha(y', x_n - z_n)}{h}\right\}dz_n\,dy'\right|,$$

$$= \left|\frac{G_\alpha g(x',\, x_n + h) - G_\alpha g(x)}{h}\right|.$$

This shows that $G_\alpha g(x', x_n + h) < \infty$ for all h and hence that no point $(x', x_n + h)$ belongs to A.

§ 7. — Differentiability of functions in P^α.

The purpose of this section is to characterize the functions in P^α by differentiability and continuity properties.

THEOREM 1. — *Let $u \in P^\alpha$ and let $|i| = m \leq \alpha$. The derivative $D_i u$ exists in the ordinary sense exc. $\mathfrak{A}_{2\alpha-2m}$ and $D_i u \in P^{\alpha-m}$. If $u = G_\alpha g$ and $m < \alpha$, then $D_i u(x) = \int \frac{\partial^m G_\alpha(x-y)}{\partial x^i}\, g(y)\,dy$ exc. $\mathfrak{A}_{2\alpha-2m}$. If j is a permutation of i, then $D_i u(x) = D_j u(x)$ exc. $\mathfrak{A}_{2\alpha-2m}$. If $m \leq \alpha - 1$, $D_i u$ is absolutely continuous on all lines in any given direction except those contained entirely in a set $\in \mathfrak{A}_{2\alpha-2m-2}$ [36]. Finally, the direct formulas (1, 5) and (1, 10) for $d_\alpha(u)$ and $\|u\|_\alpha^2$ respectively, are valid for all $u \in P^\alpha$ and*

$$(7,\ 1) \qquad \begin{cases} d_\alpha(u) = \displaystyle\sum_{\substack{i \\ |i|=m}} d_{\alpha-m}(D_i u), \\ \|u\|_\alpha^2 = \displaystyle\sum_{k=0}^{m} \binom{m}{k} \sum_{\substack{j \\ |j|=\kappa}} \|D_j u\|_{\alpha-m}^2. \end{cases}$$

[36] A function is absolutely continuous on a straight line if it is absolutely continuous on each finite interval of the line.

PROOF. — By using Fourier transforms we deduce that there are potentials $u_j \in P^{\alpha-|j|}$ such that $\hat{u}_j = i^{|j|}\xi^j \hat{u}$ and, when replacing $D_j u$, they satisfy (7. 1). Furthermore, we see immediately (by passing to Fourier transforms) that the difference quotient $\frac{1}{h}[u_j(x_1, \ldots, x_k + h, \ldots, x_n) - u_j(x)]$ converges in $P^{\alpha-|j|-1}$ to $u_{j,k}$, $(|j| \leqq \alpha - 1)$. An easy induction shows then that the theorem reduces to the following two statements: 1° the absolute continuity of u_j (replacing $D_j u$) and the pointwise convergence of the difference quotients exc. $\mathfrak{A}_{2\alpha-2|j|-2}$, and 2° $D_j u = (D_j G_\alpha) * g$ exc. $\mathfrak{A}_{2\alpha-2|j|}$. We prove first 1°; it is clear that we can restrict ourselves to the consideration of u (i.e. $|j| = 0$), $\alpha \geqq 1$ and the derivative $\frac{\partial}{\partial x_n}$.

We start with the case $\alpha = 1$ (which is treated separately since there is no kernel G_0). Let $\{u_k\}$ be a sequence in $C_0^\infty(R^n)$ which converges in P^1 to the given function u. By (7, 1), which obviously holds for functions in $C_0^\infty(R^n)$, the sequence $\left\{\frac{\partial u_k}{\partial x_n}\right\}$ converges in $P^0 = L^2$ to some function $\varphi \in P^0$. By picking a subsequence if necessary, it can be assumed that for fixed x' outside a set $E' \subset R^{n-1}$ of (n-1)-dimensional measure 0, $\frac{\partial u_k(x', x_n)}{\partial x_n} \to \varphi(x', x_n)$ in L^2 with respect to the variable x_n. By using Lemma 1 at the end of the last section, and again picking a subsequence if necessary, it can be assumed that if $x' \notin E'$, then $u_k(x', x_n) \to u(x', x_n)$. Then if $x' \notin E'$ we have

$$\begin{aligned} u(x', b) - u(x', a) &= \lim \left[u_k(x', b) - u_k(x', a)\right] \\ &= \lim \int_a^b \frac{\partial u_k(x', t)}{\partial x_n}\, dt = \int_a^b \varphi(x', t)dt. \end{aligned}$$

It follows that if $x' \notin E'$, then $u(x', x_n)$ is absolutely continuous in x_n so that $\frac{\partial u}{\partial x_n}(x', x_n)$ exists and is equal to $\varphi(x', x_n)$ for almost all x_n. Hence $\frac{\partial u}{\partial x_n}$ exists and is equal to φ almost everywhere.

Now suppose that $\alpha > 1$, and consider $u = G_\alpha g$, with $g \in L^2$. If we put

$$\bar{g}(x', x_n) = \sup_{h \neq 0} \frac{1}{h} \int_0^h |g(x', x_n + t)|dt$$

then, by the Hardy-Littlewood inequality [12], $\bar{g} \in L^2$. Let

$$A_1 = \mathop{E}_{x}\left[\int \left|\frac{\partial G_\alpha(x-y)}{\partial x_n}\right| \bar{g}(y)dy = +\infty\right].$$

Since

$$\left|\frac{\partial G_\alpha(x-y)}{\partial x_n}\right| \left|\frac{1}{h}\int_0^h |g(y', y_n + t)|\,dt\right| \leqq \left|\frac{\partial G_\alpha(x-y)}{\partial x_n}\bar{g}(y)\right|,$$

it follows that if $x \notin A_1$, then

$$(7,2)\qquad \lim_{h\to 0}\int \frac{\partial G_\alpha(x-y)}{\partial x_n}\frac{1}{h}\int_0^h g(y', y_n+t)\,dt\,dy = \frac{\partial G_\alpha}{\partial x_n} * g(x),$$

and also that

$$(7,3)\qquad \int \frac{\partial G_\alpha(x-y)}{\partial x_n}\frac{1}{h}\int_0^h g(y', y_n + t)\,dt\,dy$$

$$= \frac{1}{h}\int \{G_\alpha(x'-y', x_n+h-y_n) - G_\alpha(x'-y', x_n - y_n)\}\, g(y)\,dy.$$

Now let $u = G_\alpha g$ exc. $\mathfrak{A}_{2\alpha}$ and let A_2 be the union of all lines parallel to the x_n-axis which contain some point of $\mathop{E}_{x}[u(x) \neq G_\alpha g(x)] \cup \mathop{E}_{x}[G_\alpha|g|(x) = +\infty]$. Then if $x \notin A_1 \cup A_2$, (7, 2) and (7, 3) give

$$(7,3')\qquad \lim_{h\to 0}\frac{u(x', x_n+h) - u(x', x_n)}{h} = \frac{\partial G_\alpha}{\partial x_n} * g(x),$$

so that $\dfrac{\partial u(x)}{\partial x_n}$ exists provided that $x \notin A_1 \cup A_2$.

By (4, 5), $A_1 \in \mathfrak{A}_{2\alpha-2}$, and by the lemma at the end of the last section, $A_2 \in \mathfrak{A}_{2\alpha-2}$. Hence, for every $u \in P^\alpha$ the derivative $\dfrac{\partial u}{\partial x_n}$ exists in the pointwise sense except on a set of (2α-2)-capacity 0.

We prove now the statement about absolute continuity. If l is any line parallel to the x_n-axis and not contained entirely in $A_1 \cup A_2$, then there is a point $x \in l - (A_1 \cup A_2)$, and, as A_2 is a union of lines, all points $(x', x_n + h) \in l - A_2$. It follows that the right hand side of (7, 3) can be written as

$$\frac{1}{h}[u(x', x_n + h) - u(x', x_n)]$$

and hence $u(x', x_n + h)$, as a function of h, is an integral, and therefore absolutely continuous.

Finally, to prove that $D_j u = (D_j G_\alpha) * g$ exc. $\mathfrak{A}_{2\alpha - 2|j|}$ we proceed by induction with respect to the number of indices in the system j. We use an argument completely similar to the one which led to (7,2), (7,3) and (7,3'). The kernel G_α is now replaced by $D_j G_\alpha$ and we use the inequality

$$|D_j G_\alpha(x)| \leqq c[G_\alpha(x) + G_{\alpha - |j|}(x)],$$

with a constant c depending only on α, $|j|$, and n. Thisi nequality is deduced in the same way as (4,5), from (3,7), (4,1), (4,2) and (4,3).

As corollaries of Theorem 1 we can now prove.

Corollary 1. — *If $\alpha > 2m$, where m is a positive integer, and if $g \in L^2$,*

$$(1 - \Delta)^m G_\alpha g(x) = G_{\alpha - 2m} g(x).$$

We apply here formula (4, 11).

Corollary 2. — *If m is a positive integer and $g \in L^2$,*

$$(1 - \Delta)^m G_{2m} g(x) = g(x), \text{ almost everywhere.}$$

Proof. — The formula for derivatives of a potential in Theorem 1 is not valid for orders $|i| = \alpha$ (37). Since we have to prove equality of two functions in L^2, the simplest is to compare their Fourier transforms, both of which turn out to be $\hat{g}$.

Corollary 2 shows that $G_{2m}(x - y)$ is a fundamental solution corresponding to $(1 - \Delta)^m$. As was already mentioned in § 6 (see (6, 6)), $G_{2\alpha}$ is the pseudo-reproducing kernel of P^α. For $\alpha = m$ the reproducing property can be put in a form avoiding the use of measures:

Corollary 3. — *For $u \in P^m$, we have*

$$u(x) = \sum_{k=0}^{m} \binom{m}{k} \sum_{|i|=k} \int \frac{\partial^k G_{2m}(x - y)}{\partial y^i} \frac{\partial^k u(y)}{\partial y^i} \, dy \quad \text{exc. } \mathfrak{A}_{2m} \text{ (38)}.$$

(37) The formula $\frac{\partial^{|i|}}{\partial x^i} G_\alpha g(x) = \int \frac{\partial^{|i|} G_\alpha(x - y)}{\partial x^i} g(y)\, dy$ for $|i| = \alpha$ can be made valid if we consider the integral as a singular integral.

(38) A similar direct formula for arbitrary α is more complicated; double integrals must be used.

PROOF. — Since $\frac{\partial^k u}{\partial y^i}$ for $|i| = k \leqq m$ is at least in L^2 and $\frac{\partial^k G_{2m}}{\partial y^i} = (-1)^k \frac{\partial^k G_{2m}}{\partial x^i}$, each term in the sum is a potential of order $\geqq m$. Furthermore the Fourier transforms of both sides are obviously equal. Hence the equation is true almost everywhere and since both sides are in P^α our statement follows.

In proving a converse of Theorem 1 we shall use a generalization of the Frostman mean value theorem, 1), § 4. It will be recalled that a part of the Frostman theorem asserts that for each positive measure μ the value of a potential $G_{2\alpha}\mu$ at a point x_0 is the limit of the averages of $G_{2\alpha}\mu$ over spheres with center x_0 and radius converging to 0. In the generalization of this theorem the spheres are replaced by much more general closed sets, the essential point being that the closed sets can be quite thinly distributed.

1) *Let x_0 be a given point, and for each positive integer k let A_k be a closed set contained in $S(x_0, \rho_k)$, where $\rho_k \to 0$. If for some $\beta \leqq \alpha$ there is a constant $c > 0$ such that*

$$(7, 4) \qquad \gamma_{2\beta}(A_k) \geqq c\gamma_{2\beta}[S(x_0, \rho_k)],$$

then each A_k supports a measure ν^k of total mass 1 such that

$$G_{2\alpha}\mu(x_0) = \lim_{k\to\infty} \int G_{2\alpha}\mu(y)\, d\nu^k(y)$$

for every positive Borel measure μ. A suitable choice for ν^k is the normalised capacitary distribution for A_k.

A similar result holds for each function u in P^α exc. $\mathfrak{A}_{2\alpha}$.

2) *For each point x and each positive integer k let $A_k(x)$ be a closed set contained in $S(x, \rho_k(x))$ where $\rho_k(x) \to 0$. If for each x there is a $\beta \leqq \alpha$ and a constant $c > 0$ (both may depend on x) such that (7, 4) holds, then each $A_k(x)$ supports a mearure ν_x^k of total mass 1 such that for every function $u \in P^\alpha$*

$$u(x) = \lim_{k\to\infty} \int u(y)\, d\nu_x^k(y) \text{ exc. } \mathfrak{A}_{2\alpha}.$$

A suitable choice for ν_x^k is the normalized capacitary distribution for $A_k(x)$.

The proofs of 1) and 2) are given in [15] for the special case

$\beta = \alpha$. In view of proposition 22) of the last section, however, if the hypothesis (7, 4) holds for some $\beta \leqq \alpha$, then it holds also for $\beta = \alpha$ [39]. Indications of how thinly the sets $A_k(x)$ can be distributed are given by the results on the nature of the capacity of a set near the end of the last section. Additional results related to 1) and 2) can be found in [15].

REMARK 1. — It is not necessary to require the sets A_k to be closed, provided the outer capacity $\gamma_{2\alpha}$ is replaced by the inner capacity. Of course, either the outer or the inner capacity can be used if the A_k are capacitable, in particular, if they are analytic. In this case ν^k cannot necessarily be taken to be the normalized capacitary distribution for A_k, but rather can be taken to be the normalized capacitary distribution of some closed subset of A_k.

3) *If* $u \in P^\alpha$, *then for each* $\varepsilon > 0$ *there is a set* B_ε *such that* $\gamma_{2\alpha}(B_\varepsilon) < \varepsilon$ *and such that the restriction of* u *to* $R^n - B_\varepsilon$ *is continuous. Conversely, if a function* u *has this continuity property and is equal almost everywhere to a function in* P^α, *then* $u \in P^\alpha$.

PROOF. — The first part of the proposition is evident from proposition 2), § 2, chapter I. The second part is proved by showing that if $v \in P^\alpha$ and if $u = v$ almost everywhere, then $u = v$ except on a set of 2α-capacity 0.

To see this, let $\varepsilon > 0$ be given, and let B_ε be a set such that $\gamma_{2\alpha}(B_\varepsilon) < \varepsilon$ and such that on $R^n - B_\varepsilon$ both u and v are continuous. We may obviously increase B_ε to be an open set with the same properties; we may therefore assume that B_ε is an open set. Choose $g_\varepsilon \geqq 0$ such that $\|g_\varepsilon\|^2_{L^2} < \varepsilon$ and such that $G_\alpha g_\varepsilon(x) \geqq 1$ everywhere on B_ε and let

$$D_\varepsilon = \mathop{E}_{x}\left[G_\alpha g_\varepsilon(x) > \frac{1}{2}\right].$$

Then clearly $D_\varepsilon \supset B_\varepsilon$ and $\gamma_{2\alpha}(D_\varepsilon) < 4\varepsilon$. Moreover, if $x_0 \notin D_\varepsilon$, then $G_\alpha g_\varepsilon(x_0)$ is not the limit of mean values of $G_\alpha g_\varepsilon(y)$ over the sets $S(x_0, \rho_n) \cap B_\varepsilon$ for any sequence $\rho_n \to 0$. Therefore

[39] The proof of 1) is rather simple, the proof of 2) rather delicate. In many applications the special case $\beta = 0$ is sufficient, and this special case is an almost immediate consequence of the Frostman theorem itself.

it follows from 1) (with $\beta = 0$) and the Remark 1 above, that

$$\frac{|S(x_0, \rho) \cap B_\varepsilon|}{|S(x_0, \rho|)|} \to 0,$$

so that in particular, $|S(x_0, \rho) - B_\varepsilon| \neq 0$.

Since $u = \varphi$ almost everywhere, we have

$$\int_{S(x_0, \rho) - B_\varepsilon} u(y)\, dy = \int_{S(x_0, \rho) - B_\varepsilon} \varphi(y)\, dy,$$

and, as both u and φ are continuous on $R^n - B_\varepsilon$ and as $|S(x_0, \rho) - B_\varepsilon| \neq 0$, we can divide both sides by $|S(x_0, \rho) - B_\varepsilon|$ and let $\rho \to 0$ and conclude that $u(x_0) = \varphi(x_0)$. This shows that $u = \varphi$ outside D_ε, and since $\gamma_{2\alpha}(D_\varepsilon) < 4\varepsilon$, it follows that $u = \varphi$ except on a set of 2α-capacity 0.

We can now state the converse of Theorem 1.

THEOREM 2. — *Let $u \in L^2$ and m be an integer, $0 \leq m \leq \alpha$. The function u belongs to P^α if the following conditions are satisfied:*

a) u is defined except on a set of 2α-capacity 0 and for each $\varepsilon > 0$ there exists a set $B_\varepsilon \subset R^n$ with $\gamma_{2\alpha}(B_\varepsilon) < \varepsilon$ such that u, restricted to $R^n - B_\varepsilon$ is continuous.

b) All derivatives $D_j u$ of orders $|j| \leq m$ exist when determined successively pointwise in the ordinary sense each one except on a set of corresponding $(2\alpha\text{-}2|j|)$-capacity 0; each derivative of order $|j| < m$ is absolutely continuous on all lines in the directions of coordinate axes except a set of such lines forming a union of $(2\alpha\text{-}2|j|\text{-}2)$-capacity 0.

c) All derivatives of order $|j| = m$ are in $P^{\alpha-m}$.

PROOF. — One could give different proofs of this theorem (for instance by using the theory of distributions and proposition 3). The most direct, perhaps, is the one using regularization. If $u_\rho = e_\rho * u$, then, by using partial integration one gets successively for all $|j| \leq m$

$$D_j u_\rho = (D_j e_\rho) * u = e_\rho * D_j u.$$

Hence, for $|j| = m$, since $D_j u \in P^{\alpha-m} \subset L^2$,

$$\widehat{D_j u_\rho} = (2\pi)^{n/2}(i)^m \xi^j \hat{e}_\rho \hat{u} = (2\pi)^{n/2} \hat{e}_\rho \widehat{D_j u}.$$

It follows that $\widehat{D_j u} = (i)^m \xi^j \hat{u}$ and, $D_j u \in P^{\alpha-m}$, $u \in L^2$ gives (since $(1+|\xi|^2)^\alpha < 2^\alpha[1+|\xi|^{2m}(1+|\xi|^2)^{\alpha-m}]$

$$\int |\hat{u}|^2(1+|\xi|^2)^\alpha d\xi \leq 2^\alpha \int |\hat{u}|^2[1+|\xi|^{2m}(1+|\xi|^2)^{\alpha-m}]\, d\xi$$
$$= 2^\alpha \int |\hat{u}|^2\Big[1+\sum_{|j|=m}(\xi^j)^2(1+|\xi|^2)^{\alpha-m}\Big]\, d\xi < \infty.$$

Therefore u is equal to some $u' \in P^\alpha$ almost everywhere and by proposition 3) and condition a), $u \in P^\alpha$.

Remark 2. — Our proof shows that condition b) can be considerably weakened. In this condition it is enough to assume that for $k = 1, \ldots, n$, the pure derivatives $\frac{\partial^j u}{\partial x_k^j}$, $j < m$, are equal a.e. to functions absolutely continuous on almost all lines parallel to the x_k- axis. Correspondingly, condition c) can be (and should be) relaxed as follows: all derivatives $\frac{\partial^m u}{\partial x_k^m}$, $k = 1, \ldots, n$ are equal a.e. to functions in $P^{\alpha-m}$. In the last formula of the proof the inequality would be

$$\int |u|^2(1+|\xi|^2)^\alpha d\xi \leq 2^\alpha n^{m-1} \int |\hat{u}|^2\Big[1+\sum_{k=2}^{n} \xi_k^{2m}(1+|\xi|^2)^{\alpha-m}\, d\xi\Big] < \infty.$$

Remark 3. — Theorems 1 and 2 and the preceding remark allow a simple direct characterization of functions $u \in P^\alpha$ without using the Fourier transforms. To this effect we take $m =$ the largest integer $\leq \alpha$. The function u should be in L^2 and satisfy condition a) of Theorem 2, the derivatives $\frac{\partial^j u}{\partial x_k^j}$, $1 \leq k \leq n$, $|j| \leq \alpha - 1$ should exist pointwise and be equivalent to absolutely continuous funtions except on a set of lines of measure 0 and $\frac{\partial^m u}{\partial x_k^m}$ for each k must have a finite Dirichlet integral of order $\alpha - m$ (if $\alpha = m$ it is just $\int \left|\frac{\partial^m x}{\partial x_k^m}\right|^2 dx\Big)$. This integral, for $\alpha > m$, is given by (1, 4). The norm $\|u\|_\alpha$ is given by (7, 1) where $\|D_i u\|^2_{\alpha-m}$ are given directly by any of the formulas (1, 10), (4, 9), or (4, 10).

The next proposition is obtained immediately by using

Fourier transforms, proposition 5), § 2, and the corollary to proposition 2) § 2.

4) *If* $u \in P^{\alpha+1}$, *then for every unit vector e and every real number h,*

$$\left\|\frac{u(x+he)-u(x)}{h}\right\|_{\alpha} \leqq \|u\|_{\alpha+1}.$$

Conversely, if $u \in P^{\alpha}$ *and if for each vector e in a basis for* R^n *there is a constant* M *such that for every real h*

$$\left\|\frac{u(x+he)-u(x)}{h}\right\|_{\alpha} \leqq M,$$

then u is equal except on a set of 2α*-capacity* 0 *to a function in* $P^{\alpha+1}$.

§ 8. — Restrictions to subspaces.

The purpose of the section is to characterize the restrictions of the functions in P^{α} to a subspace $R^k \subset R^n$. In accordance with conventions, quantities associated with R^k are primed. In addition, if u is a function defined on R^n, u' denotes its restriction to R^k.

THEOREM 1 *a.* — *If* $u \in P^{\alpha}$, $2\alpha > n-k$, *then* $u' \in P^{\alpha-\frac{n-k}{2}}(R^k)$ *and*

$$(8,1)\quad \begin{cases} \|u'\|'^2_{\alpha-\frac{n-k}{2}} \leqq \dfrac{\Gamma\left(\alpha-\frac{n-k}{2}\right)}{2^{n-k}\pi^{\frac{n-k}{2}}\Gamma(\alpha)}\|u\|^2_{\alpha} \\ \hat{u}'(\xi') = (2\pi)^{\frac{k-n}{2}}\displaystyle\int_{R^{n-k}}\hat{u}(\xi',\xi'')d\xi'' \text{ almost everywhere.} \end{cases}$$

PROOF. — For $u \in P^{\alpha}$, let Tu denote the function on the right side of the second formula in (8, 1). We shall show first that T$u(\xi')$ is defined almost everywhere and that if

$$|||w|||^2 = \int (1+|\xi'|^2)^{\alpha-\frac{n-k}{2}}|w|^2 d\xi'$$

then

$$(8,2)\qquad |||Tu|||^2 \leqq \frac{\Gamma\left(\alpha-\frac{n-k}{2}\right)}{2^{n-k}\pi^{\frac{n-k}{2}}\Gamma(\alpha)}\|u\|^2_{\alpha}.$$

If we apply the Schwarz inequality to the product $(1+|\xi|^2)^{-\alpha/2}\cdot(1+|\xi|^2)^{\alpha/2}\,\hat{u}(\xi)$ we obtain (at first with $\hat{u}$ replaced by $|\hat{u}|$ in order to show the absolute integrability of the functions involved)

$$|\mathrm{T}u(\xi')|^2 \leq (2\pi)^{k-n}\int_{\mathrm{R}^{n-k}}\frac{d\xi''}{(1+|\xi|^2)^\alpha}\int_{\mathrm{R}^{n-k}}(1+|\xi|^2)^\alpha|\hat{u}(\xi)|^2 d\xi''$$
$$= \frac{(2\pi)^{k-n}}{(1+|\xi'|^2)^{\alpha-\frac{n-k}{2}}}\int_{\mathrm{R}^{n-k}}\frac{d\eta''}{(1+|\eta''|^2)}\int_{\mathrm{R}^{n-k}}(1+|\xi|^2)^\alpha|\hat{u}(\xi)|^2 d\xi''.$$

Multiplying by $(1+|\xi'|^2)^{\alpha-\frac{n-k}{2}}$ and integrating with respect to ξ', we get (8, 2) and also the fact that the integral for $\mathrm{T}u(\xi')$ is absolutely convergent for almost all ξ'.

Now, if $u \in \mathrm{C}_0^\infty(\mathrm{R}^n)$, then, as is well known, $\mathrm{T}u$ is simply $\hat{u}'$, Let u be an arbitrary function in P^α and let $\{u_m\}$ be a sequence in $\mathrm{C}_0^\infty(\mathrm{R}^n)$ which converges to u in P^α and pointwise except on a set of 2α-capacity 0. The inequality (8, 2) shows that $\{u'_m\}$ is Cauchy in $\mathrm{P}^{\alpha-\frac{n-k}{2}}(\mathrm{R}_k)$, and the second formula in (6, 10) shows that $u'_m \to u'$ pointwise in R_k exc. $\mathfrak{A}'_{2\alpha-n+k}$. This proves that $u' \in \mathrm{P}^{\alpha-\frac{n-k}{2}}(\mathrm{R}^k)$ and also the inequality in (8, 1).

From the fact that $\mathrm{T}u_m \to \mathrm{T}u$ relative to the norm $|||w|||$, and therefore relative to the L^2 norm, it follows that for some subsequence $\{u_{m_i}\}$, $\mathrm{T}u_{m_i}(\xi') \to \mathrm{T}u(\xi')$ pointwise almost everywhere. On the other hand, since $u'_{m_i} \to u'$ in $\mathrm{P}^{\alpha-\frac{n-k}{2}}$, $\hat{u}_{m_i} \to \hat{u}'$ in L^2, and hence some subsequence converges to $\hat{u}'$ almost everywhere. Therefore, since $\mathrm{T}u_{m_i} = \hat{u}'_{m_i}$, we have $\mathrm{T}u = \hat{u}'$ almost everywhere.

THEOREM 1 *b.* — *If* $u' \in \mathrm{P}^{\alpha-\frac{n-k}{2}}(\mathrm{R}^k)$, $2\alpha > n-k$, *then the restriction of the function* $u \in \mathrm{P}^\alpha$ *whose Fourier transform is given by*

$$(8,\,3)\qquad \hat{u}(\xi) = \frac{2^{\frac{n-k}{2}}\Gamma(\alpha)}{\Gamma\left(\alpha-\frac{n-k}{2}\right)}\frac{(1+|\xi'|^2)^{\alpha-\frac{n-k}{2}}}{(1+|\xi|^2)^\alpha}\hat{u}'(\xi')$$

is u', *and for this function* u *equality holds in* (8, 1).

PROOF. — Inspection shows that any function whose Fourier transform is given by (8, 3) is equal almost everywhere to a function $u \in P^\alpha$. The second formula (in 8, 1) shows that the Fourier transform of the restriction of u to R^k is $\hat{u}'$, and hence that the restriction of u to R^k is equal to u' almost everywhere, and therefore, since both functions belong to $P^{\alpha-\frac{n-k}{2}}(R^k)$ except on a set of $\mathfrak{A}'_{2\alpha-n+k}$. Computation shows that for these u and u' equality holds in (8, 1).

REMARK 1. — Formulas (6, 11) and (6, 10) show that if $u' = G'_{2\alpha-n+k}\mu$ for some measure $\mu \in \Omega'_{2\alpha-n+k}$ (μ may even be a signed measure, i.e. in $\Omega'_{2\alpha-n+k} - \Omega'_{2\alpha-n+k}$) then the function u defined by (8, 3) is simply

$$\frac{2^{n-k}\pi^{\frac{n-k}{2}}\Gamma(\alpha)}{\Gamma\left(\alpha-\frac{n-k}{2}\right)}G_{2\alpha}\mu.$$

Next we give a generalization of Theorem 1 *b* to the case in which not only the function but also certain of its normal derivatives are given on a hyperplane R^{n-1} (in the previous notations we now put $k = n - 1$). In the formula we make use of a system of functions $\varphi_p(t)$ biorthogonal to the powers of t on $(-\infty, +\infty)$ with respect to the weight function $(1+t^2)^{-\alpha}$. To be explicit, let r be an integer $< \alpha - \frac{1}{2}$ and let φ_p be the polynomial of degree $\leq r$ which satisfies

$$(8, 4) \qquad \int_{-\infty}^{\infty} \frac{t^q\varphi_p(t)dt}{(1+t^2)^\alpha} = \delta_{pq}, \qquad 0 \leq p, q \leq r.$$

THEOREM 1 *c*. — *Let r be an integer $< \alpha - \frac{1}{2}$ and let $\nu_p \in P^{\alpha-p-\frac{1}{2}}(R^{n-1})$ for $p = 0, 1, \ldots, r$. If u is the function in P^α whose Fourier transform is given by*

(8, 5)

$$\hat{u}(\xi) = (2\pi)^{1/2}\sum_{p=0}^{r} i^{-p}\frac{(1+|\xi'|^2)^{\alpha-\frac{p+1}{2}}}{(1+|\xi|^2)^\alpha}\varphi_p\left(\frac{\xi_n}{\sqrt{1+|\xi'|^2}}\right)\hat{\nu}_p(\xi')$$

then

$$(8, 6) \qquad \frac{\partial^p u(x', 0)}{\partial x_n^p} = \nu_p(x') \quad \text{for} \quad p = 0, 1, \ldots, r,$$

and there is a constant c (depending only on α and r) such that

$$(8,7) \qquad ||u||_\alpha^2 \leq c \sum_{p=0}^{r} ||\varphi_p||'^2_{\alpha-p-\frac{1}{2}}.$$

Moreover, u is the function in P^α *with minimum norm satisfying* (8, 6).

PROOF. — From the fact that $\varphi_p \in P^{\alpha-p-\frac{1}{2}}(R^{n-1})$ it follows that the product of each summand in (8, 5) by $(1+|\xi|^2)^{\alpha/2}$ is square integrable over R^n, so that by proposition 5) § 2, there does exist a function $u \in P^\alpha$ whose Fourier transform is given by (8, 5) and whose α-norm satisfies (8, 7). The constant c can be taken to be $2\pi(r+1)\max_p \int_{-\infty}^{\infty} \frac{\varphi_p(t)^2 dt}{(1+t^2)^\alpha}$. By Theorem 1 *a* (and Theorem 1 of the last section) the Fourier transform of $\frac{\partial^p u}{\partial x_n^p}(x', 0)$ is

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty} (i\xi_n)^p\, \hat{u}(\xi)\, d\xi_n,$$

which, by (8, 4) is easily shown to be $\hat{\varphi}_p(\xi')$. Therefore $\frac{\partial^p u(x', 0)}{\partial x_n^p} = \varphi_p(x')$ almost everywhere, and since both are in $P^{\alpha-p-\frac{1}{2}}(R^{n-1})$, $\frac{\partial^p u(x', 0)}{\partial x_n^p} = \varphi_p(x')$ exc. $\mathfrak{A}'_{2\alpha-2p-1}$.

In order to prove that u is the function with minimum α-norm among all functions in P^α which satisfy (8, 6), we have to prove that u is orthogonal to all $\varphi \in P^\alpha$ which satisfy $\frac{\partial^p \varphi}{\partial x_n^p}(x', 0) = 0$ for $p = 0, \ldots, r$. In terms of the Fourier transform, the problem is to show that

$$\int (1+|\xi|^2)^\alpha \hat{u}(\xi)\overline{\hat{\varphi}(\xi)}d\xi = 0$$

for all φ which satisfy

$$\int (i\xi_n)^p \hat{\varphi}(\xi) d\xi_n = 0 \qquad \textit{for} \qquad p = 0, \ldots, r.$$

This is immediate, since φ_p is a polynomial of degree $\leq r$.

REMARK 2. — It is easy to see by the same argument that if $r + \frac{1}{2} < \beta \leqq \alpha$, and u is determined by (8, 5), then

$$(8, 8) \qquad ||u||_\beta^2 \leqq c_1 \sum_{p=0}^{r} ||\varphi_p||^2_{\beta-p-\frac{1}{2}}.$$

where c_1 depends on α, β, and r. This shows (interchanging the roles of α and β) that Theorem 1 c can be strengthened in the following way:

Suppose that $\beta \geqq \alpha > r + \frac{1}{2}$ *and that for* $p = 0, \ldots, r$, $\varphi_p \in P^{\beta-p-\frac{1}{2}}(R^{n-1})$. *Then there exists* $u \in P^\beta$ *such that* (8, 6) *and* (8, 7) *hold.*

In the next propositions we will use the following notations: as before x' denotes the first k coordinates of x as well as the corresponding point of R^k; x'' denotes the last $(n-k)$ coordinates of x as well as the corresponding point of R^{n-k} — the subspace where $x' = 0$. If E is a subset of R^n, $E_{x'}$ denotes the set of all x'' such that $(x', x'') \in E$. If f is a function in R^n, $f_{x'}$ is the function on R^{n-k} defined by $f_{x'}(x'') = f(x', x'')$. By $\mathfrak{A}''_{2\alpha}$, $\gamma''_{2\alpha}$, etc., are denoted the classes, functions, etc., corresponding to R^{n-k}.

1) *If* $A \subset R^n$ *and* $A \in \mathfrak{A}_{2\alpha}$ *and if* $0 \leqq \beta \leqq \alpha$ *then, for all* x' *exc.* $\mathfrak{A}'_{2\alpha-2\beta}$, $A_{x'} \in \mathfrak{A}''_{2\beta}$.

2) *If* $u \in P^\alpha(R^n)$ *and* $0 \leqq \beta \leqq \alpha$, *then* $u_{x'} \in P^\beta(R^{n-k})$ *for all* x' *exc.* $\mathfrak{A}'_{2\alpha-2\beta}$.

PROOFS. — We assume first that $0 < \beta < \alpha$. Consider $u \in P^\alpha(R^n)$. The function $\hat{h}(\xi) = (1 + |\xi'|^2)^{\frac{\alpha-\beta}{2}} (1 + |\xi''|^2)^{\frac{\beta}{2}} \hat{u}(\xi)$ is in L^2 and $||\hat{h}||_{L^2} \leqq ||(1 + |\xi|^2)^{\frac{\alpha}{2}} \hat{u}||_{L^2}$, hence its inverse Fourier transform $h(x)$ satisfies

$$(8, 9) \qquad ||h(x)||_{L^2(R^n)} \leqq ||u||_\alpha.$$

Since $\hat{u}(\xi) = (1 + |\xi'|^2)^{-\frac{\alpha-\beta}{2}} (1 + |\xi''|^2)^{-\frac{\beta}{2}} \hat{h}(\xi)$, by using the kernels $G'_{\alpha-\beta}$ and G''_β corresponding to spaces R^k and R^{n-k} we can write

$$(8, 10) \qquad u(x', x'') = \int_{R^k}\int_{R^{n-k}} G'_{\alpha-\beta}(x' - y') G''_\beta(x'' - y'') h(y', y'')\, dy'\, dy''.$$

This equality is at first valid only almost everywhere. However, if we apply the above formulas to the regularized functions $u_\rho = e_\rho * u$, the corresponding h_ρ is obviously $e_\rho * h$ and the equation

(8, 10')
$$u_\rho(x', x'') = \int_{R^k}\int_{R^{n-k}} G'_{\alpha-\beta}(x' - y')\, G''_\beta(x' - y'')\, h_\rho(y', y'')\, dy'\, dy''$$

is valid everywhere. We put now

$$w_{(\rho)}(x', y'') = \int_{R^k} G'_{\alpha-\beta}(x' - y')\, h_\rho(y', y'')dy',$$

$$h'_{(\rho)}(y') = \left[\int_{R^{n-k}} |h_\rho(y', y'')|^2 dy''\right]^{\frac{1}{2}}$$

and get

$$u_\rho(x', x'') = \int_{R^{n-k}} G''_\beta(x'' - y'') w_{(\rho)}(x', y'')dy'',$$

$$\|u_{\rho x'}\|_\beta^2 = \int_R |w_{(\rho)}(x', y'')|^2\, dy''$$

$$= \int_{R^{n-k}}\int_{R^k}\int_{R^k} G'_{\alpha-\beta}(x'-y')G'_{\alpha-\beta}(x'-z')\overline{h_\rho(y', y'')}h_\rho(z', y'')\, dy'\, dz'\, dy''$$

$$\leqq \left[\int_{R^k} G'_{\alpha-\beta}(x' - y')h'_{(\rho)}(y')\, dy'\right]^2,$$

(8, 11) $$\|u_{\rho x'}\|_\beta \leqq \int_R G'_{\alpha-\beta}(x' - y')h'_{(\rho)}(y')\, dy'.$$

Similarly, for $u_\rho - u_{\rho_1}$, putting

$$h'_{\rho,\rho_1}(y') = \left[\int_{R^{n-k}} |h_\rho(y', y'') - h_{\rho_1}(y', y'')|^2 dy''\right]^{\frac{1}{2}}$$

we get

(8, 11') $$\|u_{\rho x'} - u_{\rho_1 x'}\|_\beta \leqq \int_{R^k} G'_{\alpha-\beta}(x' - y')h'_{\rho,\rho_1}(y')\, dy'.$$

Since h'_{ρ,ρ_1} converges to 0 in $L^2(R^k)$ when ρ and $\rho_1 \to 0$, we can choose a sequence $\rho_m \searrow 0$ such that the series

$$h'_{\rho_1}(x') + \sum_{m=1}^{\infty} h'_{\rho_{m+1},\rho_m}(x') = H'(x')$$

converges strongly in $L^2(R^k)$ and hence the sequence $\{u_{\rho_m x'}\}$ converges in $P^\beta(R^{n-k})$ for all $x' \in R^k$ except where $G'_{\alpha-\beta}H'(x') = +\infty$ i.e. except a set $B' \in \mathfrak{A}'_{2\alpha-2\beta}$. For each x' outside of this set we can then choose a subsequence $\{u_{\rho_{m_i} x'}\}$ converging pointwise in R^{n-k} except on a set $B''_{(x')} \in \mathfrak{A}''_{2\beta}$.

To prove proposition 1) we replace u by $v = G_\alpha g$, $g \in L^2(R^n)$, $g \geq 0$, such that $v(x) = +\infty$ for $x \in A$. Then $v_\rho(x) \to v(x)$

everywhere in extended sense and now, denoting the above sets B' and $B''_{(x')}$ by A' and $A''_{(x')}$ we get $A_{x'} \subset A''_{(x')}$, and hence $A_{x'} \in \mathfrak{A}''_{2\beta}$ except for $x' \in A' \in \mathfrak{A}'_{2\alpha-2\beta}$.

To prove proposition 2), denote by A the set where $u_\rho(x)$ does not converge to $u(x)$. Using proposition 1) we find a set $A' \in \mathfrak{A}'_{2\alpha-2\beta}$ outside of which $A_{x'} \in \mathfrak{A}''_{2\beta}$. Then, for $x' \notin A' \cup B'$, $u_{\rho_{mi}x'}(x'')$ converges pointwise outside of $A_{x'}$ to $u_{x'}(x'')$ and hence $u_{x'}(x'')$ coincides with the limit of $\{u_{\rho_{m_i}x'}\}$ in

$$P^\beta(R^{n-k}) \text{ exc. } \mathfrak{A}''_{2\beta}.$$

We still have to settle the extreme cases: 1) $0 = \beta = \alpha$, 2) $0 = \beta < \alpha$, and 3) $0 < \beta = \alpha$. The first is trivial. The last two cases are treated like the general one: we must remember only that the operator G_θ reduces to identity for $\theta = 0$. For instance,

for $0 = \beta < \alpha$ (8, 10) becomes

$$u(x', x'') = \int_{R^k} G'_\alpha(x' - y')h(y', x'')\,dy'$$

for $0 < \beta = \alpha$ (8, 11) becomes $\|u_{\rho x'}\|_\beta \leqq h'_{(\rho)}(x')$.

As special cases or corollaries of proposition 2) we mention the following:

2 *a*) $0 < \beta = \alpha$. Almost everywhere in R^k, $u_{x'} \in P^\alpha(R^{n-k})$.

2 *b*) $0 = \beta < \alpha$. Except on a set of 2α-capacity 0 in R^k, $u_{x'} \in L^2(R^{n-k})$.

2 *c*) If $\alpha > \frac{n-k}{2}$, $u_{x'}$ is continuous in R^{n-k} except for x' in a set of 2δ-capacity 0 in R^k for any $\delta < \alpha - \frac{n-k}{2}$.

It should be noticed, however, that if $\alpha \geqq \frac{k}{2}$ we obtain from 2) that $u_{x'} \in P^{\alpha-\frac{k}{2}}(R^{n-k})$ except on a set of logarithmic capacity 0 in R^k whereas we know from Theorem 1 *a* that $u_{x'} \in P^{\alpha-\frac{k}{2}}(R^{n-k})$ for all $x' \in R^k$.

§ 9. Functions locally in P^α on an open set.

If D is an open set, $P^\alpha_{\text{loc}}(D)$ denotes the class of all functions on D which belong to P^α locally, that is, the class of all functions u defined on D exc. $\mathfrak{A}_{2\alpha}$ such that each point of D has a neighborhood on which u coincides with some function in P^α. Many results about functions in $P^\alpha_{loc}(D)$ are immediate consequences of results already proved about functions in P^α. For example, if $u \in P^\alpha_{loc}(D)$ and if $|i| \leq \alpha$, then $D_i u$ exists in the ordinary sense exc. $\mathfrak{A}_{2\alpha-2|i|}$ and belongs to $P^{\alpha-|i|}_{loc}(D)$. Or, if $u \in P^\alpha_{loc}(D)$ and if $2\alpha > n-k$, then the restriction of u to $D \cap R^k$ belongs to $P^{\alpha-\frac{n-k}{2}}_{loc}(D \cap R^k)$. In this section we give a few results about $P^\alpha_{loc}(D)$ which are not so obviously covered by the earlier theorems.

1) $u \in P^\alpha_{loc}(D)$ *if and only if for each compact* $K \subset D$ *there exists a function* $u_K \in P^\alpha(R^n)$ *which coincides with* u *on the set* K.

Proof. — The first half follows directly from the definition of P^α_{loc}. To prove the second part we use the method of partition of unity.

We choose a locally finite covering $\{U_i\}$ of D by open sets such that $\overline{U}_i \subset D$ and that each U_i be sufficiently small so that there exists a function $u_i \in P^\alpha(R^n)$ coinciding with u on U_i. We take then a partition of unity corresponding to the covering $\{U_i\}$ i.e. functions $\varphi_i \in C_0^\infty(R_n)$ with values between 0 and 1, such that each φ_i vanishes outside of U_i and that $\Sigma\varphi_i(x) = 1$ for each $x \in D$.

We take now those $U_{i_1}, \ldots, U_{i_l}$ which intersect the compact K; there is only a finite number of them. The function

$$\varphi_{i_1} u_{i_1} + \ldots + \varphi_{i_l} u_{i_l}$$

is then the desired function u_K. In fact, by proposition 6) § 2, each $\varphi_i u_i \in P^\alpha(R_n)$. Then for $x \in D$, $\varphi_i u = \varphi_i u_i$ and for $x \in K$, $\varphi_{i_1}(x)u_{i_1}(x) + \ldots + \varphi_{i_l}(x)u_{i_l}(x) = u(x)\Sigma\varphi_i(x) = u(x)$.

As corollaries from 1) we get

1′) *If* $u \in P^\alpha_{loc}(D)$ *and* $\varphi \in C^{(\alpha^*, 1)}$ *and has a compact support in* D *then* φu, *when extended by* 0 *outside of* D, *belongs to* $P^\alpha(R^n)$.

Here again we use 6) § 2.

1″) *If* u *has a compact support in* D *and we extend* u *by* 0 *outside of* D, *then* $u \in P^\alpha_{loc}(D)$ *if and only if* $u \in P^\alpha(R^n)$.

As we already mentioned before, Theorem 1, § 7 on differentiation has an obvious extension in a localized form to $P^\alpha_{loc}(D)$. The extension, however, of the converse, Theorem 2, is not so immediate. We prove it here under the weak hypotheses stated in Remark 2, section 7.

2) *Let* m *be an integer* $0 \leqq m \leqq \alpha$. *The function* u *belongs to* $P^{\alpha}_{loc}(D)$ *if*

a) u *is defined in* D *except a set of* 2α*-capacity* 0 *and for each* $\epsilon > 0$ *there exists* $B_\epsilon \subset D$ *with* $\gamma_{2\alpha}(B_\epsilon) < \epsilon$, *so that* u *restricted to* $D\text{-}B_\epsilon$ *is continuous.*

b) All pure derivatives $\dfrac{\partial^j u}{\partial x^k_j}$, $k = 1, \ldots, n$, $j = 0, 1, \ldots, m$, *exist pointwise almost everywhere and all those of order* $j < m$ *are equivalent to absolutely continuous functions on almost all lines parallel to the corresponding* x_k*-axis* [40].

c) The derivatives $\dfrac{\partial^m u}{\partial x^m_k}$, $k = 1, \ldots, n$, *are almost everywhere equal to functions in* $P^{\alpha-m}_{loc}(D)$.

Proof. — The first step is to prove that all the derivatives $\dfrac{\partial^j u}{\partial x^k_j}$, $j = 1, \ldots, m$ are locally L^2 in D. This is true for $j = m$ by *c*). To show it for $j < m$ we have to extend a lemma by Nikodym [13 *b*]. We introduce the following notations: Let Q be the closed cube $0 \leqq x_k \leqq a$, $k = 1, \ldots, n$, Q_k the face of Q lying in the coordinate hyperplane orthogonal to the x_k-axis, $x^{(k)}$ a variable point in Q_k. A point x in Q will be written $(x^{(k)}, x_k)$ for any k.

Lemma. — *Let* u *be defined almost everywhere in the n-dimensional cube* Q. *Suppose furthermore that for each* k, $1 \leqq k \leqq n$, *and for almost all* $x^{(k)} \in Q_k$, *the derivatives* $\dfrac{\partial^j u(x^{(k)}, x_k)}{\partial x^j_k}$, $j = 0, 1, \ldots, m-1$, *are equivalent to absolutely continuous functions in* $0 \leqq x_k \leqq a$. *Then, if each* $\dfrac{\partial^m u}{\partial x^m_k}$, $k = 1, 2, \ldots, n$ *is in* $L^2(Q)$, *all the derivatives* $\dfrac{\partial^j u}{\partial x^j_k}$, $k = 1, 2, \ldots, n$. $j = 0, 1, \ldots, m-1$ *are in* $L^2(Q)$.

Proof. — We use induction with respect to the dimension n. For $n = 1$, the theorem is obviously true. Suppose that it is true for dimension $n-1$ and fix an index k, $1 \leqq k \leqq n$.

[40] The last condition means that for almost all such lines l intersecting D the derivatives are absolutely continuous on every *closed* segment contained in $l \cap D$.

By our assumptions it is clear that we can find a function $p_k(x^{(k)}, x_k)$ defined for almost all $x^{(k)} \in Q_k$ and which is a polynomial of order $\leq m-1$ in x_k such that if $u_k(x)$ is defined by

$$(9,0)\quad u_k(x^{(k)}, x_k) = p_k(x^{(k)}, x_k) + \int_0^{x_k} \frac{\partial^m u(x^{(k)}, t)}{\partial x_k^m} \frac{(x_k - t)^{m-1}}{(m-1)!} dt$$

then

$$(9,0')\quad \frac{\partial^j u_k(x)}{\partial x_k^j} = \frac{\partial^j u(x)}{\partial x_k^j} \text{ almost everywhere in } Q \text{ for } j = 0, 1, \ldots, m.$$

Consider the $(n-1)$-dimensional cubes obtained by intersecting Q with the hyperplanes $x_k = \text{const.}$, $0 \leq x_k \leq a$. It is clear that by the hypotheses of our lemma, for almost all x_k in $[0, a]$ the function u satisfies in these $(n-1)$-cubes the conditions of our lemma with respect to the $n-1$ remaining variables. Hence we can find m distinct values a_i, $0 \leq a_1 < \ldots < a_m \leq a$ such that

$$(9,1)\quad \sum_{i=1}^m \int_{Q_k} |u_k(x^{(k)}, a_i)|^2 dx^{(k)} = \sum_{i=1}^m \int_{Q_k} |u(x^{(k)}, a_i)|^2 dx^{(k)} < \infty.$$

For almost all $x^{(k)}$ in Q_k, $p_k(x^{(k)}, t)$ can be determined by its values at $t = a_1, \ldots, a_m$ (we use the Lagrange interpolation formula). Therefore, from (9. 0) we get

$$(9,2)\quad u_k(x^{(k)}, x_k) = \int_0^{x_k} \frac{\partial^m u(x^{(k)}, t)}{\partial x_k^m} \frac{(x_k - t)^{m-1}}{(m-1)!} dt$$
$$+ \sum_{i=1}^m \frac{q(x_k)}{(x_k - a_i) q'(a_i)} \left[u_k(x^{(k)}, a_i) - \int_0^{a_i} \frac{\partial^m u(x^{(k)}, t)}{\partial x_k^m} \frac{(a_i - t)^{m-1}}{(m-1)!} dt \right],$$

where $q(t)$ is the polynomial $\prod_{i=1}^m (t - a_i)$ and q' is its derivative. By taking in (9, 2) the derivatives $\frac{\partial^j}{\partial x_j^k}$ for $j = 1, \ldots, m-1$ we derive without trouble an evaluation

$$(9,3)\quad \sum_{j=0}^{m-1} \int_0^a \left| \frac{\partial^j u_k(x^{(k)}, x_k)}{\partial x_k^j} \right|^2 dx_k \leq$$
$$c \left[\int_0^a \left| \frac{\partial^m u(x^{(k)}, x_k)}{\partial x_k^m} \right|^2 dx_k + \sum_{i=1}^m |u_k(x^{(k)}, a_i)|^2 \right]$$

with constant $c > 0$ depending only on the $a_i's$, a, m, and n. Integration in (9, 3) with respect to $x^{(k)}$ over Q_k and (9, 1) together with (9, 0') gives then the lemma.

Going back to the proof of proposition 2), consider a compact $K \subset D$ and a bounded open set $U \supset K$ such that $\overline{U} \subset D$. Take $\varphi \in C_0^\infty(R^n)$ such that $\varphi(x) = 1$ on K and $\varphi(x) = 0$ outside of U. We write $v = \varphi u$ extending this function by 0 outside of D. It is obvious that $v \in L^2(R^n)$ by the above lemma, that condition *a*) of theorem 2, § 7 is satisfied and that condition *b*) of this theorem in the weakened form of remark 2 is satisfied also. However, condition *c*) of the theorem, even in the weakened version of remark 2 presents still a problem. Let us write

$$(9, 4) \qquad \frac{\partial^m v}{\partial x_k^m} = \frac{\partial^m \varphi u}{\partial x_k^m} = \varphi \frac{\partial^m u}{\partial x_k^m} + \binom{m}{1} \frac{\partial \varphi}{\partial x_k} \frac{\partial^{m-1} u}{\partial x_k^m} + \cdots + \frac{\partial^m \varphi}{\partial x_k^m} u.$$

The first term is equivalent to a function in $P^{\alpha - m}$ but the best we know about the remaining terms, by the above lemma, is that they are in L^2. Hence $\frac{\partial^m v}{\partial x_k^m} \in L^2 = P^0$ and by theorem 2 (for $\alpha = m$) $v \in P^m$ and so $u \in P^m_{\text{loc}}(D)$.

Suppose that we know already that $u \in P^\beta_{\text{loc}}(D)$ for some β with $m \leqq \beta < \alpha$. Then by (9, 4) and theorem 1, § 7 (applied to $\frac{\partial^j u}{\partial x_k^j}$ with $j = m - 1$) we get that $\frac{\partial^m v}{\partial x_k^m}$ is equivalent to a function in P^{β_1} with $\beta_1 = \min(\alpha - m, \beta - m + 1)$. Again, by theorem 2, remark 2, it follows that v is a potential of order $\min(\alpha, \beta + 1)$ and hence u is locally such a potential. This procedure allows us to reach stepwise (in a number of steps smaller than $\alpha - m + 1$) the stage where $\min(\alpha, \beta + 1) = \alpha$, when the proof will be done.

For later use we record a similar proposition (with a similar proof) which gives sufficient conditions in order that a function on D be equal almost everywhere to a function in $P^\alpha_{\text{loc}}(D)$.

2') *If a measurable function u defined almost everywhere on* D *satisfies conditions b) and c) of proposition* 2, *then u is equal almost everywhere to a function in* $P^\alpha_{\text{loc}}(D)$.

A transformation is said to be of class $C^{(m, 1)}$ on an open set if each of its coordinate functions is of class $C^{(m, 1)}$ on the open set. A transformation is a homeomorphism of class $C^{(m, 1)}$ if both the transformation and its inverse are of class $C^{(m, 1)}$.

3) *If* T *is a transformation of* D^* *into* D *which is locally a homeomorphism of class* $C^{(\alpha^*, 1)}$ *then for each* $u \in P^{\alpha}_{loc}(D)$ *the function* $T^*u(x) = u(Tx)$ *belongs to* $P^{\alpha}_{loc}(D^*)$. *If* $\alpha \geq 1$, *the partial derivatives of* T^*u *are calculated by the usual formulas for the partial derivatives of composite functions.*

Proof. — By a classical theorem of topology, T transforms open sets on open sets [41]. Therefore, it can be assumed (by restricting ourselves to a subdomain of D^*) that T is a homeomorphism of class $C^{(\alpha^*, 1)}$. Also, by multiplying u by a function $\varphi \in C_0^\infty(D)$, it can be assumed that u has compact support in D. Hence it is sufficient to prove the following statement (which is a special case of the proposition).

4) *Let* T *be a homeomorphism of class* $C^{(\alpha^*, 1)}$ *of* D^* *onto* D, *and let* U *be a relatively compact open subset of* D. *Then there is a constant* c *such that if* $u \in P^\alpha$ *and* u *vanishes outside* U, *then* $T^*u \in P^\alpha$ *and* $||T^*u||_\alpha \leq c||u||_\alpha$.

Proof. — If u is of class $C^{(\alpha^*, 1)}$ then T^*u is also of class $C^{(\alpha^*, 1)}$, so, by proposition 5) § 2, $T^*u \in P^\alpha$. In evaluating the norm of T^*u, formula (1, 5) is used. For integral values of α the existence of the constant c is obvious from the classical formula for the transformation of multiple integrals. For non-integral values of α the double integral over R^n is dominated by a sum of integrals

$$\int_{R^n-U}\int_{R^n-U}, \quad \int_{R^n-D}\int_U \quad \text{and} \quad \int_D\int_D$$

The first of these is 0, since u vanishes outside U. The second is easily seen to be dominated by $|T^*u|_{\alpha^*}$ (the constant depends on the distance from U to the boundary of D), and hence by $|u|_\alpha$. Finally, the evaluations that were used in the proof of proposition 6) §2, and the classical formula for the transformation of integrals show that the third is also dominated by $|u|_\alpha$. Thus, the proposition is proved for u of class $C^{(\alpha^*, 1)}$. It is proved for arbitrary u by using approximations of class $C^{(\alpha^*, 1)}$. The approximating functions can

[41] If $\alpha^* > 0$, the implicit function theorem can be used here, but if $\alpha^* = 0$ the theorem of Brouwer is needed.

be chosen to vanish outside an arbitrary neighborhood of U, and the above results applied to this neighborhood.

Proposition 3) shows that $P^{\alpha}_{\text{loc}}(D)$ is defined not only when D is an open subset of R^n, but actually when D is an open subset of any differentiable manifold of class $C^{(\alpha^*, 1)}$. This fact will be important in chapter IV, part. II.

To finish this section we prove that some general potentials are locally in P^{α}.

Consider a measure μ (in general a signed measure) for which the potential $G_\alpha\mu$ has a meaning. Following (6, 4) this requires

$$(9,5) \qquad \int (1+|x|)^{\frac{\alpha-n-1}{2}} e^{-|x|} |d\mu(x)| < \infty.$$

In general $G_\alpha\mu(x)$ is defined and finite exc. $\mathfrak{A}_{2\alpha}$ (see 18), § 6). However, more can be said under additional assumptions.

5) *If* μ *satisfies* (9, 5) *and in some domain* D, $d\mu(x) = g(x)\,dx$ *with* $g \in P^{\beta}_{\text{loc}}(D)$, *then* $u = G_\alpha\mu$ *restricted to* D *is in* $P^{\alpha+\beta}_{\text{loc}}(D)$.

PROOF. — Take any bounded open set U with $\overline{U} \subset D$ and take $\varphi \in C^\infty_0(R^n)$ and such that $\varphi(x) = 1$ on U and φ vanishes outside of a compact lying in D. Then $d\mu = \varphi g\,dx + (1-\varphi)\,d\mu$ and

$$(9,6) \qquad G_\alpha\mu(x) = \int G_\alpha(x-y)\,\varphi(y)\,g(y)\,dy + \int G_\alpha(x-y)\,(1-\varphi(y))\,d\mu(y).$$

Since $\varphi g \in P^{\beta}(R^n)$, the first potential $\in P^{\alpha+\beta}(R^n)$. In the second, there are no masses in the open set U and hence the second integral is an analytic regular function of the n variables $x = (x_1, \ldots, x_1)$ in U ([42]). It follows from 2) that the second integral belongs to all classes $P^{\nu}_{\text{loc}}(U)$ and thus $u \in P^{\alpha+\beta}_{\text{loc}}(U)$. Since u is an arbitrary open bounded subset of D we get our statement.

REMARK. — A statement similar 5) can be proved concerning more classical potentials such as newtonian or more generally Riesz potentials $R_\alpha\mu$ corresponding to the kernel $R_\alpha(r) = C_\alpha r^{\alpha-n}$ ([43]).

([42]) To see this, in the integral replace the variables x_k by complex variables z_k, which is possible since $G_\alpha(r)$ is an analytic function of r regular outside $r = 0$.

([43]) For $\alpha \geq n$, and $\alpha - n$ an even integer, this definition of R_α should be changed by putting a factor $\lg \frac{2}{r}$ and adding similar terms of lower order in r.

Since for $\alpha \leq n$, up to a constant coefficient, $R_\alpha(r)$ is the principal term of the development of $G_\alpha(r)$ around 0, both kernels together with all their derivatives behave alike in any fixed bounded set ([44]). It follows that when the measure μ (or density g) have *compact support* the potentials $R_\alpha\mu$ and $G_\alpha\mu$ behave alike in every *bounded* domain as concerns sets where they are defined, or differentiability or the transfer of differentiation under the sign of integral. The essential difference between the potentials $R_\alpha\mu$ and $G_\alpha\mu$ is in their global behavior; $R_\alpha\mu$ is much less manageable than $G_\alpha\mu$. This is due to the fact that the kernel $R_\alpha(r)$ is never in L^2 or in L^1. However, for $\alpha < \frac{n}{2}$, the kernel is L^1 inside a sphere $r < r_0$ and L^2 outside of the sphere, which allows the use of classical theorems in Fourier transforms for the corresponding potentials of functions in L^2.

Coming back to the extension of 5) to $R_\alpha\mu$ we notice that the condition (9, 5) should be replaced now by

$$(9,7) \qquad \int (1 + |x|)^{\alpha - n} d\mu(x) < \infty \text{ ([45])}.$$

Replacing (9, 5) by (9, 7) in the statement of 5), and $u = G_\alpha\mu$ by $u = R_\alpha\mu$, the proof proceeds in the same way (with G replaced by R). The second integral in (9,6) is again analytic in U. The first, however, causes more trouble, and we must use proposition 2). To this effect we differentiate (which we may do) m times under the integration sign, m being such that $0 < \alpha - m < \frac{n}{2}$. The resulting differentiated kernels $D_j R_\alpha$ are then in L^1 in a sphere and in L^2 outside of a sphere which allows the application of classical theorems to the Fourier transform of $D_j R_\alpha(\varphi g)$ and we obtain

$$\widehat{D_j R_\alpha(\varphi g)} = (2\pi)^{n/2}\, \widehat{D_j R_\alpha}\; \widehat{(\varphi g)} = (i)^m \xi^j |\xi|^{-\alpha}\, \widehat{(\varphi g)}.$$

Since $\widehat{\varphi g}$ is an entire function and $(1 + |\xi|^2)^{\beta/2} \widehat{\varphi g}$ is in L^2 and since $|j| = m$, $|\xi^j| \leq |\xi|^m$, $|\xi^j |\xi|^{-\alpha}| < |\xi|^{m-\alpha}$ it follows that

([44]) For $\alpha \geq n$, these facts are true only for derivatives of orders $\geq \alpha - n$ but those of lower orders are continuous at 0.

([45]) Or the corresponding expression when $\alpha - n$ is an even non-negative integer.

$(1+|\xi|^2)^{\frac{\alpha+\beta-m}{2}}\widehat{D_jR_\alpha(\varphi g)}$ is in L^2, hence $D_jR_\alpha(\varphi g)\in P^{\alpha+\beta-m}$ which makes it possible to apply 2).

This line of proof leaves out two exceptional cases: $n=2$, α an integer and $n=1$, $\alpha=k+\alpha'$ with k integer and

$$1/2<\alpha'<1.$$

Another way of proving that $R_\alpha(\varphi g)\in P^{\alpha+\beta}_{\text{loc}}(U)$ which avoids the exceptional cases is to prove it first for small α, $0<\alpha<\frac{n}{2}\cdot$ We then use the composition formula $R_\alpha=R_{\alpha_1}*R_{\alpha_2}*\cdots*R_{\alpha_m}$ with $\alpha=\alpha_1+\cdots+\alpha_m$, $0<\alpha_k<\frac{n}{2}\cdot$ We have successively $R_{\alpha_m}(\varphi g)\in P^{\alpha_m+\beta}_{\text{loc}}(U)$, $R_{\alpha_{m-1}}*R_{\alpha_m}(\varphi g)$ $P^{\alpha_{m-1}+\alpha_m+\beta}_{\text{loc}}(U)$ and so on till $R_\alpha(\varphi g)\in P^{\alpha+\beta}_{\text{loc}}(U)$. Since the composition formula is valid only for $\alpha<n$, for $\alpha\geq n$ we have to replace it by an approximate composition formula where the composition is taken not over the whole space but over a sufficiently large sphere containing U. The result of the composition differs then from R_α by a function regular in U and thus the proof can be achieved.

§ 10. — Relations between the classes P^α and L^q.

In this section we establish the L^q class to which a potential $G_\alpha f$, $f\in L^p$, $p\geq 1$, belongs. When $p=2$ we obtain the L^q class to which the functions in P^α belong.

1) *If $f\in L^p$, $p\geq 1$, then $G_\alpha f\in L^q$ for every $q\geq p$ satisfying*

$$\frac{1}{q}\geq\frac{1}{p}-\frac{\alpha}{n}\quad \textit{if}\quad p>1\quad\textit{and}\quad \frac{1}{p}-\frac{\alpha}{n}\neq 0,$$
$$\frac{1}{q}>\frac{1}{p}-\frac{\alpha}{n}\quad \textit{if}\quad p=1\quad\textit{or}\quad \frac{1}{p}-\frac{\alpha}{n}=0,$$

and there is a constant M *depending only on α, n, p, and q such that*

$$||G_\alpha f||_{L^q}\leq M||f||_{L^p}.$$

Proof. — A classical theorem of W. H. Young states that if $\frac{1}{r}+\frac{1}{p}\geq 1$, $1\leq r\leq\infty$, $1\leq p\leq\infty$, $\frac{1}{q}=\frac{1}{r}+\frac{1}{p}-1$ and if $u\in L^r(R^n)$, $v\in L^p(R^n)$ then

$$||u*v||_{L^q}\leq||u||_{L^r}||v||_{L^p}.$$

Since $G_\alpha \in L^r(R^n)$ for every $r \geqq 1$ with $\frac{1}{r} > 1 - \frac{\alpha}{n}$ we get immediately for any $q \geqq p$ satisfying $\frac{1}{q} > \frac{1}{p} - \frac{\alpha}{n}$

$$||G_\alpha F||_{L^q} \leqq ||G_\alpha||_{L^r} ||f||_p \qquad \textit{with} \qquad \frac{1}{r} = \frac{1}{q} - \frac{1}{p} + 1.$$

This gives the statement (with $M = ||G_\alpha||_{L^r}$) except when $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} > 0$ and $p > 1$. In this case, however, Soboleff's theorem, [15 b] or [14], states that

$$||R_\alpha f||_{L^q} \leqq M||f||_{L^p}.$$

Since $\alpha < \frac{n}{p} < n$ and $R_\alpha(x) \geqq G_\alpha(x)$, it follows that

$$||G_\alpha f||_{L^q} \leqq M||f||_{L^p},$$

which finishes the proof.

Corollary. — $P^\alpha \subset L^q$ *for every* $q \geqq 2$ *satisfying*

$$\frac{1}{q} \geqq \frac{1}{2} - \frac{\alpha}{n} \quad \textit{if} \quad \alpha \neq \frac{n}{2}, \qquad \frac{1}{q} > 0 \quad \textit{if} \quad \alpha = \frac{n}{2}.$$

§ 11. — Comparison of the class P^α with various other classes.

In the present section we are going to compare our potentials of order α with other classes of functions introduced and used previously by different authors. These are essentially the classes of Riesz potentials of order α $\left(\text{for } \alpha < \frac{n}{2}\right)$, the (BL)-classes and the classes $H^\alpha = W^\alpha = W_2^\alpha$ for α a non-negative integer. We will not give proofs in the remarks which follow; most of these proofs rely on arguments similar to those used in preceding sections.

1. *The Riesz potentials of order* α. — These were introduced by the present authors in [1] for $\alpha < n/2$ as the perfect functional completion of the class C_0^∞ with respect to the Dirichlet norm of order α, $\sqrt{d_\alpha(u)}$. They were also introduced by J. Deny [8] as potentials of magnetic distributions of order

2α which form the completion of the class of signed measures of finite 2α-energy with respect to the energy norm. For $\alpha \geqq n/2$, C_0^∞ does not have a functional completion with respect to the norm $\sqrt{d_\alpha(u)}$ (see 2), § 1).

By using the same kind of computations as those which lead in § 1 to (1, 5) (or (1, 10)) we can give a direct formula for $d_\alpha(u)$ without using derivatives, for $u \in C_0^\infty$. To this effect we introduce the k-th differences of $u(x)$, $\Delta^k u(x, z_1, \ldots, z_k)$ as follows.

$$(11, 1) \quad \begin{aligned} &\Delta^0 u(x) = u(x), \\ &\Delta^{k+1} u(x; z_1, \ldots, z_k, z_{k+1}) = \Delta^k u(x; z_1, \ldots, z_k) \\ &\qquad\qquad - \Delta^k u(x + z_{k+1}; z_1, \ldots, z_k). \end{aligned}$$

We then take any decomposition $\alpha = \alpha_1 + \cdots + \alpha_k$, $0 < \alpha_i < 1$ and obtain

(11, 2)
$$d_\alpha(u) = \frac{1}{\Pi C(n, \alpha_i)} \int_{R^n} \cdots \int_{R^n} \frac{|\Delta^k u(x; z_1, \ldots, z_k)|^2}{|z_1|^{n+2\alpha_1} \ldots |z_k|^{n+2\alpha_k}} dx\, dz_1 \cdots dz_k \text{ [46]}.$$

In this way $d_\alpha(u)$ has a meaning for all measurable functions u. We could be tempted to consider those u for which (11, 2) is finite as Riesz potentials of order α. However, on the class of all such functions $\sqrt{d_\alpha(u)}$ is a pseudo-norm; it is 0 if (and only if) u is equivalent to *a* polynomial of order $< k$. It can be proved that this class is independent of the decomposition $\alpha = \alpha_1 + \ldots + \alpha_k$ except for the adjunction of additional polynomials when we increase k.

Let us call the class of u with (11, 2) finite $F^{\alpha,k}$. Obviously $\Delta^k u(x, z_1, \ldots, z_k)$ is L^2 for almost all systems $(z_1, \ldots, z_k)$. From this, by an inductive argument as in Nikodym's lemma (see § 9), we can prove that $u \in L^2_{loc}(R^n)$.

If $\alpha < n/2$ it can be proved that $F^{\alpha,k}$ admits of a direct decomposition,

$$(11, 3) \quad F^{\alpha, k} = [\textit{polynomials of order} < k] \dotplus \overline{C_0^\infty},$$

the second class being the functional completion of C_0^∞ under the norm (11, 2) relative to the class of sets of measure 0,

[46] A similar formula, using higher differences instead of derivatives can be obtained for $\|u\|^2_\alpha$. The reason such formulas are not given in the main text is that they do not lead in a simple manner to norms in subdomains of R^n.

i.e. the class of all functions equivalent to Riesz potentials of order α. Such decomposition is no longer possible for $\alpha \geqq \frac{n}{2}$.

Our space P^{α} is a subclass of $F^{\alpha, k}$. A function u in $F^{\alpha, k}$ which is equivalent to a function in P^{α} is characterized by the simple fact that $u \in L^2$. Each function in $F^{\alpha, k}$ can be proved to be equivalent to a function in $P^{\alpha}_{\mathrm{loc}}(R^n)$. We can consider as the essential part of $F^{\alpha, k}$ the subclass $\check{F}^{\alpha, k}$ of $P^{\alpha}_{\mathrm{loc}}(R^n)$ with finite (11, 2). If we add $\int_S |u|^2\, dx$ to (11, 2) for a fixed bounded set of positive Lebesgue measure, we obtain a quadratic norm which makes $\check{F}^{\alpha, k}$ into a complete functional space. This space is the perfect functional completion of $C^{\infty}(R^n) \cap F^{\alpha, k}$ with respect to this new norm. For $\alpha > n/2$ it will be a proper functional space.

2. *(BL)-classes of order* α. — These classes were introduced first for $\alpha = 1$ by O. Nikodym [13 *b*], for α a positive integer by J. Deny [8], and for arbitrary α by J. Deny and J. L. Lions [8 *a*] as a special case of much more general (BL)-classes which, besides those which are akin to our spaces P^{α}, contain many other important classes. They are divided into two categories : those « in large sense » and those in « precise sense ».

Those in large sense akin to our P^{α} (which we will denote by $(BL)_{\alpha}$) were formed explicitly for α an integer by assuming that the derivatives exist pointwise a.e. (original definition) or that they are taken in the sense of distributions and that the last derivatives (of order α when α integer, of order α^* when α is not an integer) have a finite $\sqrt{d_{\alpha - \alpha_*}}$ norm. These classes are essentially the same as the above classes $F^{\alpha,k}$ except that the latter contain polynomials of higher order and that in the original definition the exceptional class of sets was somehow smaller than the sets of measure 0 (see [8 *a*]).

The classes in precise sense differ from those in large sense only by the fact that their exceptional sets are taken smaller than in the original, namely they are sets of 2-capacity 0.

Spaces $H^m = W^m = W^2_m$, m positive integer. — These spaces are used by many authors working in partial differential equations (the notation depending on the author). A function u

belongs to $H^m(R^n)$ if it is L^2 and has strong derivatives in L^2 sense of all orders $\leq m$.

The space H^m is exactly the functional completion of C_0^∞ with respect to our norm $|u|_m$ or $\|u\|_m$ relative to the class of all sets of Lebesgue measure 0. It is therefore an « imperfect » version of P^m.

GENERAL REMARK. — All functions in the above considered spaces are equivalent to functions in the corresponding P^α or P^α_{loc}. We can always replace the former by the latter for which all our results concerning differentiability and restrictions to lower dimensional spaces, etc., are valid. This replacement can be very easily achieved by considering as the corrected value of the function u at a point x the mean value limit

$$\lim \frac{1}{|S|}\int_S u(y)\,dy = u'(x)$$

for spheres S with center at x, with radius converging to 0. The corrected function u' is defined wherever the limit exists (in any case a.e. if u is locally integrable), it is equivalent to u (when $u \in L^1_{\text{loc}}$) and belongs to P^α or P^α_{loc} whenever u is equivalent to such a function.

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