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METRIC TRANSITIVITY AND INTEGER-VALUED FUNCTIONS [1]

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Let X be a measure space with measure μ satisfying

$$\mu(X) = 1.$$

Suppose φ is a measurable map of X onto itself such that $\mu(\varphi^{-1}(s)) = \mu(s)$ for every measurable set S. Throughout this paper B will denote the additive group of bounded measurable integer-valued functions. We will denote by H_p the subset of B consisting of all functions $f(x)$ in B such that

$$0 \leqq f(x) \leqq p - 1$$

for all x with the exception that we exclude the function which is identically equal to $p - 1$ from H_p. We will follow the convention that two functions are to be regarded as identical if they differ only set of measure zero.

THEOREM. — *Statements* 1,2, *and* 3_p (*p any integer greater than* 2) *are equivalent.*

1) φ is metrically transitive.

2) Every $f(x)$ in B has a unique representation of the form $k + \alpha_0 + (2 - T)\alpha_1 + \cdots + (2 - T)^n\alpha_n$ where k is an integral constant, $\alpha_i \in H_2$, and α_n is not identically zero.

3_p) Every non-negative $f(x)$ in B has a unique representation of the form $\alpha_0 + (p - T)\alpha_1 + \cdots + (p - T)^n\alpha_n$ where $\alpha_i \in H_p$ and α_n is not identically zero.

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In the above statements, $(p - \mathrm{T})^k$ is the k^{th} iterate of the operator which sends $f(x)$ into $pf(x) - f(\varphi(x))$.

Notice that if we take our measure space to consist of a single point and our transformation φ is just the identily map, statement 3_p simply tells us that any non-negative integer can be expressed uniquely in a « decimal » expansion to the base $p - 1$.

It is easy to see that metric transitivity follows from any one of the statements 2 or 3_p. Suppose 2 holds and φ is not metrically transitive. Let $f(x)$ denote the characteristic function of a proper invariant subset of X. If

$$f(x) = k + \alpha_0 + \cdots + (2 - \mathrm{T})^n \alpha_n,$$

then $(2 - \mathrm{T})f(x) = k + (2 - \mathrm{T})\alpha_0 + \cdots + (2 - \mathrm{T})^{n+1}\alpha_n$. Since $f(x)$ is invariant, $(2 - \mathrm{T})f(x) = f(x)$ so we get two expansions for $f(x)$. By the uniqueness part of 2), the expansions must be identical and therefore $f(x)$ must be a constant, which contradicts the fact that $f(x)$ is the characteristic function of a proper subset of X. If 3_p holds let $f(x)$ be as above. Since $f(x)$ is invariant, $(p - \mathrm{T})f(x) = (p - 1)f(x)$. On the left side of the equation we have an expansion of type 3_p with $\alpha_1 = f(x)$ and all other $\alpha_i = 0$, while on the right hand side of the equation we have an expansion of type 3_p with $\alpha_0 = (p - 1)f(x)$ and all other $\alpha_i = 0$. Thus we again have a contradiction of uniqueness. The non-trivial part of our theorem therefore comes in showing that metric transitivity implies the existence and uniqueness of the expansions described in 2 and 3_p. To prove this we need a preliminary result. In all that follows we assume that φ is metrically transitive.

LEMMA. — *If $f(x) \in \mathrm{B}$ and p is any integer greater than or equal to two, there exists one and only one pair of functions $(\alpha(x), \mathrm{I}(x))$ such that $\alpha(x) \in \mathrm{H}_p$, $\mathrm{I}(x) \in \mathrm{B}$ and $f = \alpha + (p - \mathrm{T})\mathrm{I}$. Moreover the following relations hold between f, α and* I.

a) Ess sup $f \geqq (p - 1)$ ess sup I, with equality holding if and only if $\alpha = 0$, $\mathrm{I} =$ constant.

b) Ess inf $f < (p - 1)\,(1 +$ ess inf I).

Once we know that a representation of the kind indicated exists it is a straightforward verification to show that relations

a) and b) must hold. In fact, suppose that $f = \alpha + (p - \mathrm{T})\mathrm{I}$, $\alpha \in \mathrm{H}_p$, $\mathrm{I} \in \mathrm{B}$. Let U be the set on which I assumes its essential supremum. For every $x \in \mathrm{U}$, $(p - \mathrm{T})\mathrm{I}$ evaluated at x is obviously $\geqq (p - 1)$ ess sup I, with equality holding if and only if $\varphi(x) \in \mathrm{U}$. Since $\alpha(x) \geqq 0$, $\alpha + (p - \mathrm{T})\mathrm{I}$ evaluated at x is $\geqq (p - 1)$ ess sup I, and equality holds if and only if $\alpha(x) = 0$ and $\varphi(x) \in \mathrm{U}$. If these conditions held for almost all points in U we would have $\varphi(\mathrm{U}) \leqq \mathrm{U}$ and therefore by metric transitivity U would equal X; moreover $\alpha(x)$ would then have to be zero almost everywhere. In that case, we would have $f(x)$ and $\mathrm{I}(x)$ constants with $f = (p - 1)\mathrm{I}$ so our inequality would in fact be an equality. If these conditions de not hold almost everywhere in U, then there is a subset of U of positive measure on which $f(x) > (p - 1)$ ess sup I and therefore ess sup $f > (p - 1)$ ess sup I.

Similarly, in order to prove b) let L denote the subset of X consisting of all points x for which $\mathrm{I}(x) =$ ess inf I. For any $x \in \mathrm{L}$, $\alpha + (p - \mathrm{T})\mathrm{I}$ evaluated at x is less than or equal to $(p - 1)(1 +$ ess inf I), since $\alpha(x) \leqq p - 1$. Moreover, equality holds if and only if $\alpha(x) = p - 1$ and $\varphi(x)$ belongs to L. If these conditions held for almost all points of L, by metric transitivity L would be almost all of X and so α would equal $p - 1$ almost everywhere. However this contradicts the assumption that α is in H_p. Therefore, there exists a subset of L of positive measure such that $\alpha + (p - \mathrm{T})\mathrm{I}$ evaluated at any point of the subset is less than $(p - 1)(1 +$ ess inf I). Thus ess inf $f < (p - 1)(1 +$ ess inf I).

To complete the proof of the lemma we must now show that every $f \in \mathrm{B}$ can be expressed in one and only one way in the form stated in the lemma. We begin by assigning to each f in B a function F whose value at any point x in X is given by the formula

$$\mathrm{F}(x) = \exp 2\pi i \left(\frac{f(x)}{p} + \frac{f(\varphi(x))}{p^2} + \cdots + \frac{f(\varphi^n(x))}{p^{n+1}} + \cdots \right).$$

Clearly $\mathrm{F}(x)$ is a measurable function of absolute value one satisfying the equation $\mathrm{F}(\varphi(x)) = [\mathrm{F}(x)]^p$. If we denote by G_p the collection of all measurable functions of absolute value one satisfying this functional equation, it is clear that G_p forms a group under multiplication. As usual we identify two functions in G_p which agree except on a set of measure zero.

Now let ρ stand for the transformation which assigns to each f in B the function F in G_p defined above. It is trivial to verify that ρ is homomorphism of the additive group of B into the multiplicative group G_p. We wish to determine the kernel of this homomorphism.

We regard B as a subset of $L^2(X)$ and let T denote, as usual, the transformation sending $f(x)$ into $f(\varphi(x))$. Since T is a transformation of norm one, the operator series $I/p + T/p^2 + \cdots + T^n/p^{n+1} + \cdots$ converges to the operator $(p - T)^{-1}$. A function in B belongs to the kernel of ρ if and only if $f(x)/p + \cdots + f(\varphi^n(x))/p^{n+1} + \cdots$ is integer-valued; that is, if and only if $(p - T)^{-1}f$ belongs to B. Thus every f in the kernel of ρ can be expressed in one and only one way in the form $(p - T)I$, where I belongs to B. Thus the lemma will be proved if we can show that each coset of the homomorphism contains exactly one element of H_p.

Now let g be any function in G_p. There is one and only one function $\alpha(x)$ such that $0 \leqq \alpha(x) < 1$ and $g(x) = \exp 2\pi i\alpha(x)$. The set S of points at which $\alpha(x)$ is a p-adic rational is clearly invariant under φ because of the equation $g(\varphi(x)) = [g(x)]^p]$. In fact, if 0 denotes the set of points at which $\alpha(x) = 0$, it is clear that $S = \sum_{n=0}^{\infty} \varphi^{-n}(0)$. It is obvious that $\varphi^{-(n+1)}(0) \geqq \varphi^{-n}(0)$. Since φ is measure preserving $\mu(0) = \mu(S)$ because S is defined the union of an increasing sequence of sets, with all sets in the sequence having the same measure and therefore differing by a null set from the set 0 which begins the sequence. Since φ is metrically transitive and S is invariant, S is either the whole space or a null set. Since S differs from 0 by a null set it follows that if g is not the function which is identically 1, the set of points x at which $\alpha(x)$ is a p-adic rational is a null set; i.e., $\alpha(x)$ admits of a unique expansion as a p-adic decimal almost everywhere. Thus, if $\alpha(x)$ is not identically zero there exists a unique sequence of integer-valued functions $e_1(x), \ldots, e_k(x), \ldots$ taking values from zero through $p - 1$ and satisfying the equation $\alpha(x) = e_1(x)/p + e_2(x)/p^2 + \cdots$. Since $\exp 2\pi i\alpha(\varphi(x)) = \exp 2\pi i p\alpha(x)$ it follows that $e_1(\varphi(x))/p + e_2(\varphi(x))/p^2 + \cdots$ differs from $e_1(x) + e_2(x)/p + \cdots + e_{k+1}(x)/p^k + \cdots$ and hence from

$e_2(x)/p + \cdots + e_{k+1}/p^k + \cdots$ by an integer. Since, however, both these expression represent functions taking on values greater than or equal to zero but less than one it follows that they must be equal. Since, moreover, the set of points x for which $\alpha(x)$ is a p-adic rational is a null set not only the functions but the expansions as p-adic decimals must be identical almost everywhere; i.e., $e_n(\varphi(x)) = e_{n+1}(x)$ and therefore $e_k(x) = e_1(\varphi^{k-1}(x))$. Clearly $e_1(x)$ belongs to H_p and we have $g(x) = \exp 2\pi i(e_1(x)/p + \cdots + e_1(\varphi^k(x))/p^{k+1} + \cdots)$. Moreover no other function in H_p would serve in place of e_1 since that would give a second p-adic decimal expansion for $\alpha(x)$. Thus for each g in G_p there is one and only one function in H_p which gets sent into g by the homomorphism ρ. This completes the proof of the lemma.

We now proceed to the proof of our main theorem. Let f be any function in B. Then $f = \alpha_0 + (p - T)I_0$ where $\alpha_0 \in H_p$, $I_0 \in B$. Inductively we define α_n and I_n in H_p and B respectively by the equation $I_{n-1} = \alpha_n + (p - T)I_n$. Then by successive substitutions we see that

$$\begin{aligned} f = \alpha_0 + (p - T)I_0 &= \alpha_0 + (p - T)\alpha_1 + (p - T)^2 I_1 = \cdots \\ &= \alpha_0 + (p - T)\alpha_1 + \cdots + (p - T)^n \alpha_n + (p - T)^{n+1} I_n. \end{aligned}$$

Applying the two inequalities of our lemma to the representation $I_{n-1} = \alpha_n + (p - T)I_n$ we get ess sup $I_{n-1} \geqq (p - 1)$ ess sup I_n and ess inf $I_{n-1} < (p - 1)(1 + \text{ess inf } I_n)$. Next let $a_n =$ ess sup I_n — ess inf I_n. By subtracting our two inequalities we see that $a_{n-1} > (p - 1)(a_n - 1)$. Therefore, $a_{n-1} > a_n - 1$, and since the numbers a_n are integers, $a_{n-1} \geqq a_n$. Thus the sequence $\{a_n\}$ is a decreasing sequence of non-negative integers and so must eventually equal some non-negative integer k.

We wish to show that k must be zero. Suppose this is not the case. Then in the inequality ess sup $I_n \geqq (p - 1)$ ess sup I_{n+1} we could not have equality holding, since the complete statement of inequality a) of our main lemma tells us that if equality holds I_{n+1} must be a constant. If that were so a_{n+1} and therefore k would have to be zero.

In the remaining case, since we are dealing with integers, our inequalities become ess sup $I_n \geqq 1 + (p - 1)$ ess sup I_{n+1} and ess inf $I_n \leqq -1 + (p - 1)(1 + \text{ess inf } I_{n+1})$. Subtracting and taking n sufficiently large we get

$$k \geqq 2 + (p - 1)(k - 1)$$

or equivalently $(2 - p)(k - 1) \geqq 1$. Since p is greater than or equal to two this can only be the case if k is zero. Thus for sufficiently large values of n we get ess sup I_n = ess inf I_n; i.e., I_n is a constant. If we let N be the smallest integer for which I_N is a constant, then clearly $n \geqq N$ implies I_n is a constant.

Let us next confine our attention to the case $p = 2$. Since I_N is a constant, $I_N = 0 + (2 - T)I_N$; i.e., $\alpha_{N+1} = 0$ and $I_{N+1} = I_N$. Proceeding inductively we see that $\alpha_n = 0$ for $n \geqq N + 1$. Now the equation

$$f = \alpha_0 + (2 - T)\alpha_1 + \cdots + (2 - T)^N\alpha_N + I_N$$

holds, since $(2 - T)^{N+1}I_N = I_N$. This shows the existence of the representation described in 2) of our main theorem. Moreover, if $f = \bar{\alpha}_0 + (2 - T)\bar{\alpha}_1 + \cdots + (2 - T)^r\bar{\alpha}_r + k$ is any representation for f of the type described, it is clear that $f = \bar{\alpha}_0 + (2 - T)(\bar{\alpha}_1 + \cdots + (2 - T)^{r-1}\bar{\alpha}_r + k)$ and so $\bar{\alpha}_0 = \alpha_0$ by the uniqueness part of our main lemma. Proceeding inductively, we see that $\bar{\alpha}_i = \alpha_i$ for all i. Finally, solving for k and I_N in each of the equations giving the two representations for f shows that $k = I_N$. This shows the existence and uniqueness in the case $p = 2$.

For p greater than 2 the proof that there is at most one representation of the type described proceeds exactly as above. To complete the proof we therefore need only show that there exists a represehtation of the type described and for this we need only show that I_n is eventually zero. Since we are now in the case 3_p we have available the assumption that $f(x)$ is non-negative. Thus using our inequality b we see that $0 \leqq$ ess inf $f < (p - 1)(1 +$ ess inf $I_0)$ and so ess inf $I_0 > -1$; or, since we are dealing with integers, ess inf $I_0 \geqq 0$. Proceeding inductively we see that ess inf $I_n \geqq 0$ for all n. On the other hand, by inequality a) ess sup $I_k \leqq$ ess sup $I_0/(p - 1)^k$. Thus for sufficiently large values of n, ess sup $I_n \leqq 0$ and ess inf $I_n \geqq 0$; i.e., ess sup I_n = ess inf $I_n = I_n = 0$. This completes the proof of our theorem.