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BRUCE L. REINHART

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THE WINDING NUMBER ON TWO MANIFOLDS

par Bruce L. REINHART (R.I.A.S.) (1).

In his thesis [6], Smale has found the regular homotopy classes of regular closed curves (*i. e.*, immersed circles) on a Riemannian manifold M. His work leaves unanswered the question: Which homotopy classes contain embedded circles? We may assume tha the dimension of M is 2, since otherwise the problem is trivial. Then our question has been answered for the plane by Whitney [9] by use of the winding number. For the torus, we have extended the winding number technique to get necessary conditions for the existence of a simple closed curve in a given regular homotopy class [3, 4].

In this paper, we shall define the winding number, or more precisely the winding homomorphism, for compact orientable two manifolds; it is a homomorphism from the regular homotopy group of M into the integers modulo $\varkappa$, where $\varkappa$ is the Euler characteristic of M. We shall compute the value of w for a regular simple closed curve, assuming its homotopy class (in the usual sense) is known. As applications, we get conditions for the nonexistence of periodic solutions of differential equations, and necessary and sufficient conditions for regular homotopy of curves on the sphere and the torus.

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1. — The winding homomorphism. Axioms.

Let M be a compact, connected, oriented, two dimensional Riemannian manifold of class C^3 and Euler number $\varkappa$. A parametrized curve on M will be a mapping of the unit interval $I = \{t/0 \leqslant t \leqslant 1\}$ into M; it will be called regular if it is C^1 and has everywhere a nonzero derivative, that is, the image has a nonzero tangent vector at each point. A regular curve is an equivalence class of parametrized regular curves under the relation: two parametrized curves are equivalent if their parameters are related by a function with everywhere positive derivative. Corresponding to every regular curve C there is an induced curve $\dot{C} : I \rightarrow T(M)$ where $T(M)$ is the bundle of unit vectors tangent to M. $\dot{C}$ will be called the tangent curve to C. A homotopy of curves will be called regular if each stage is a regular curve, and there is induced a homotopy of the tangent curves. A regular curve is closed if its initial point and direction coincide with its final point and direction. The notion of regular homotopy with fixed base and direction defines an equivalence relation on the set of regular closed curves beginning at this base. By the theorem of Smale [6], the regular homotopy classes are in one one correspondence with the elements of the fundamental group $\pi_1(T(M))$. This correspondence is such that the composition of regular curves on M by joining them end to end corresponds to the multiplication in $\pi_1(T(M))$; hence the set of regular homotopy classes may be given a natural group structure. We shall call this the regular homotopy group of M and denote it by $\pi_R(M)$. The zero of $\pi_R(M)$ is the class of a nullhomotopic curve shaped like a figure 8, traced out as in ordinary writing. For a fuller discussion of the concept of regular curves, see [6].

Let M be of genus g, and $\{A_i\}$ $i = 1, \ldots, 2g$ be a generating set for $\pi_1(M)$, satisfying the relation

$$A_1 A_2 A_1^{-1} A_2^{-1} \ldots A_{2g}^{-1} = 1.$$

Let H be a fibre of T(M). Then it is proved by Seifert [5] that $\pi_1(T(M))$ is generated by $\{A_i, H\}$ with the relations: H commutes with each other generator, and

$$A_1 A_2 A_1^{-1} \ldots A_{2g}^{-1} H^{2g-2} = 1$$

(see also § 2 below). In the case of the torus, the methods of [4] show that there is defined a homomorphism

$$w : \pi_R(M) \to Z$$

such that the value of w on any regular curve with a finite number of selfintersections is equal to the number of nullhomotopic loops which it contains. Here Z is the ring of integers. For general M, we might seek a similar homomorphism. Such cannot be found, however, since it would depend only upon the abelianized form of $\pi_R(M)$, and in this form H is of order $|\chi|$. This suggests using instead a homomorphism into Z_χ, the integers modulo χ.

It will be convenient to introduce the notion of a regular generating system for $\pi_1(M)$ at the point Q; this will be a set of regular simple closed curves $\{A_i\}$ $i = 1, \ldots, 2g$, on M tangent to a fixed direction at Q. Let $D_1 \subset D_2$ be two discs about Q of small diameter. We shall assume that the curves A_i do not meet outside D_2; then from the structure of the 4 g sided polygon we see that their crossings of its boundary are arranged in the order.

$$1/o,\ 2/i,\ 1/i,\ 2/o,\ \ldots,\ (2g-1)/o,\ (2g)/i,\ (2g-1)/i,\ (2g)/o$$

where $1/o$ is the point where A_1 leaves D_2, $2/i$ the point where A_2 enters, etc. Moreover, we shall assume that the A_i meet within D_1 only at Q, and that their crossings of its boundary are arranged in the order $1/o$, $3/o$, $\ldots$, $(2g-1)/o$, $2/i$, $1/i$, $4/i$, $\ldots$, $(2g)/i$, $(2g-1)/i$, $2/o$, $4/o$, $\ldots$, $(2g)/o$. Finally, we shall assume that in $D_2 - D_1$, only those crossings occur which are necessitated by the relative order of the points on the two boundary curves; for this purpose, the ordering will be understood to be linear, not circular. The properties of a regular generating system will be examined more closely in § 3.

We may now define precisely the object of our study, the winding homomorphism.

Definition. — *The winding homomorphism w is a homomorphism of $\pi_R(M)$ into Z_χ (the integers modulo χ) such that.*

(*i*) *w has the value O on the regular homotopy class of each of the curves of a regular generating system for $\pi_1(M)$.*

(ii) w has the value 1 on any positively oriented contractible regular simple closed curve passing through the base direction at Q.

Definition. — *The winding number of a regular closed curve* C *is the value of w on the regular homotopy class of* C; *by abuse of language, we shall denote this value by w* (C).

2. — Existence and uniqueness of the winding homomorphism.

In order to show the existence and uniqueness of w, we use the techniques of obstruction theory, as exposed in Steenrod [7]. It is known that there exists a cross section F of the unit tangent bundle T(M) defined on M minus one point P. Given any two such sections with the same singular point, hence the same primary obstruction, there is defined a difference cocycle, which is a coboundary if and only if the two sections are homotopic. Hence, the homotopy classes of such sections correspond one one to the elements of $H^1(M, P; Z)$, which is isomorphic to $H^1(M; Z)$.

Definition. — *If* C *is a closed curve on* M *and* F *and* F′ *are vector fields defined along* C, *we shall denote by* $d(C; F, F')$ *the value of the difference cohomology class of* F *and* F′ *on the homology class determined by* C. $d(C; F, F')$ *will be called the difference number of* C *with respect to* F *and* F′.

At any point of M — P, there is a signed angle from F′ to F defined by the Riemannian metric; denote this angle by F — F′.

Proposition 1. — *If* F *and* F′ *are of class* C^1,

$$d(C; F, F') = \frac{1}{2\pi}\int_C d(F - F').$$

Proof. The function F — F′ is a map from C onto the unit circle, considered as the set of angles. As shown in [4, Proposition 1], the degree of this mapping is given by the indicated integral. On the other hand, from obstruction theory it is clear that this degree is equal to the difference number.

DEFINITION. — *The winding number of a regular closed curve* C *with respect to a vector field* F *which has no singularity on* C *is*

$$w(C;\, F) = d(C;\, \dot{C},\, F)$$

where $\dot{C}$ *is the tangent vector field to* C.

If C is of class C^2, then by Proposition 1

$$w(C;\, F) = \frac{1}{2\pi}\int_C d(\dot{C} - F).$$

It follows from Smale's theorem that any regular curve of class C^2 with tangent vectors close to those of a given curve C is regularly homotopic to C. On the other hand, the winding number is clearly unchanged by a small deformation. Hence, we shall allow ourselves approximations by C^2 regular curves whenever it is convenient.

Let A_i be a regular generating system for $\pi_1(M)$, so that the homology classes of these curves form a basis for $H_1(M;\, Z)$. Let H be a regular simple closed curve through the base direction at Q, contained in D_1 and positively oriented. Let F′ be a vector field with one singularity, located at a point P not lying on any of the curves in question. Finally, let $w(A_i;\, F') = \alpha_i$. Define a vector field F by the requirement that $d(A_i;\, F,\, F') = \alpha_i$. F is unique up to homotopy, and

$$w(A_i;\, F) = w(A_i;\, F') + d(A;\, F',\, F) = \alpha_i - \alpha_i = 0.$$

Let $\varkappa_*(z)$ for $z \in Z$ denote the class modulo $\varkappa$ to which z belongs.

DEFINITION. — *If* C *is any regular closed curve,*

$$w(C) = \varkappa_* w(C;\, F).$$

LEMMA 1. — *Let* D *be a closed disk about* P *and let* $N = M - D$. *Then* $w(C;\, F)$ *depends only upon the regular homotopy class of* C *in* N, *so defines a homomorphism* w': $\pi_R(N) \to Z$ *such that*

(*i*) w' *vanishes on the classes determined by* A_i, $i = 1, \ldots, 2\,g$.

(*ii*) $w' = 1$ *on the class of any nullhomotopic positively oriented regular simple closed curve through the base direction.*

Proof. Consider a regular homotopy $C(t,\, \tau)$ on N, where t is the parameter along the regular curves. Expressing the integral for $w(C; F)$ in terms of t and τ, it is clearly a continuous function of τ. Since it is integer valued, it must be constant. Thus we may define for $\xi \in \pi_R(N)$, $C \in \xi$, $w'(\xi) = w(C; F)$. w' is a homomorphism; this follows easily from the integral formula. (*i*) follows from the choice of F. To show (*ii*) let C be a curve of the kind required. Lifting this curve up into the universal covering space of N, we again get a closed curve, which is regularly homotopic to a small simple closed curve by the Whitney-Graustein theorem [9]. This homotopy projects into a regular homotopy in N; we may suppose that the final curve is small enough to lie on a given disc about the base point. Since any two nonsingular vector fields on a disc are homotopic, the winding number of the curve is the same as it would be in the plane, that is, $+1$.

It will be of interest later to note that the winding number of a regular « figure 8 » curve enclosing D, but nullhomotopic on M, is $2g - 2$. This follows by lifting the situation onto the universal covering space of M, then computing $w(C; F) = d(C; \dot{C}, F)$ by comparing each field with the field of constant direction.

In order to relate the results of lemma 1 to regular homotopy on M, we consider the injection map of N into M, and the induced injection of T(N) into T(M). The latter is consistent with the representation of regular homotopy classes by the homotopy of the tangent bundle. Using this fact, we may compute explicitly the map $i_{\#} : \pi_R(N) \to \pi_R(M)$ induced by the injection : $i : N \to M$. For this purpose, we use the notion of C W complexes, an exposition of which may be found in [8]. Our proof is simply a modernization of the work of Seifert [5]. We may give a C W decomposition of T(N) as follows :

Let Q be the base point on N, $A'_i (i = 1, \ldots, 2g)$ simple closed curves bounding a $4g$ sided polygon, and E a simple closed curve through Q, otherwise interior to the polygon, and enclosing D. T(N) has a product structure, given explicitly by the fact that F and its orthogonal field define a parallelization. Let H be the fibre through Q (this makes sense since N may be embedded as a cross section in T(N)). There is a deformation retract of N onto a manifold bounded by E, which gives rise to a deformation retract of T(N) onto a manifold bounded by $H \times E$. Thus, we get the following decomposition of space or the homotopy type of $T(N)$:

I. Zero cell: Q.

II. One cells: A'_i, H, and E.

III. Two cells: the $4g$ sided polygon minus a disc, and the products of H with of each the other one cells.

IV. One three cell.

In order to get a decomposition of $T(M)$, we need to add a one cell E' on $E \times H$, such that E' is contractible as a curve on the surface of the solid torus filling $E \times H$; the homology class of this in terms of E and H may be computed by using the known index of the singularity at P. We need also to add a two cell spanning E', and a three cell which is the rest of the solid torus. Then $\pi_1(T(N))$ has generators A'_i, H, and E and relations:

$$A'_1A'_2A'^{-1}_1A'^{-1}_2 \ldots A'^{-1}_{2g}E^{-1} = 1.$$

H commutes with all other generators

while $\pi_1(T(M))$ has generators A'_i, H, E, and E' and all the same relations, plus $EH^{2g-2} = E' = 1$. This may be rewritten so that we use the same generators as $\pi_1(T(N))$, but one additional relation;

$$EH^{2g-2} = 1.$$

Hence, the map on the fundamental group induced by the injection map becomes simply the quotient map by the least normal subgroup generated by EH^{2g-2}.

Proposition 2. — *There exists a unique homomorphism* w *of* $\pi_R(M)$ *into* z_χ *such that.*

(i) w *vanishes on the classes determined by* A_i, $i = 1, \ldots, 2g$ *(where these are the* A_i *of lemma 1).*

(ii) $w = 1$ *on the class of any nullhomotopic positively oriented regular simple closed curve through the base direction.*

Proof. Let $\xi \in \pi_R(M)$ and $C \in \xi$. Define

$$w(\xi) = w(C) = \chi_* w'(C; F).$$

It is clear from the definitions of w and w' that $w = \chi_* w' i_{\#}^{-1}$, when evaluated on a particular regular curve C. We need to show that w depends only upon the regular homotopy class of C in M. Let us represent the generating elements of $\pi_1(T(M))$ and $\pi_1(T(N))$ by regular curves on M. Since $w'(A_i) = 0$, we may take A_i to represent the class of A'_i. Represent H by a positively oriented contractible regular simple closed curve on N. Finally, since E winds about the fibre over P $2g-2$ times, we represent it by a contractible « figure 8 » curve enclosing P. The kernel of $i_{\#}$ consists of products of conjugates of EH^{2g-2}; since

$$\varkappa_* w'(EH^{2g-2}) = 0,$$

it follows that w is a homomorphism on $\pi_R(M)$. Properties (i) and (ii) follow from lemma 1. These properties define w on a generating system for $\pi_R(M)$; hence they determine it uniquely.

3 — The winding number of a regular simple closed curve.

Heretofore, we have defined the winding homomorphism and proved its exsitence and uniqueness. There remains the crucial question of computing the winding number for simple curves C. To do this, we represent the homotopy class of C in terms of a regular generating system for $\pi_1(M)$, then use a covering space argument to show that the winding number depends only upon the way the curves of the regular generating system used to represent C cross in D_2. Finally, we give an algorithm for computing the winding number.

We begin by extending the regular generating system to include curves representing A_i^{-1}. Each such curve will agree with the corresponding A_i outside D_1, except for the direction of motion. The curves A_{2i-1}^{-1} must cross the A_{2j-1}, $i < j$, in order to approach Q in the correct direction. The curves A_{2i}^{-1} must cross A_{2j}, $i < j$, and A_{2k-1}, all k, in order to approach Q in the correct direction. In leaving Q, A_{2i}^{-1} must cross A_{2i-1} and A_i^{-1} must cross A_j, provided there exists a k such that $i \leqslant 2k < j$. Only those intersections mentioned above will be permitted to occur. It is easily seen that $A_i\ A_i^{-1}$ is a nullhomotopic « figure 8 » curve; this justifies the notation A_i^{-1} and shows that $w(A_i^{-1}) = 0$.

Let the homotopy class of C be given by $X_1 \ldots X_q$, where X_i is one of the classes $A_j^{\pm 1}$. We shall assume that no subsequence

$X_k X_{k+1} \ldots X_{k+r}$ is nullhomotopic; the total sequence can reduce to the empty sequence only in case C is nullhomotopic. In that case, the winding number is $+1$ if C is positively oriented and -1 if C is negatively oriented, by arguments used before.

LEMMA 2. — *On any orientable 2 manifold covered by the plane, every element of the fundamental group is of infinite order.*

Proof. The fundamental group acts on the plane as covering transformations. Thus, any element of finite order would generate a finite cyclic subgroup acting on the plane, such that only the identity has fixed points. But this is impossible, since by a theorem of Brouwer [1] and Kérékjarto [2], every such transformation is topologically équivalent to an orientation preserving linear map.

LEMMA 3. — *If* C *is a simple closed curve homotopic to* $X_1 \ldots X_q$, *the winding number of* C *is the sum of* q *integers, each associated to one of the ordered pairs* $X_1X_2, \ldots, X_{q-1}X_q$, X_qX_1.

Proof. Let $(\tilde{M}, p)$ be the universal covering space of M, that is, $p : \tilde{M} \to M$ and $\tilde{M}$ is either the sphere or the plane. In the first case, C must be nullhomotopic, so we may assume $\tilde{M}$ is the plane.

Let $\tilde{C}$ be a component of $p^{-1}(C)$; it is infinite in both directions because C is of infinite order, and it separates the plane. Let $\tilde{Q}_0$ be a point of $p^{-1}(Q)$ not lying on $\tilde{C}$. Let $\tilde{X}_1$ be a connected subset of $p^{-1}(X_1)$ beginning at $\tilde{Q}_0$ and covering every point of X_1 except Q exactly once, and let $\tilde{Q}_1$ be its endpoint.

By induction, define curves $\tilde{X}_i$ and points $\tilde{Q}_i$, $i = 2, \ldots, q$. The result is a curve X^* joining $\tilde{Q}_0$ to $\tilde{Q}_q$, whose projection is homotopic to C.

By proper choice of $\tilde{Q}_0$, we may assume that X^* does not meet $\tilde{C}$; then the same is true of the curve $\tilde{X}$ (infinite in both directions) made by duplicating X^* by the operation of the covering transformation $\bar{C}$ corresponding to C. $\tilde{X}$ is not necessarily a simple curve, since $\tilde{Q}_i$ has a connected neighborhood in $p^{-1}(D_2)$ which may contain some points of intersection associated to the sequence $X_i X_{i+l}$; we may assume that the neighborhoods are also disjoint from C. Let P_1 be a point on $\tilde{X} \notin p^{-1}(D_2)$, R_1 a point on $\tilde{C}$, and P_2 and R_2 their transforms by $\overline{C}$. Let $\tilde{Y}$ be a regular simple closed curve tangent to $\tilde{C}$ at P_1 and $\tilde{X}$ at R_1, not otherwise meeting them, and not passing througt any singular point in $p^{-1}(P)$. By translating half of $\tilde{Y}$ to a curve P_2R_2, we may construct a regular closed curve $B(= P_1P_2R_2R_1)$ which, without loss of generality, may be assumed positively oriented. Then we have the following equations, in which K is the vector field of constant direction in the plane:

$$(I) \quad \frac{1}{2\pi}\int_B d(\dot{B} - F) = \frac{1}{2\pi}\int_B d(\dot{B} - K) + \frac{1}{2\pi}\int_B d\,(K - F)$$
$$\equiv \frac{1}{2\pi}\int_B d(\dot{B} - K)$$
$$\equiv 1 + \text{algebraic number of crossings in B} \pmod{x}.$$

(II)
$$\frac{1}{2\pi}\int_B d(\dot{B} - F) = \frac{1}{2\pi}\left(\int_{P_1P_2} + \int_{P_2R_2} + \int_{R_2R_1} + \int_{R_1P_1}\right) d(\dot{B} - F)$$
$$\equiv 1 + \frac{1}{2\pi}\int_{P_1P_2} d(\dot{C} - F) \equiv 1 + w(C) \pmod{x},$$

since the second and fourth integrals add up to 1, while the third integral is $w(X_1 \ldots X_q) = 0$, and the first integral is along the portion of B which covers C.

We conclude that $w(C)$ is equal to the algebraic number of crossings in B, which may be found by adding up the number of loops associated to each sequence $X_i\, X_{i+1}$. This proves the lemma.

THEOREM. — *Let* C *be a regular simple closed curve homotopic in the usual sense to* $X_1 \ldots X_q$, *where* $X_i = A_j^{\pm 1}$ *and no subsequence* $X_k X_{k+1} \ldots X_{k+r}$ *is nullhomotopic. Then* $w(C)$ *is the sum of the integers associated to the sequences*

$$X_1\, X_2, \ldots, X_{q-1}\, X_q, X_q\, X_1$$

by the following algorithm:

Consider the schema

$$\begin{gathered}1/o,\quad 2/i,\quad 1/i,\quad 2/o,\quad 3/o,\quad 4/i,\quad 3/i,\quad 4/o,\quad \ldots,\\ 2g-1/o,\quad 2g/i,\quad 2g-1/i,\quad 2g/o\\ 1/o,\quad 3/o,\quad \ldots,\\ 2g-1/o,\quad 2/i,\quad 1/i,\quad 4/i,\quad 3/i,\quad \ldots,\\ 2g/i,\quad 2g-1/i,\quad 2/o,\quad 4/o,\quad \ldots,\quad 2g/o.\end{gathered}$$

Then to any sequence $A_j^e A_k^f (e,\ f = \pm 1)$ *we associate the two by two subschema formed by choosing the terms* j/x *and* k/y; *here*

$$x = \begin{cases} i \\ o \end{cases} \quad \text{if} \quad e = \begin{cases} 1 \\ -1 \end{cases}$$
$$y = \begin{cases} i \\ o \end{cases} \quad \text{if} \quad f = \begin{cases} 1 \\ -1 \end{cases}$$

Convert this schema into an integer two by two matrix by the substitution of 0 *for* j/x *and* 1 *for* k/y; *let* s *denote the determinant of this matrix. If* $e = -f$, *define another integer* t *by the rule:*

$$t = 1 \text{ for the sequences} \begin{cases} A_{2j-1}^{-1} A_{2k-1} & j < k \\ A_{2j} A_{2k} & j < k \\ A_{2j}^{-1} A_{2k-1} & \end{cases}$$

$$t = -1 \text{ for the sequences} \begin{cases} A_{2j-1}\ A_{2j}^{-1} & \\ A_j A_k & k \leq 2l < j \text{ for some } l \end{cases}$$

$t = 0$ *otherwise.*

Then the integer associated to $A_j^e\ A_k^f$ *is the sum* $s + t$.

Proof. By lemma 3, we have reduced the problem to computing the number of loops produced by the crossings of A_j^e and A_k^f in the region D_2. Examination of the construction of the regular generating system for $\pi_1(M)$ reveals that these crossings give rise to loops of positive orientation in the cases indicated by $+ 1$, to loops of negative orientation in the cases indicated by $- 1$, and to no loops in the cases indicated by 0 above. (The integer s expresses the number of loops necessitated by crossings in $D_2 - D_1$, and the integer t the number of loops necessitated by crossing in D_1).

Corollary 1. — *On the sphere and torus, the winding number of a regular simple closed curve is* ± 1 *if nullhomotopic, and* 0 *otherwise. Moreoever, two regular closed curves which* *are homotopic are regularly homotopic if and only if they have the same winding number. Both these results are false for surfaces of higher genus.*

Proof. On the sphere, all curves are nullhomotopic. Delete the singular point of the standard vector field F; then we have the plane situation, in which a simple closed curve has winding ± 1. Hence, its winding number on the sphere is 1(modulo 2). On the torus, we deal with nullhomotopic curves easily by referring to the covering by the plane, since the standard vector field may be chosen to be that one covered by the field of constant direction. Curves which are not contractible we represent in the form $A_1^k A_2^l$ (using the abelian character of the fundamental group); thus we need only know that for each of the four pairs of sequences

$$A_1^e A_2^f \quad \text{and} \quad A_2^f A_1^e,$$

with $e, f = \pm 1$, the sum of the associated integers is zero. This proves the first statement.

Consider the presentation of $\pi_R(M)$ given in § 2. For the sphere or torus, the powers of H form a direct summand of an abelian group, and w is an isomorphism on this subgroup; this proves the second statement. On a surface of genus greater than 1, there is a simple closed curve in the homotopy class $A_1 A_4^{-1}$. By the algorithm, 0 is associated to $A_1 A_4^{-1}$ and 1 to $A_4^{-1} A_1$. Hence, the winding number is 1, showing that the first statement is false for such a surface. The second is disproved by the fact that the powers of H form an infinite cyclic subgroup of $\pi_R(M)$, hence cannot be distinguished by a homomorphism into a finite cyclic group.

Corollary 2. — *Let* F_1 *be a vector field on* M *having unique trajectories through nonsingular points and let* S *be its locus of singular points. Let* C *be a noncontractible closed curve and* w_0 *the winding number of any simple closed curve homotopic to* C. *Suppose*

$$\frac{1}{2\pi}\int_C d(F_1 - F) \not\equiv w_0 \mod \chi.$$

Then C *is not homotopic on* $M - S$ *to any integral curve of* F_1

Proof. If C′ is any closed integral curve homotopic to C on M — S, then C′ is a regular simple closed curve. Hence we have

$$\frac{1}{2\pi}\int_C d(F_1 - F) \equiv \frac{1}{2\pi}\int_{C'} d(F_1 - F)$$
$$\equiv \frac{1}{2\pi}\int_{C'} d(\dot{C}' - F) \equiv w_0 \pmod{\gamma}.$$

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