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THE REPRESENTATIONS OF LINEAR FUNCTIONALS BY MEASURES ON SETS OF EXTREME POINTS

by Errett BISHOP and Karel de LEEUW [1]

1. — INTRODUCTION

Let X be a compact Hausdorff space. We shall denote by $C_r(X)$ the real Banach space of continuous real valued functions on X, and by $C_c(X)$ the complex Banach space of continuous complex valued functions on X, supplied with norms denoted by $\|.\|$ and defined by

$$\|f\| = \operatorname*{Sup}_{x \in X} |f(x)|.$$

Suppose that B is some linear subspace of either $C_r(X)$ or $C_c(X)$ that distinguishes points of X and that contains the constant functions. We are concerned in this paper with the problem of representing the linear functionals in the dual space B^* of B by measures on X. It is well known that such representations are always possible; if B is a linear subspace of $C_r(X)$, by the Hahn-Banach theorem, any L in B^* extends to a continuous linear functional of $C_r(X)$ and thus by the Riesz representation theorem, there will be some signed Baire measure μ on X so that

$$(1.1) \qquad L(f) = \int f \, d\mu, \quad \text{all } f \text{ in B}.$$

If B is a linear subspace of $C_c(X)$, a similar argument shows

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that each L in B^* has a representation of the form (1. 1) for μ a complex valued Baire measure.

There are cases in which it is possible to find a subset Y of X which is such that each L in B^* has a representation of the form (1. 1) for some μ concentrated on Y. In this paper we introduce such a subset, the *Choquet boundary* of B.

The Choquet boundary of B is denoted by M(B) and consists of all points x in X having the following property: there is a *unique positive* Baire measure μ that represents, in the sense of (1. 1), the linear functional L_x defined by

$$(1.\ 2) \qquad L_x(f) = f(x), \qquad \text{all } f \text{ in B.}$$

This unique μ will of course be the unit point mass at x. In the case that B is a uniformly closed subalgebra of $C_c(X)$ and X is metrizable, M(B) is the *minimal boundary* of [3] and [4].

We show in Section 4 that the extreme points of the subset

$$\{L : L \in B^*, \qquad L(1) = ||L|| = 1\}$$

of B^* are those L_x défined by (1. 2) for x in M(B).

From this and the Krein-Milman theorem it follows easily that any L in B^* has a representation of the form (1. 1) with μ a measure concentrated on the *closure* of M(B).

The question that concerns us is whether it is possible to choose the measure μ so that it is concentrated on M(B) itself. If X is metrizable, an application of the theorem of Choquet in [7] shows that this is indeed possible. We proceed in the reverse direction, showing directly that the measure can be concentrated on M(B); this leads to a relatively simple proof of the Choquet theorem.

In the case that X is not metrizable the situation is much more complicated. We give examples in the concluding section of the paper to show that M(B) need not even be a Borel set. We prove nevertheless that each L in B^* has a representation of the form (1. 1) for a measure μ that is « concentrated on M(B) » in following sense: it is a measure on the σ-ring generated by M(B) and the Baire sets of X, and is zero on each set in this σ-ring which is disjoint from M(B). Furthermore if $L(1) = ||L||$, the measure μ can be chosen to be non-negative. This leads to an extension of the theorem of Choquet to convex sets that are not metrizable.

The existence of measures concentrated on the Choquet boundary is obtained roughly as follows. An ordering relation on the class of non-negative Baire measures on X is introduced. We say that μ is a B-cover of η if

$$\int f\, d\eta = \int f\, d\mu, \qquad \text{all } f \text{ in B},$$

and

$$(1.3) \qquad \int f^2\, d\eta \leqq \int f^2\, d\mu, \qquad \text{all } f \text{ in B}.$$

We say that μ is a *proper B-cover* of η if μ is a B-cover of η and furthermore the inequality in (1. 3) is strict for some f in B. η is called *B-maximal* if it has no proper B-cover. A simple argument using Zorn's Lemma and weak* compactness assures for any given non-negative Baire measure η, the existence of a B-maximal μ that is a B-cover for η. The crucial result now is theorem 5. 3 which shows that $\mu(S) = 0$ if μ is B-maximal and S is disjoint from M(B). From this it follows simply that any B-maximal μ can be extended to a measure « concentrated on M(B) » in the sense described above. Since each linear functional in B* has a representation of the form (1. 1) for some Baire measure μ on X, and such a μ is a linear combination of non-negative Baire measures, and each non-negative Baire measure has a B-maximal B-cover that is concentrated on M(B), it follows that any linear functional in B* has a representation of form (1. 1) for some μ concentrated on M(B).

Section 6 is devoted to uniformly closed subalgebras of $C_c(X)$. We give somewhat simpler proofs of some of the results of [3] and [4] and remove the hypothesis of metrizability of X imposed there. We show that if A is a uniformly closed subalgebra of $C_c(X)$ that distinguishes points of X and contains the constant functions, the points x of M(A) can be characterized by either of the following conditions:

I. For each neighborhood U of x there is a function f in A with $\|f\| \leqq 1$, $f(x) > \frac{3}{4}$ and $|f(y)| < \frac{1}{4}$ for all y not in U.

II. If S is a closed G_δ containing x, then there is some f in A with $|f(x)| = \|f\|$ and $\{y : |f(y)| = \|f\|\} \subset S$.

A subset Y of X is said to be a *boundary for A* if for each f

in A there is some y in Y with $|f(y)| = ||f||$. It is a simple consequence of the Krein-Milman theorem that the Choquet boundary of A is a boundary for A. Condition II shows that in addition, any Baire subset of X that is a boundary for A must contain M(A). Furthermore if X has the property that each point is a G_δ (in particular if X is metrizable), condition II shows that for each point x of M(A) there is some f in A that « peaks » at x, so that M(A) is the smallest boundary for A, that is, the *minimal boundary* (for X metrizable, this was established in [3] and [4]). If not every point of X is a G_δ, there may be no smallest boundary, even if A is all of $C_c(X)$.

Since each L in A^* has a representation of the form (1. 1) for μ a measure concentrated on M(A), it is reasonable to inquire whether for any set Y that is a boundary for A, a measure can be found that represents L and is concentrated on Y. We show by example in Section 7 that this cannot be done for linear subspaces of $C_c(X)$ that are not subalgebras. Nevertheless we are able to show that for subalgebras such measures can always be found. The existence of these measures was suggested to us by Irving Glicksberg, who studied essentially the same problem for the case $A = C_c(X)$ in [8].

In all that follows, by « subspace of $C_r(X)$ » (or « of $C_c(X)$ ») we shall mean a linear subspace *containing the constant functions*, but not necessarily closed or distinguishing points of X. For applications it is useful to have results concerning not necessarily closed linear subspaces; furthermore it is necessary for technical reasons for us to consider linear subspaces that do not distinguish points of X. By « measure » in the following, we shall always mean finite measure.

II. — THE BASIC DEFINITIONS

In the following X is a fixed compact Hausdorff space. We shall denote by H the class of all non-negative Baire measures on X. This class H will be identified in the usual manner with a subset of the dual space of $C_r(X)$. By the weak* topology of H we shall mean the restriction to H of the weak* topology on the dual space of $C_r(X)$. The basic fact concerning the weak* topology that we shall need is that bounded closed subsets of H are weak* compact.

The following allows us to reduce questions concerning linear functionals on subspaces of $C_r(X)$ or $C_c(X)$ to questions about the Baire measures in H.

LEMMA 2. 1. — *Let* B *be a subspace of* $C_r(X)$ *or* $C_c(X)$. *Then each* L *in* B^* *has a representation of the form*

$$(2.1) \qquad L(f) = \int f\, d\mu, \qquad \text{all } f \text{ in B};$$

where μ *is a Baire measure which is of the form*

$$(2.2) \qquad \mu_1 - \mu_2, \qquad \mu_1, \mu_2 \textit{ in } H,$$

if B *is a subspace of* $C_r(X)$, *and of the form*

$$\mu_1 - \mu_2 + i\mu_3 - i\mu_4, \qquad \mu_1, \ldots, \mu_4 \text{ in H},$$

if B *is a subspace of* $C_c(X)$. *Furthermore the* μ *in* (2. 1) *can be chosen to be in* H *if and only if* $L(1) = ||L||$.

PROOF. — We shall treat the case of B a subspace of $C_r(X)$; the proof for $C_c(X)$ is completely analogous. By the Hahn-Banach theorem, L can be extended to all of $C_r(X)$ with preservation of norm; i.e. there is a linear functional L′ on $C_r(X)$ with

$$L'(f) = L(f), \quad \text{all } f \text{ in B},$$

and $||L'|| = ||L||$. By the Riesz representation theorem, there is a signed Baire measure μ on X so that

$$L'(f) = \int f\, d\mu, \quad \text{all } f \text{ in } C_r(X).$$

Each signed Baire measure on X is of the form (2. 2). If $L(1) = ||L||$, then

$$||\mu|| = ||L'|| = ||L|| = L(1) = \mu(X),$$

so μ must be non-negative and thus in H. Conversely if μ is in H, the L defined by (2. 1) clearly satisfies $L(1) = ||L||$.

If B is a subspace of $C_r(X)$ or $C_r(X)$, and x is a point of X, we define $H_x(B)$ to be the subset of H consisting of all μ with

$$(2.3) \qquad \int f\, d\mu = f(x), \quad \text{all } f \text{ in B}.$$

$H_x(B)$ is always non-empty since it must contain at least the unit point mass at x. Since subspaces are assumed to contain the constant functions, by (2. 3) each μ in $H_x(B)$ satisfies $\mu(X) = 1$.

If S is any subset of X and B is any collection of functions on X, we define $i_B(S)$ to be the subset of X consisting of those points that cannot be distinguished from points of S by the functions in B; i.e. $i_B(S)$ is

$$\{y : f(y) = f(x) \quad \text{for some} \quad x \in S \text{ and all} \quad f \in B\}.$$

If μ is any Baire measure in H, we denote by $\hat{\mu}$ its regular Borel extension. This is the unique regular Borel measure on X that agrees with μ on the Baire sets of X. It is defined by

$$\hat{\mu}(S) = \text{Inf}\ \mu(U),$$

where U runs over all open Baire sets that contain S.

If B is a subspace of either $C_r(X)$ or $C_c(X)$, the *Choquet boundary* of B, denoted by $M(B)$, is defined to consist of those points x in X which are such that any μ in $H_x(B)$ satisfies $\hat{\mu}(i_B(x)) = 1$. If B contains sufficiently many functions to distinguish a point x from all other points of X, $i_B(x) = \{x\}$, so that x will be in the Choquet boundary of B if and only if the unit point mass at x is the only μ in $H_x(B)$.

If B is a subspace of $C_c(X)$, we shall denote by B_r the subspace of $C_r(X)$ consisting of real parts of the functions in B. The following is immediate.

LEMMA 2. 2. — $M(B_r) = M(B)$, *and for each* x *in* X, $H_x(B_r) = H_x(B)$.

We next introduce some ordering relations on H which are basic to the constructions that follow.

If B is a subspace of $C_r(X)$ and μ and η are in H, we shall say that μ is a B-*cover* of η if

$$\int f\, d\eta = \int f\, d\mu, \qquad \text{all } f \text{ in B},$$

and

$$\int f^2\, d\eta \leqq \int f^2\, d\mu, \qquad \text{all } f \text{ in B}. \tag{2. 4}$$

We shall say that μ is a *proper B-cover* of η if μ is a B-cover of η and furthermore the inequality (2. 4) is strict for some f in B.

A measure η will be called *B-maximal* if it has no proper *B-cover.*

Much of the work done in the remainder of the paper goes into demonstrating that any B-maximal measure must be concentrated on the Choquet boundary of B.

The following two lemmas will be applied later.

LEMMA 2. 3. — *Let x be a point of* X *and η in* H *be the unit point mass at x. Let* B *be a subspace of* $C_r(X)$. *Then any μ in* $H_x(B)$ *is a* B-*cover of η.*

PROOF. — By the Schwarz inequality, for each f in B,

$$\int f^2\, d\mu = \int f^2\, d\mu \int 1^2\, d\mu \geq \left|\int f\, d\mu\right|^2 = (f(x))^2 = \int f^2\, d\eta.$$

Furthermore

$$\int f\, d\mu = f(x) = \int f\, d\eta$$

for all f in B.

LEMMA 2. 4. — *Let* B *be a subspace of* $C_r(X)$. *Then each η in* H *has a B-cover that is B-maximal.*

PROOF. — Consider subsets $\{\mu_\alpha\}_{\alpha \in J}$ of H indexed by totally ordered sets J, where the ordering is such that μ_β is a B-cover of μ_α if $\alpha < \beta$. By Zorn's lemma there is a maximal such subset $\{\mu_\alpha\}_{\alpha \in J}$ that contains η. Since B contains the constant functions, each μ_α is in

$$\{\nu : \nu \in H, \quad \nu(X) = \eta(X)\}. \tag{2. 5}$$

(2. 5) is compact in the weak* topology and thus contains a weak* cluster point μ for the net $\{\mu_\alpha\}_{\alpha \in J}$. It is clear that μ is a B-cover of η and that μ is B-maximal.

III. — REPRESENTATION OF LINEAR FUNCTIONALS IN THE SEPARABLE CASE.

The main result in this section is Theorem 3. 2. Using it we establish in Theorem 3. 4 the possibility of representing linear functionals on B by measures on the Choquet boundary of B in the case that B is separable, that is, has a countable dense subset. Theorem 3. 2 will also be applied later in the non-separable case.

We shall need the following well known lemma.

LEMMA 3. 1. — *Let* S *be a closed subset of* X. *Then for each positive real number c, the subset*

$$\{\mu : \mu \in \mathrm{H}, \quad \mu(\mathrm{X}) = \hat{\mu}(\mathrm{T}) = c, \quad \textit{for some finite} \quad \mathrm{T} \subset \mathrm{S}\}$$

of H *is dense in*

$$\{\mu : \mu \in \mathrm{H}, \quad \mu(\mathrm{X}) = \hat{\mu}(\mathrm{S}) = c\}$$

in the weak topology.*

We can now prove

THEOREM 3. 2. — *Let* B *be a subspace of* $\mathrm{C}_r(\mathrm{X})$, *and* μ *in* H *be a B-maximal measure. Let* S *be a closed subset of* X *which has the following property: there is a separable subspace* D *of* B *so that for each* x *in* S *there is some* σ *in* $\mathrm{H}_x(\mathrm{B})$ *with* $\sigma(i_{\mathrm{D}}(x)) < 1$. *Then* $\hat{\mu}(\mathrm{S}) = 0$.

The statement of this theorem is necessarily complicated as it must be applied later to the situation where B is not separable. In that case D will necessarily be a proper subspace of B and will not distinguish points of X. In the application in this section however D = B, and in this case the hypothesis on S in the theorem becomes simply that it be disjoint from the Choquet boundary of B.

PROOF OF THEOREM 3. 2. — Let $\{f_n : n = 1, 2, \ldots\}$ be a countable subset of D that is dense in D. For each pair of positive integers n and m define L_{nm} to be the subset of X consisting of all x for which there is some μ in $\mathrm{H}_x(\mathrm{B})$ with.

$$\int f_n^2 \, d\mu \geq \frac{1}{m} + (f_n(x))^2. \tag{3. 1}$$

L_{nm} is closed. For if $\{x_\alpha\}$ is a net of points in L_{nm} converging to a point x in X, and if for each α a measure μ_α is chosen in $\mathrm{H}_{x_\alpha}(\mathrm{B})$ so that

$$\int f_n^2 \, d\mu_\alpha \geq \frac{1}{m} + (f_n(x_\alpha))^2,$$

by the weak* compactness of

$$\{\mu : \mu \in \mathrm{H}, \qquad \mu(\mathrm{X}) = 1\}$$

the net $\{\mu_\alpha\}$ will have a weak* cluster point μ in $H_x(B)$, and μ will satisfy (3. 1).

We will show next that

$$S \subset \bigcup L_{nm}. \tag{3. 2}$$

Let x be a point of X that is in none of the L_{nm}. Let σ be in $H_x(B)$. By the definition of the L_{nm},

$$\int f_n^2 \, d\sigma \leqq (f_n(x))^2 = \left|\int f_n \, d\sigma\right|^2. \tag{3. 3}$$

On the other hand, by the Schwarz inequality,

$$\int f_n^2 \, d\sigma = \int f_n^2 \, d\sigma \int 1^2 \, d\sigma \geqq \left|\int f_n \, d\sigma\right|^2, \tag{3. 4}$$

with equality if and only if f_n is constant a.e. with respect to σ. Comparing (3. 3) and (3. 4) we see that each f_n must be constant a.e. with respect to σ. Since also $f_n(x) = \int f_n \, d\sigma$, it follows that f_n is equal to $f_n(x)$ a.e. with respect to σ for each n. Thus the set

$$i_D(x) = [y : f_n(y) = f_n(x), \quad n = 1, 2, \ldots]$$

has σ measure 1. Since this holds for each σ in $H_x(B)$, x cannot be in S. This completes the proof of the inclusion (3. 2).

We now show that $\hat{\mu}(S) > 0$ contradicts the B-maximality of μ. Suppose that $\hat{\mu}(S) > 0$. Then by (3. 2), $\hat{\mu}(L_{nm}) > 0$ for some L_{nm}. Let ν be the measure in H that is the restriction of μ to L_{nm} : i.e., $\nu(T) = \hat{\mu}(T \cap L_{nm})$ for all Baire sets T. Now $\nu \not\equiv 0$ since $\nu(X) = \hat{\mu}(L_{nm}) \neq 0$. By Lemma 3. 1, ν is the limit in the weak* topology of a net $[\nu_\alpha]$ of measures in H with $\nu_\alpha(X) = \nu(X)$ and each ν_α concentrated on a finite subset of L_{nm}. By Lemma 2. 3 and the definition of L_{nm} it follows that there exists a corresponding net $[\eta_\alpha]$ of measures in H such that each η_α is a B-cover of ν_α and in addition that

$$\int f_n^2 \, d\eta_\alpha \geqq \frac{\nu(X)}{m} + \int f_n^2 \, d\nu_\alpha.$$

If η is any weak* cluster point of the net $[\eta_\alpha]$, η is a B-cover of ν and

$$\int f_n^2 \, d\eta \geqq \frac{\nu(X)}{m} + \int f_n^2 \, d\nu.$$

Thus η is a *proper* B-cover of ν and it follows that $(\mu - \nu) + \eta$

is a proper B-cover of μ. This contradicts the B-maximality of μ and shows that $\hat{\mu}(S)$ must be 0. This completes that proof of Theorem 3. 2.

COROLLARY 3. 3. — *Let* D *be a separable subspace of* $C_r(X)$. *Then the Choquet boundary of* D *is a* G_δ. *If* η *is any measure in* H, *then there is some measure* μ *in* H *with*

$$\int f\,d\mu = \int f\,d\eta, \quad \text{all } f \text{ in } D, \tag{3. 5}$$

and which is concentrated on M(D); *i.e. satisfies* $\hat{\mu}(M(D)) = \mu(X)$.

PROOF. — Let $\{f_n : n = 1, 2, \dots\}$ be a countable subset of D that is dense in D. Define the closed subsets L_{nm} of X as in the proof of theorem 3. 2. We shall show that M(D) is the complement of $\bigcup L_{nm}$ and is thus a G_δ. Let x be any point which is not in M(D). Then that part of the hypothesis of theorem 3. 2 which concerns B, D, and S is satisfied if B is taken to be D and S to be $\{x\}$. The proof of Theorem 3. 2 shows that x is in $\bigcup L_{nm}$. Conversely, from the definition of the L_{nm} it is clear that no point in $\bigcup L_{nm}$ can be in M(D). Thus M(D) is a G_δ as claimed.

Now let η be any measure in H. Let μ be a D-maximal measure in H that is a D-cover of η. The existence of such a μ is guaranteed by Lemma 2. 4. Equality (3. 5) holds since μ is a D-cover of η. To show that $\hat{\mu}(M(D)) = \mu(X)$, by regularity of $\hat{\mu}$ it suffices to show that $\hat{\mu}(S) = 0$ for each closed S that is disjoint from M(D). That this holds is a consequence of Theorem 3. 2 for the special case D = B. This completes the proof of Corollary 3. 3.

The following is an immediate consequence of Lemma 2. 1 and Corollary 3. 3.

THEOREM 3. 4. — *Let* D *be a separable subspace of* $C_r(X)$ *or* $C_c(X)$. *Then any linear functional* L *in* D^* *has a representation of the form*

$$L(f) = \int f\,d\mu, \quad \text{all } f \text{ in } D, \tag{3. 6}$$

for μ a Baire measure on X that is concentrated on M(D) in the sense that $\hat{\mu}(S) = 0$ for each Borel set S disjoint from M(D). Furthermore the μ in (3. 6) can be chosen to be in H if and only if $L(1) = \|L\|$.

IV. — EXTREME POINTS AND THE CHOQUET THEOREM

Let E be a real locally convex topological linear space with dual space E^*. Let X be a compact convex subset of E and B the linear subspace of $C_r(X)$ consisting of all functions f having the form

$$f(x) = F(x) + c, \qquad \text{all } x \text{ in } X,$$

for some F in E^* and some real constant c.

If μ is a real-valued measure on X whose domain includes the Baire sets of X and if S is a Baire subset of X, we shall denote by

$$(4.1) \qquad \int_S y \, d\mu\,(y)$$

the unique element v of E that satisfies $F(v) = \int_S F \, d\mu$ for each F in E^*. We shall use standard properties of the vector valued integral (4. 1) that are discussed for example in [6]. The following lemma was announced by Bauer in [2].

Lemma 4. 1. — *Let x be a point of* X. *Then the following are equivalent:*

1. *x is an extreme point of* X.
2. *x is in* M(B).

Proof. — To show that 2 implies 1, let x be a point of X that is not extreme. Then $x = \frac{1}{2}(u + v)$ for some u and v in X with $u \neq v$. Thus the measure μ in H that satisfies $\hat{\mu}(\{u\}) = \hat{\mu}(\{v\}) = \frac{1}{2}$ and $\hat{\mu}(X) = 1$ is in $H_x(B)$ by the definition of B. Since the functions of B distinguish points of X, $i_B(x) = \{x\}$ so $\mu(i_B(x)) = 0$ and x is not in M(B).

To show that 1 implies 2, suppose that x is an extreme point of X. Let μ be in $H_x(B)$. We shall show that μ is the unit point mass at x so that x is in the Choquet boundary of B. Since μ is in $H_x(B)$, $x = \int y \, d\mu(y)$. If S is any Baire subset of X with $0 < \mu(S) < 1$, and $T = X - S$, x has a representation as a convex combination

$$(4.\ 2)\quad x = \mu(S)\left(\frac{1}{\mu(S)}\int_S y\, d\mu(y)\right) + \mu(T)\left(\frac{1}{\mu(T)}\int_T y\, d\mu(y)\right)$$

of points in X. Since x is an extreme point of X,

$$(4.\ 2)\quad \mu(S)x = \int_S y\, d\mu(y), \qquad \text{all Baire} \qquad S \subset X.$$

Equivalently,

$$(4.\ 3)\quad \mu(S)F(x) = \int_S F\, d\mu, \qquad \text{all Baire} \qquad S \subset X,\ F \in E^*.$$

(4. 3) is possible only if

$$(4.\ 4)\quad \mu(\{y : F(y) = F(x)\}) = 1, \qquad \text{all} \qquad F \in E^*.$$

If $C(\mu)$ is the carrier of μ, that is, the smallest closed subset of X whose complement has $\hat{\mu}$ measure 0, (4. 4) shows that

$$C(\mu) \subset \{y : F(y) = F(x)\}, \qquad \text{all } F \in E^*.$$

But since the functions in E^* distinguish points of X, $C(\mu)$ must be $\{x\}$ and μ must be the unit point mass at x. Thus x is in M(B) and the proof of Lemma 4. 1 is completed.

We can now establish the Choquet theorem by using our Theorem 3. 4.

THEOREM 4. 2. — *Let* X *be a compact convex* metrizable *subset of a locally convex topological linear space.* E. *Let* X_e *be the set of extreme points of* X. *Then* X_e *is a* G_δ *in* X *and every* x *in* X *has a representation of the form*

$$(4.\ 5)\qquad x = \int y\, d\mu\, (y)$$

for some non-negative Baire measure μ *on* X *satisfying*

$$\mu(X_e) = \mu(X) = 1.$$

PROOF. — Let B be the subspace of $C_r(X)$ defined earlier. Since X is metrizable, B is separable, so $X_e = M(B)$ is a G_δ in X. Let x be a point in X and L the linear functional of B defined by

$$L(f) = f(x), \qquad \text{all } f \text{ in B}.$$

Then $L(1) = ||L|| = 1$, so by Theorem 3. 4 there is a non-negative Baire measure μ with $\mu(M(B)) = \mu(X) = 1$ and

$$(4.6) \qquad L(F) = \int F\, d\mu, \qquad \text{all } F \text{ in } E^*.$$

But (4. 6) is simply a restatement of (4. 5). And $\mu(X_e) = 1$ since by Lemma 4. 1, $X_e = M(B)$. This completes the proof.

Lemma 4. 1. gives a characterization of the Choquet boundary in terms of extreme points in a very special situation. It will be necessary for our later work to have a suitable replacement of this in the general case.

Lemma 4. 3. — *Let* X *be a compact Hausdorff space,* B *a subspace of* $C_r(X)$. *Let* L_0 *be a linear functional in* B^*. *Then the following are equivalent:*

1. L_0 *is an extreme point of*

$$(4.7) \qquad \{L : L \in B^*, \qquad L(1) = ||L|| = 1\}$$

2. *There is a point* x *in* M(B) *so that*

$$L_0(f) = f(x), \quad \text{all } f \text{ in B.}$$

Proof. — To show that 2 implies 1, suppose that L_0 is not an extreme point of (4. 7). $L_0 = \frac{1}{2}(L_1 + L_2)$, with L_1 and L_2 in (4. 7), $L_1 \neq L_0$. By Lemma 2. 1 there are measures μ_1 and μ_2 in H with $\mu_1(X) = \mu_2(X) = 1$ and

$$L_i(f) = \int f\, d\mu_i, \quad \text{all } f \text{ in B}, \quad i = 1, 2.$$

Since $L_0 \neq L_1$, $\mu_1(i_B(x)) < 1$. Let $\mu = \frac{1}{2}(\mu_1 + \mu_2)$. Then μ is in $H_x(B)$ and $\mu(i_B(x)) < 1$, so x is not in M(B). This completes the proof that 2 implies 1.

To show that 1 implies 2, suppose that L_0 is an extreme point of (4. 7). By Lemma 2. 1 there is a measure μ in H so that

$$L_0(f) = \int f\, d\mu, \quad \text{all } f \text{ in B.}$$

Let S_1 be any Baire subset of X with $0 < \mu(S_1) < 1$, and $S_2 = X - S_1$. Then if the linear functionals L_1 and L_2 in (4. 7) are defined by

$$L_i(f) = \frac{1}{\mu(S_i)} \int_{S_i} f\, d\mu, \quad \text{all } f \text{ in B,}$$

we have a representation of L_0 as a convex combination

$$L_0 = \mu(S_1)\, L_1 + \mu(S_2)\, L_2$$

of points in (4. 7). Since L_0 is an extreme point of (4. 7), this must be a trivial representation, so

$$\int_S f\, d\mu = \mu(S) \int f\, d\mu,$$

for all f in B and all Baire S in X. Thus each f in B is constant almost everywhere with respect to μ, and if x is chosen to be in the carrier $C(\mu)$, of μ, that constant value must be $f(x)$. This shows that $L_0(f) = f(x)$ for all f in B and that

$$(4.\ 8) \qquad C(\mu) \subset \{y : f(y) = f(x), \quad \text{all } f \text{ in } B\} = i_B(x).$$

Since μ could have been chosen to be any measure in $H_x(B)$, (4. 8) shows that x is in M(B). This completes the proof of the lemma.

Lemma 4. 3. allows us to draw a useful conclusion concerning the relation between M(B) and (CM) where B is a subspace of $C_r(X)$ and C is a subspace of B.

For this we need the following.

Lemma 4. 4. — *Let* E_1 *and* E_2 *be real locally convex topological linear spaces and* $\varphi : E_1 \rightarrow E_2$ *a continuous linear transformation. Let* Y *be a compact convex subset of* E_1. *Then for each extreme point* v *of* $\varphi(Y)$ *there is some extreme point* u *of* Y *with* $\varphi(u) = v$.

Proof. — Let u be an extreme point of $\varphi^{-1}(v) \cap Y$. Such exist because of the Krein-Milman theorem. Then u will be extreme in Y and satisfy $\varphi(u) = v$.

Corollary 4. 5. — *Let* B *be a subspace of* $C_r(X)$ *and* C *a subspace of* B. *Then*

$$M(C) \subset i_C(M(B)).$$

Proof. — Let $\varphi : B^* \rightarrow C^*$ be the adjoint map of the natural injection of C into B; φ is continuous in the weak* topologies. Because of the Hahn-Banach theorem, the image of the weak* compact set

$$(4.\ 9) \qquad \{L : L \in B^*, \qquad L(1) = \|L\| = 1\}$$

under φ is

$$(4.\ 10) \qquad \{L : L \in C^*, \qquad L(1) = \|L\| = 1\}.$$

Let x be a point in M(C). Then by Lemma 4. 3 the linear functionnal L_x in C^* defined by

$$L_x(f) = f(x), \qquad \text{all } f \text{ in C},$$

is an extreme point of (4. 10). By Lemma 4. 4, L_x is the image under φ of an extreme point of (4. 9), which must by Lemma 4. 3 be of the form L_y,

$$L_y(f) = f(y), \qquad \text{all } f \text{ in B},$$

for some y in M(B); $\varphi(L_y) = L_x$ means simply that L_y when restricted to C agrees with L_x, that is,

$$f(x) = f(y), \qquad \text{all } f \text{ in C}. \tag{4. 11}$$

Since y is in M(B), (4. 11) shows that x is in $i_C(M(B))$. Since x was any point in M(C), it follows that $M(C) \subset i_C(M(B))$, as was to be proved.

It is worth noting that under the hypotheses of Corollary 4. 5, neither $M(C) \subset M(B)$ nor $M(B) \subset M(C)$ holds in general.

V. — REPRESENTATION OF LINEAR FUNCTIONALS IN THE GENERAL CASE

The purpose of this section is to extend our Theorem 3. 4 on the representation of linear functionals to the case of subspaces that are not separable, and to use this result to remove the hypothesis of metrizability in the Choquet theorem.

If B is a subspace of $C_r(X)$ that distinguishes points of X, the fact that any linear functional in B^* can be represented by a measure concentrated on the Choquet boundary of B follows simply from the following (which is our Theorem 5. 3) : If μ in H is B-maximal and S is a Baire set disjoint from M(B), then $\mu(S) = 0$. This in turn is an immediate consequence of Theorem 3. 2 if it is possible to find for each x in S a measure σ in $H_x(B)$ with $\sigma(S) < 1$. It is the establishment of the existence of these measures that is the main work of this section. This is accomplished by a reduction to the separable case and an application of Corollary 3. 3.

For each subset S of X, we shall denote by χ_S the characteristic function of S,

$$\chi_S(x) = \begin{cases} 1, & x \text{ in S} \\ 0, & x \text{ not in S.} \end{cases}$$

LEMMA 5. 1. — *Let* B *be a subspace of* $C_r(X)$, S *a closed subset of* X *and* x *a point of* S. *Then the following are equivalent:*

1. *Each* μ *in* $H_x(B)$ *has* $\hat{\mu}(S) = 1$,
2. *For each* g *in* $C_r(X)$ *with* $g \geqq \chi_S$,
   Sup $\{f(x) : f \in B,\ f \leqq g\} \geqq 1$.

PROOF. — Suppose that 2 holds. Let μ be in $H_x(B)$. Then for each g in $C_r(X)$ with $g \geqq \chi_S$,

$$\int g\, d\mu \geqq \operatorname{Sup}\left\{\int f\, d\mu : f \in B,\quad f \leqq g\right\} = \operatorname{Sup}\{f(x) : f \in B,\quad f \leqq g\} \geqq 1.$$

But $$\hat{\mu}(S) = \operatorname{Inf}\left\{\int g\, d\mu : g \in C_r(X),\quad g \geqq \chi_S\right\},$$

so $\hat{\mu}(S) \geqq 1$.

For the converse suppose that 2 does not hold. Let g be a function in $C_r(X)$ with $g \geqq \chi_S$ and

$$\operatorname{Sup}\{f(x) : f \in B,\quad f \leqq g\} = 1 - \varepsilon.$$

Define the linear functional L_x on B by

$$L_x(f) = f(x), \quad \text{all } f \text{ in B.} \tag{5. 1}$$

L_x is a positive linear functional on B (that is, non-negative on non-negative functions) and thus by a standard result (see [10], p. 22) on the extension of such functionals, there is a positive linear functional L on $C_r(X)$ with

$$L(f) = L_x(f), \quad \text{all } f \text{ in B,} \tag{5. 2}$$

and $L(g) = 1 - \varepsilon$. By the Riesz representation theorem, there is a μ in H with

$$L(f) = \int f\, d\mu, \quad \text{all } f \text{ in } C_r(X). \tag{5. 3}$$

Because of (5. 1), (5. 2) and (5. 3), μ is in $H_x(B)$ and

$$\hat{\mu}(S) = \int \chi_S\, d\mu \leqq \int g\, d\mu = 1 - \varepsilon < 1.$$

This completes the proof of Lemma 5. 1.

LEMMA 5. 2. — *Let* B *be a subspace of* $C_r(X)$ *and* S *a closed* G_δ *in* X. *Let* x *be a point of* S. *Suppose that each* μ *in* $H_x(B)$ *satisfies* $\mu(S) = 1$. *Then there is a* separable *subspace* C *of* B *which is such that each* μ *in* $H_x(C)$ *satisfies* $\mu(S) = 1$.

PROOF. — Since S is a closed G_δ, there is a sequence

$$\{g_n : n = 1, 2, \dots\}$$

in $C_r(X)$ decreasing point wise to χ_S. By « 1 implies 2 » of Lemma 5. 1, for each positive n and m it is possible to find some f_{nm} in B with $f_{nm} \leqq g_n$ and $f_{nm}(x) \geqq 1 - m^{-1}$. Let C be the subspace of B generated by the f_{nm} and the constant functions. C is a separable subspace. If μ is in $H_x(C)$, then

$$\mu(S) = \operatorname*{Inf}_n \int g_n \, d\mu \geqq \operatorname*{Inf}_n \left(\operatorname*{Sup}_m \int f_{nm} \, d\mu \right) = \operatorname*{Inf}_n \left(\operatorname*{Sup}_m f_{nm}(x) \right) \geqq 1,$$

so $\mu(S)$ must be 1.

THEOREM 5. 3. — *Let* B *be a subspace of* $C_r(X)$ *that distinguishes points of* X. *If* S *is a Baire set disjoint from* M(B), *and* μ *in* H *is B-maximal, then* $\mu(S) = 0$.

PROOF. — By regularity of μ we can assume S closed Baire and thus a G_δ. B distinguishes points of X so there will be a separable subspace D of B that distinguishes the points of S from those of X — S; i.e. that satisfies $i_D(S) = S$. We consider two cases.

Case 1 : For each x in S there is a σ in $H_x(B)$ with $\sigma(S) < 1$.

Case 2 : For some x in S, and all σ in $H_x(B)$, $\sigma(S) = 1$.

We shall use Theorem 3. 2 to show that $\mu(S) = 0$ follows in case 1, and then use Corollories 3. 3 and 4. 5 and Lemma 5. 2 to show that case 2 cannot occur.

Case 1 : Since $i_D(S) = S$, for each x in S, $i_D(x) \subset S$. Thus for each x in S there is some σ in $H_x(B)$ with $\sigma(i_D(x)) \leqq \sigma(S) < 1$, so by Theorem 3. 2, $\mu(S) = 0$.

Case 2 : If each σ in $H_x(B)$ satisfies $\sigma(S) = 1$, then by Lemma 5. 2 there is a separable subspace C of B (which can be chosen to contain D) so that each σ in $H_x(C)$ satisfies $\sigma(S) = 1$. We shall now use Corollary 3. 3 to contradict this by showing that there actually is a σ in $H_x(C)$ with $\sigma(S) = 0$. Since $D \subset C$ and $i_D(S) = S$, $i_C(S) = S$, and S will be disjoint from $i_C(M(B))$ since it is disjoint from $M(B)$. But by Corollary 4. 5, $M(C) \subset i_C(M(B))$, so S is disjoint from $M(C)$. By Corollary 3. 3 applied to the unit mass η at x, there is a σ in $H_x(C)$ with $\hat{\sigma}(M(C)) = 1$ and thus with $\sigma(S) = 0$. This contradicts the earlier assertion that each σ in $H_x(C)$ must satisfy $\sigma(S) = 1$. Thus Case 2 cannot occur, and Theorem 5. 3 is completely proved.

Corollary 5. 4. — *Let* B *be a subspace of* $C_r(X)$ *that distinguishes points of* X. *Let* μ *in* H *be* B-*maximal. Then* μ *can be extended to a measure that is concentrated on* M(B); *to be precise, there is a measure* $\tilde{\mu}$ *on the* σ-*ring generated by* M(B) *and the Baire sets that satisfies*

$$\tilde{\mu}(S) = \mu(S), \qquad \textit{all Baire } S,$$

and

$$\tilde{\mu}(M(B)) = \tilde{\mu}(X) = 1.$$

Proof. — Any set T in the σ-ring generated by M(B) and the Baire sets has a representation of the form

$$(5.4) \quad T = \{S_1 \cap M(B)\} \cup \{S_2 \cap (X - M(B))\}, \quad S_1, S_2 \text{ Baire}.$$

It is simple to check, using Theorem 5. 3, that if $\tilde{\mu}$ is defined by

$$\tilde{\mu}(T) = \mu(S_1), \qquad \text{with } S_1 \text{ as in (5. 4)},$$

$\tilde{\mu}$ is well defined and satisfies the conditions claimed.

The following is now an immediate consequence of Lemmas 2. 1, 2. 4 and Corollary 5. 4.

Theorem 5. 5. — *Let* B *be a subspace of* $C_r(X)$ *or* $C_c(X)$ *that distinguishes points of* X. *Let* $\mathcal{S}$ *be the* σ-*ring generated by* M(B) *and the Baire sets of* X. *Then any linear functional* L *in* B^* *has a representation of the form*

$$(5.5) \qquad L(f) = \int f\, d\mu, \qquad \text{all } f \text{ in } B,$$

for μ *a measure on* $\mathcal{S}$ *that satisfies* $\mu(T) = 0$ *for each* $T \in \mathcal{S}$ *disjoint from* M(B). *Furthermore the* μ *in* (5. 5) *can be chosen to be non-negative if and only if* $L(1) = \|L\|$.

We can now state the generalized Choquet theorem. Its proof is identical with that of Theorem 4. 2, except that Theorem 5. 5 is used instead of Theorem 3. 4.

THEOREM 5. 6. — *Let* X *be a compact convex subset of a real locally convex topological linear space. Let* X_e *be the set of extreme points of* X *and* $\mathcal{S}$ *the* σ*-ring generated by* X_e *and the Baire subsets of* X. *Then each* x *in* X *has a representation of the form*

$$\int y \, d\mu(y)$$

for some non-negative measure μ *on* $\mathcal{S}$ *that satisfies*

$$\mu(X_e) = \mu(X) = 1.$$

VI. — ALGEBRAS

If B is a subspace of $C_r(X)$ or $C_c(X)$, a subset Y of X will be called a *boundary for B* if for each f in B there is some y in Y with $|f(y)| = ||f||$.

The following lemma was announced by Bauer [2].

LEMMA 6. 1. — *If* B *is a subspace of* $C_r(X)$ *or* $C_c(X)$, M(B) *is a boundary for* B.

PROOF. — If B is a subspace of $C_c(X)$, and B_r is the subspace of $C_r(X)$ consisting of the real parts of the functions in B, then $M(B_r) = M(B)$ and also any boundary for B_r will be a boundary for B. Thus it suffices to consider the case of B a subspace of $C_r(X)$.

The subset K of B^* defined by

$$(6.1) \qquad K = \{L : L \in B^*, \quad L(1) = ||L|| = 1\}$$

is convex and weak* compact.

Choose any function h in B. Let L_0 be a point of K with $|L_0(h)| = \max\{|L(h)| : L \in K\}$, and $K_0 = \{L : L \in K, L(h) = L_0(h)\}$. By the Krein-Milman theorem, the compact convex set K_0 has an extreme point. This extreme point must also be an extreme point of K. By Lemma 4. 3, such an extreme point will be of the form L_y,

$$L_y(f) = f(y), \quad \text{all } f \text{ in B},$$

for some y in M(B). Since for each x in X, the L_x defined by

$$L_x(f) = f(x), \qquad \text{all } f \text{ in B},$$

is in K, it follows from the choice of L_y that

$$\begin{aligned}||h|| &= \max \{|h(x)| : x \in X\} \\ &= \max \{|L_x(h)| : x \in X\} = |L_y(h)| = |h(y)|.\end{aligned}$$

Since y is a point of M(B), and h is an arbitrary function in B, M(B) is a boundary for B.

Throughout the remainder of this section, A is a *uniformly closed subalgebra* of $C_c(X)$ that distinguishes the points of X and contains the constant functions. It is well known that there is a smallest *closed* boundary for the algebra A, the Silov boundary (the Silov boundary has been related to extreme points in [1], [2] and [5]). We are concerned here with boundaries that are smaller than the Silov boundary, and in particular with the question of whether if B = A, Lemma 6. 1 is the strongest result possible; i.e., whether any boundary for A must contain the Choquet boundary of A. We show that this is indeed so if each point of X is a G_δ, while we show in the general case that any boundary for A that is a Baire set must contain the Choquet boundary of A.

In order to do this we must study two properties of points of X that are equivalent to being in M(A).

We shall say that a point x of X satisfies *Condition I* if for each open neighborhood U of X there is some f in A with $||f|| \leqq 1$, $|f(x)| > \frac{3}{4}$ and $|f(y)| < \frac{1}{4}$ for all y outside of U.

We shall say that a point x of X satisfies *Condition II* if for each closed set S containing x that is a G_δ, there is some function f in A with $|f(x)| = ||f||$ and

$$\{y : |f(y)| = ||f||\} \subset S.$$

Note that if $\{x\}$ is a G_δ, Condition II simply states that there is some f in A « peaking » at x.

LEMMA 6. 2. — *If x is in* M(A), *x satisfies Condition I.*

PROOF. — Let A_r be the subspace of real parts of functions in A. Let U be any neighborhood of x. Choose a function g in $C_r(X)$ with $0 < g < 1$, $g(x) = 1$ and $g(y) = 0$ for y outside of U. Since $M(A) = M(A_r)$, any μ in $H_x(A_r)$ must satisfy $\hat{\mu}(\{x\}) = 1$. Thus by « 1 implies 2 » of Lemma 5. 1. applied to $S = \{x\}$ and $B = A_r$, there is some h_0 in A_r with $h_0 \leqq g$ and $h_0(x) > \frac{\log 6}{\log 8}$. Let $h = (\log 8)(h_0 - 1)$. Since h is in A_r, there is a k in A_r so that $h + ik$ is in A, and since A is a uniformly closed subalgebra, the function f defined by

$$f = e^{(h+ik)}$$

is in A (the use of the exponential function at this point was suggested to us by H. Royden). It is simple to check that f satisfies the conditions wanted; i.e. $||f|| \leqq 1$, $|f(x)| > \frac{6}{8} = \frac{3}{4}$, and $|f(y)| \leq \frac{1}{8} < \frac{1}{4}$ for y not in U.

Lemma 6. 3. — *If x satisfies Condition I, it must satisfy Condition II.*

Proof. — Let S be a closed G_δ containing x. Let $\{V_n\}$ be a decreasing sequence of open sets with $S = \bigcap V_n$. The construction of a function f in A with $|f(x)| = ||f||$ and

$$\{y : |f(y)| = ||f||\} \subset S$$

is identical with the construction in Theorem 2 of [4], if the sets $D_n(x)$ used in that construction are taken to be the $X - V_n$.

Lemma 6. 4. — *If x satisfies Condition II, it must be in* M(A).

Proof. — Let μ be in $H_x(A)$. Let S be any closed G_δ containing x. By Condition II there is an f in A with

$$x \in \{y : |f(y)| = ||f||\} \subset S. \tag{6. 2}$$

Since μ is in $H_x(A)$, $\int f \, d\mu = f(x)$, which by (6. 2) is possible only if $\mu(S) = 1$. Thus by the regularity of $\hat{\mu}$, $\hat{\mu}(\{x\}) = 1$, so x is in M(A).

Thus we have established.

Theorem 6. 5. — *Let x be a point of* X. *Then the following are equivalent:*

1° *x satisfies Condition I;*

2° *x satisfies Condition II;*

3° *x is in* M(A).

This equivalence for the case X *metrizable is contained in* [4].

Corollary 6. 6. — *If each point of* X *is a* G_δ, M(A) *is the smallest boundary for* A.

Proof. — By Lemma 6. 1, M(A) is a boundary for A. By Condition II, at each point x of M(A) there is some f in A with

$$|f(y)| < |f(x)|, \qquad \text{all } y \neq x.$$

Thus any boundary for A must contain M(A).
This result for X metrizable was established in [4].

Corollary 6. 7. — *Let* Y *be any Baire subset of* X *that is a boundary for* A. *Then* Y *contains* M(A).

Proof. — Suppose on the contrary that there is some point x in M(A) that is not in Y. Then there is a closed set S containing x that is a G_δ and is disjoint from Y. By Condition II there is an f in A with

$$x \in \{y : |f(y)| = \|f\|\} \subset S.$$

This f does not attain its maximum modulus on Y, contradicting the fact that Y is a boundary for A.
It is however not true that M(A) is the intersection of all of the Baire boundaries for A, as can be seen from some of the examples in the next section.

Corollary 6. 7. will now be used to show that if Y is any boundary for A, all linear functionals in A^* can be represented as measures on Y. To establish this result we need first a lemma.

Lemma 6. 8. — *Let* Y *be a boundary for* A. *Let* μ *be any* A_r*-maximal measure in* H. *If* S *is a Baire set disjoint from* Y, $\mu(S) = 0$.

Proof. — Since $Y \subset X - S$, the set $X - S$ is a Baire boundary for A. By Corollary 6. 7, $M(A) \subset X - S$, so that S is disjoint from M(A). It follows from Theorem 5. 3 that $\mu(S) = 0$.

Theorem 6. 9. below now follows from Lemma 6. 8 in the same manner that Theorem 5. 5 follows from Theorem 5. 3. We omit the details.

THEOREM 6. 9. — *Let* Y *be a boundary for* A, *and* $\mathcal{S}$ *the* σ*-ring generated by* Y *and the Baire sets of* X. *Then each linear functional* L *in* A^* *has a representation of the form*

$$(6.\ 3) \qquad L(f) = \int f\, d\mu, \qquad \textit{all } f \textit{ in } A,$$

for μ *a measure on* $\mathcal{S}$ *that satifies* $\mu(T) = 0$ *for each* T *in* $\mathcal{S}$ *that is disjoint from* Y. *Furthermore the* μ *in* (6. 3) *can be chosen to be non-negative if and only if* $L(1) = ||L||$.

VII. — EXAMPLES

We present in this section a class of examples showing that the Choquet boundary, which must be a G_δ in the separable case, can be arbitrarily bad in general. We also show that Theorems 5. 3 and 5. 5 cannot be strengthened to assertions about Borel sets rather than Baire sets. Finally there is a simple example which shows that the analogue of Theorem 6. 9 for subspaces rather than subalgebras is false.

Let $\{Y_x\}_{x \in X}$ be a family of disjoint non-empty topological spaces indexed by a topological space X. Let $Y = \bigcup_{x \in X} Y_x$ and $\pi : Y \rightarrow X$ be the projection map defined by $\pi(y) = x$ if y is in Y_x. Let $s : X \rightarrow Y$ be a cross-section; i.e., $\pi s(x) = x$ for all x in X.

We shall describe a topology (called the porcupine topology) for Y. Let $\mathcal{U}$ be the class of all subsets U of Y that satisfy the following: there is some x in X so that U is an open subset of Y_x not containing $s(x)$. Let $\mathcal{S}$ be the class of all subsets S of Y that satisfy the following: there is some x in Y so that S is a closed subset of Y_x not containing $s(x)$. Let $\mathcal{V}$ be the class of all subsets of Y of the form $\pi^{-1}(V) - (S_1 \cup \ldots \cup S_n)$, where V is an open subset of X and the S_i are in $\mathcal{S}$. The collection $\mathcal{U} \cup \mathcal{V}$ is closed under intersections and thus is the basis for a topology for Y. This is our porcupine topology. In this topology a net $\{u_\alpha\}$ of points in Y converges to a point y in $Y_x - \{s(x)\}$ if and only if the net is ultimately in Y_x and converges in the original topology of Y_x to y. If none of the u_α are in Y_x, $\{u_\alpha\}$ converges to $s(x)$ if and only if the net

$\{\pi(u_\alpha)\}$ converges to x in the topology of X. If X and the Y_x are compact Hausdorff, then Y is compact Hausdorff.

Suppose now that X and the Y_x are compact Hausdorff. Let D be a subspace of $C_r(X)$ and for each x in X, let B_x be a subspace of $C_r(Y_x)$. Let B be the subspace of $C_r(Y)$ consisting of all f in $C_r(Y)$ such that $f \circ s$ is in D, and for each x in X, f restricted to Y_x is in B_x. If D and all the B_x are closed subspaces and distinguish points, B will be a closed subspace and distinguish points. It is simple to check that the Choquet boundary M(B) of B is

$$\left\{\bigcup_{x \in X} M(B_x)\right\} - S(X - M(D)). \tag{7. 1}$$

We shall now consider a special case of the above construction. Let X be an arbitrary compact Hausdorff space and K an arbitrary subset of X. For each x in K, let Y_x consist of the one point s_x, and for each x in X-K, let Y_x be the discrete topological space consisting of the three points $\{r_x, s_x, t_x\}$. Define $s : X \to Y$ by $s(x) = s_x$, all x in X. Let D be $C_r(X)$, and if x is in K, let $B_x = C_r(Y_x)$. If x is in $X - K$, let B_x be the subspace of $C_r(Y_x)$ consisting of those f that satisfy

$$f(s_x) = \frac{1}{2}(f(r_x) + f(t_x)).$$

The construction described above applied to D and the B_x yields a closed subspace B of $C_r(Y)$ that distinguishes points. Its Choquet boundary is (7. 1) and is therefore easily seen to satisfy $Y - M(B) = s(X - K)$. Since K was an arbitrary subset of X, this shows that the Choquet boundary can be arbitrarily bad. An example of a bad boundary has also been given by Choquet in [8].

Suppose now that in this example we take X to be the unit interval with the usual topology and K to be the void set. Let ν be Lebesgue measure on X, and let μ be the Baire measure on Y defined by

$$\mu(S) = \nu(s^{-1}(s(X) \cap S))$$

for all Baire subsets S of Y. Then μ is B-maximal. Nevertheless its regular Borel extension $\hat{\mu}$ satisfies $\hat{\mu}(M(B)) = 0$.

This is in contrast to Theorem 5. 3 which shows that a B-maximal measure must be « concentrated on M(B) » in the sense that $\mu(S) = 0$ for each Baire set S disjoint from M(B). The example shows that the conclusion of the theorem cannot be strengthened to $\hat{\mu}(S)$ being 0 for each Borel set S disjoint from M(B), even if M(B) itself is Borel. It also shows that the measures μ appearing in Theorem 5. 5 may not be regular.

In order to obtain a more striking example, we take X to be the subset of the complex plane

$$X = \{ z : | z | \leq 1 \}$$

in the usual topology, and take K to be the set

$$K = \{ z : | z | < 1 \}.$$

For each x in K, let Y_x consist of one point s_x and for each x in X — K, let Y_x be the discrete topological space consisting of the three points $\{ r_x, s_x, \text{and } t_x \}$.

Define $s : X \to Y$ by $s(x) = s_x$, all x in X. Let D consist of all functions in $C_r(X)$ which are harmonic on K. If x is in K, let $B_x = C_r(Y_x)$.

If x is in X — K, let B_x be the subspace of $C_r(Y_x)$ consisting of those f that satisfy

$$f(s_x) = 1/2\,(f(r_x) + f(t_x)).$$

The construction above applied to D and the B_x yields a closed subspace B of $C_r(Y)$ that distinguishes points. Its Choquet boundary is easily seen by (7.1) to be

$$M(B) = Y - s(X).$$

Let y_0 be the point $s(0) = s_0$ of Y. We shall prove that

$$\hat{\mu}(M(B)) = 0$$

for all μ in $H_{y_0}(B)$.

To this end, we first note that the only compact subsets of M(B) are finite, by the definition of the topology on Y. It follows that in order to show that $\hat{\mu}(M(B)) = 0$, it will be sufficient to show that $\hat{\mu}(\{ r_x \}) = \hat{\mu}(\{ t_x \}) = 0$ for each x in X — K. Assume that this is not the case, so that

$\hat{\mu}(\{ r_{x_0} \}) = C > 0$, say, for some x_0 in $X - K$. Let h be any non negative harmonic function on X with $h(x_0) > 1$ and $h(0) = C$. Define the function f in B by

$$f(y) = h(x), \text{ for all } y \text{ in } Y_x.$$

Since $\mu \in H_{y_0}$, it follows that

$$f(y_0) = \int f d\mu \geq f(r_{x_0}) \mu(\{r_{x_0}\}) = h(x_0) \hat{\mu}(\{r_{x_0}\}) > C.$$

Since also

$$f(y_0) = h(0) = C,$$

this gives a contradiction. Thus

$$\hat{\mu}(M(B)) = 0.$$

This points up Theorem 5. 5, which states that there exists μ in H_{y_0} which can be extended to a measure on the σ-ring generated by M(B) and the Baire sets so as to have $\mu(M(B)) = 1$. The point is that the extension of μ does not agree with the regular Borel extension $\hat{\mu}$ of μ.

The next example demonstrates that the analogue of Theorem 6. 9 for subspaces is false. Let X be the subset

$$\{z : |z - 1| = 1\} \cup \{z : |z + 1| = 1\}$$

of the plane. Let B be the space of all functions f on X of the form

$$f(x, y) = ax + by + c, \quad (x, y) \text{ in } X,$$

for a, b and c real. Then each f in B that attains its maximum on X at $p = (1, 1)$ also attains its maximum at $(-1, 1)$, so $X - \{p\}$ is a boundary for B. However the linear functional L in B^* defined by

$$L(f) = f(p), \qquad f \text{ in B},$$

has no representation of the form

$$L(f) = \int f d\mu, \quad f \text{ in B},$$

for μ a non-negative measure on $X\text{-}\{p\}$.

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1. The topological terminology is that of [11], and the measure theoretic terminology is that of [9].