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An elementary proof of the Briançon-Skoda theorem


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Abstract. — We give an elementary proof of the Briançon-Skoda theorem. The theorem gives a criterion for when a function \( \phi \) belongs to an ideal \( I \) of the ring of germs of analytic functions at \( 0 \in \mathbb{C}^n \); more precisely, the ideal membership is obtained if a function associated with \( \phi \) and \( I \) is locally square integrable. If \( I \) can be generated by \( m \) elements, it follows in particular that \( \overline{I}^{\min(m,n)} \subset I \), where \( J \) denotes the integral closure of an ideal \( J \).

Résumé. — Nous proposons une démonstration élémentaire du théorème de Briançon-Skoda. Ce théorème donne un critère d’appartenance d’une fonction \( \phi \) à un idéal \( I \) de l’anneau des germes de fonctions holomorphes en \( 0 \in \mathbb{C}^n \); plus précisément, l’appartenance est établie sous l’hypothèse qu’une fonction dépendante de \( \phi \) et \( I \) soit de carré localement sommable. En particulier, si \( I \) est engendré par \( m \) éléments, alors \( \overline{I}^{\min(m,n)} \subset I \), où \( J \) dénote la clôture intégrale d’un idéal \( J \).

1. Introduction

Let \( \mathcal{O}_n \) be the ring of germs of holomorphic functions at \( 0 \in \mathbb{C}^n \). The integral closure \( \overline{I} \) of an ideal \( I \) is the set of all \( \phi \in \mathcal{O}_n \) such that

\[
\phi^N + a_1\phi^{N-1} + \ldots + a_N = 0,
\]

(1.1)

for some integer \( N \geq 1 \) and some \( a_k \in I^k \), \( k = 1, \ldots, N \).
By a simple estimate, (1.1) implies that there exists a constant $C$ such that

$$|\phi| \leq C|f|,$$

where $|f|$ is defined as $\sum |f_i|$ for any generators $f_i$ of $I$. It is easy to see that the choice of generators $f_i$ does not affect whether $\phi$ satisfies (1.2) for some $C$ or not.

Conversely, (1.2) implies that $\phi \in \bar{I}$ (however, we do not need this in the present paper), which is a consequence of Skoda’s theorem, [S72] and a well-known determinant trick, see for example [D07], (10.5), Ch. VIII. Another proof is given in (the republication) [LTR08].

**Theorem 1.1 (Briançon-Skoda).** — Let $I$ be an ideal of $\mathcal{O}_n$ generated by $m$ germs $f_1, \ldots, f_m$. Then $\overline{I^{\min(m,n)+l-1}} \subset I^l$ for all integers $l \geq 1$.

As noted above, $\phi \in \overline{I^{\min(m,n)+l-1}}$ implies that $|\phi| \leq C|f|^{\min(m,n)+l-1}$. Thus it suffices to show that any $\phi \in \mathcal{O}_n$ that satisfies this size condition belongs to $I^l$, in order to prove Theorem 1.1.

Another ideal that is common to consider is $\hat{I}^{(k)}$ which consists of all $\phi \in \mathcal{O}_n$ such that

$$\int_U |\phi|^2 |f|^{-2(k+\varepsilon)}dV < \infty,$$

for some neighbourhood $U$ of $0 \in \mathbb{C}^n$ and some (sufficiently small) $\varepsilon > 0$, where $dV$ is the Lebesgue measure.

Lemma 2.3 implies that $\overline{I^k} \subset \hat{I}^{(k)}$. The following theorem is thus a stronger version of Theorem 1.1:

**Theorem 1.2.** — For an ideal $I$ as in Theorem 1.1, we have

$$\hat{I}^{(\min(m,n)+l-1)} \subset I^l,$$

for all integers $l \geq 1$.

In 1974 Briançon and Skoda, [BS74], showed Theorem 1.2 as an immediate consequence of Skoda’s $L^2$-division-theorem, [S72]. Usually Theorem 1.1 is the one referred to as the Briançon-Skoda theorem.

An algebraic proof of Theorem 1.1 was given by Lipman and Tessier in [LT81]. Their paper also contains a historical summary. An account of
more recent developments and an elementary algebraic proof of the result is found in Schoutens [Sc03].

Berenstein, Gay, Vidras and Yger [BGVY93] proved Theorem 1.1 for \( l = 1 \) by finding a representation \( \phi = \sum u_i f_i \) with \( u_i \) as explicit integrals. However, some of their estimates rely on Hironaka’s theorem on resolutions of singularities.

In this paper, we provide a completely elementary proof along these lines. The key point is an \( L^1 \)-estimate (Proposition 2.1), which will be used in Section 4.

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2. The Main Estimate

In order to state Proposition 2.1, we will first recall the notion of the (standard) norm of a differential form in \( \mathbb{C}^n \). If \( x_i \) and \( y_i, 1 \leq i \leq n \), are standard coordinates for \( \mathbb{C}^n = \mathbb{R}^{2n} \), this norm is uniquely determined by demanding that the forms \( dx_i \wedge \ldots \wedge dx_j \wedge dy_{i+1} \wedge \ldots \wedge dy_k \) constitute an orthonormal basis (over \( \mathbb{C} \)) of \( \bigwedge^k T_p^* \mathbb{C}^n \).

**Proposition 2.1.** Let \( f_1, f_2, \ldots, f_m \) be generators of an ideal \( I \subset O_n \), and assume that \( \phi \in \hat{I}^{(k)} \). Then for any integer \( 1 \leq r \leq m \),

\[
\frac{|\phi| \cdot |\partial f_1 \wedge \ldots \wedge \partial f_r|}{|f|^{k+r}}
\]

is locally integrable at the origin.

**Remark 2.2.** Using a Hironaka resolution, the proof of Proposition 2.1 can be reduced to the case when every \( f_i \) is a monomial, and then the proof becomes much easier. We proceed however with elementary arguments.

**Lemma 2.3.** For any ideal \( I = (f_1, \ldots, f_m) \neq (0) \), there is a positive number \( \delta \) such that \( 1/|f|^{\delta} \) is locally integrable at the origin.

**Proof.** By considering \( F = f_1 \cdot f_2 \cdot \ldots \cdot f_m \) (remove any \( f_j \) that are identically zero), it suffices to show that \( 1/|F|^{\delta} \) is locally integrable. We can
assume that \( F \) is a Weierstrass polynomial and we consider the integral of \( 1/|F|^\delta \) on \( \Omega = D \times \Delta \), where \( D \) is a disk and \( \Delta = D^{n-1} \). By choosing \( D \) small enough, Rouché’s theorem gives that \( F \) has the same number of roots, \( s \), on each slice \( S_p = D \times \{ p \} \), \( p \in \Delta \). We partition \( S_p \) into sets \( E_p^j \), one for each root \( \alpha_j(p) \in S_p \), such that \( E_p^j \) consists of those points which are closer to \( \alpha_j(p) \) than to the other roots. We have

\[
F(z,p) = \prod_{s=1}^{s} (z - \alpha_j(p)),
\]

so on \( E_p^j \) we get

\[
\frac{1}{|F|^\delta} \leq |z - \alpha_j(p)| - \delta s.
\]

If \( \delta \) is sufficiently small, we thus get a uniform bound for the (one variable) integral of \( 1/|F|^\delta \) on \( S_p \). Fubini’s theorem then gives the integrability on \( \Omega \).

**Proof of Proposition 2.1.**— We assume for the sake of simplicity that \( r = m \), but the proof works for the other cases as well. We begin by applying Hölder’s inequality to the product of \( |\phi|/|f|k+\delta'/2 \) and \( |\partial f_1 \wedge \ldots \wedge \partial f_m|/|f|^{m-\delta'/2} \). Assume that \( \delta' \) is small enough to make the first factor \( L^2 \)-integrable. It thus suffices to show that

\[
F = \frac{|\partial f_1 \wedge \ldots \wedge \partial f_m|^2}{\prod_{s=1}^{m} |f_j|^{2-\delta}}
\]

is locally integrable for any \( \delta > 0 \). We will proceed to show that this is a consequence of the Chern-Levine-Nirenberg inequalities. The special case of these inequalities that is needed here will be proved without explicitly relying on facts about positive forms or plurisubharmonic functions. For a shorter proof of the Chern-Levine-Nirenberg inequalities, which involves these notions, see [D07] (3.3), Ch. III.

Let us first set

\[
\beta = \frac{i}{2} \partial \bar{\partial} |\zeta|^2 = \frac{i}{2} \sum d\zeta_j \wedge d\zbar_j, \quad \text{and} \quad \beta_k = \frac{\beta^k}{k!}.
\]

Then \( \beta_n \) is the Lebesgue measure \( dV \). A simple argument gives that for any \((1,0)\)-forms \( \alpha_j \),

\[
\frac{i}{2} \alpha_1 \wedge \bar{\alpha_1} \wedge \ldots \wedge \frac{i}{2} \alpha_p \wedge \bar{\alpha_p} \wedge \beta_{n-p} = |\alpha_1 \wedge \ldots \wedge \alpha_p|² dV.
\]

(2.1)

Fix a sufficiently small \( \delta > 0 \) as in Lemma 2.3. We will need at least \( \delta < 2 \) in the sequel. We now compute

\[
\partial \bar{\partial} (|f_j|^2 + \varepsilon)^{\delta/2} = \frac{\delta}{2} \left( 1 + \frac{(\delta - 1)}{|f_j|^2 + \varepsilon} \right) (|f_j|^2 + \varepsilon)^{\delta/2 - 1} \partial f_j \wedge \bar{\partial} f_j
\]

which yields that

\[
\frac{i \partial f_j \wedge \bar{\partial} f_j}{(|f_j|^2 + \varepsilon)^{1-\delta/2}} = G_j i \partial \bar{\partial} (|f_j|^2 + \varepsilon)^{\delta/2},
\]

(2.2)
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where

\[ G_j = \frac{2}{\delta} \left[ 1 + \left( \frac{\delta}{2} - 1 \right) \frac{|f_j|^2}{|f_j|^2 + \varepsilon} \right]^{-1}. \]

Observe that

\[ \left( \frac{2}{\delta} \right) \leq G_j \leq \left( \frac{2}{\delta} \right)^2. \tag{2.3} \]

We introduce forms \( F^\varepsilon_k dV \) by setting

\[ F^\varepsilon_k dV = \frac{|\partial f_k \wedge \ldots \wedge \partial f_m|^2}{\prod_k (|f_j|^2 + \varepsilon)^{1-\delta/2}} = \prod_k G_j \frac{i}{2} \partial \bar{\partial} (|f_j|^2 + \varepsilon)^{\delta/2} \wedge \beta_{n+k-m-1}. \tag{2.4} \]

Note that \( F^1 dV \) is a regularization of \( F dV \). From the equality \(|w \wedge \bar{w}| = 2^p |w|^2\), that holds for all \((p,0)\)-forms \( w \), and (2.2), we get

\[ F^\varepsilon_k dV = \frac{|\prod_k (\frac{i}{2} \partial f_j \wedge \partial f_j)|}{\prod_k (|f_j|^2 + \varepsilon)^{1-\delta/2}} = \prod_k G_j \frac{i}{2} \partial \bar{\partial} (|f_j|^2 + \varepsilon)^{\delta/2} \wedge \beta_{n+k-m-1}. \tag{2.5} \]

Comparing (2.4) with (2.5), we get

\[ H^\varepsilon_k dV := \prod_k i \partial \bar{\partial} (|f_j|^2 + \varepsilon)^{\delta/2} \wedge \beta_{n+k-m-1} = \prod_k i \partial \bar{\partial} (|f_j|^2 + \varepsilon)^{\delta/2} \wedge \beta_{n+k-m-1}. \tag{2.6} \]

Let \( B \) be a ball about the origin and let \( \chi_B \) be a smooth cut-off function supported in a concentric ball of twice the radius. We now use (2.5), (2.6) and (2.3) and integrate by parts (going from the second to the third line below) to see that

\[ \int_B F^\varepsilon_1 dV \leq C_\delta \int \chi_B \left| i \partial \bar{\partial} (|f_1|^2 + \varepsilon)^{\delta/2} \wedge \ldots \wedge i \partial \bar{\partial} (|f_m|^2 + \varepsilon)^{\delta/2} \right| dV \]

\[ = C_\delta \int \chi_B i \partial \bar{\partial} (|f_1|^2 + \varepsilon)^{\delta/2} \wedge \ldots \wedge i \partial \bar{\partial} (|f_m|^2 + \varepsilon)^{\delta/2} \wedge \beta_{n-m} \]

\[ = C_\delta \left| \int (\partial \bar{\partial} \chi_B) (|f_1|^2 + \varepsilon)^{\delta/2} \wedge \ldots \wedge i \partial \bar{\partial} (|f_m|^2 + \varepsilon)^{\delta/2} \wedge \beta_{n-m} \right| \]

\[ \leq C_1 C_\delta \sup_{2B} |f_1|^\delta \int_{2B} \left| i \partial \bar{\partial} (|f_2|^2 + \varepsilon)^{\delta/2} \wedge \ldots \wedge i \partial \bar{\partial} (|f_m|^2 + \varepsilon)^{\delta/2} \right| dV \]

\[ \leq C_1 C_\delta \sup_{2B} |f_1|^\delta \int_{2B} \chi_{2B} H^\varepsilon_{\delta} dV, \]
where $C_\delta = 2^m / \delta^{2m}$ and $C_1 = \sup \chi_B$. Should the reader have any doubts about the integration by parts, note that $d(\alpha \wedge \beta \wedge \gamma) = \partial \alpha \wedge \beta \wedge \gamma + \alpha \wedge \partial \beta \wedge \gamma$, for any function $\alpha$ and forms $\beta$ and $\gamma$ such that $\gamma$ is a closed $(n-1, n-1)$-form and $\beta$ is a $(0,1)$-form. A similar relation holds for the $\bar{\partial}$-operator. Since the second integral on the first line in the calculation above is nothing but $\int \chi_B H_\varepsilon^1 dV$, we can proceed by induction over $k$ to obtain
\[
\int_B |F_\varepsilon| dV \leq \frac{C}{\delta^{2m}} \sup_{2^{m+1}B} |f_1 \cdots f_m|^\delta < \infty,
\]
so if we let $\varepsilon$ tend to zero, we get the desired bound. □

Remark 2.4. — It is not hard to see that essentially the same proof gives that $|\partial f_1 \wedge \cdots \wedge \partial f_r| / \prod_i |f_i|$ is locally integrable.

3. Division by weighted integral formulas

We will use a division formula introduced in [B83], but for convenience, we use the formalism from [A03] to describe it.

Consider a fixed point $z \in \mathbb{C}^n$ and define the operator $\nabla_{\zeta - z} = \delta_{\zeta - z} - \bar{\partial}$, where $\delta_{\zeta - z}$ is contraction with the vector field
\[
2\pi i \sum_{k=1}^n (\zeta - z_k) \frac{\partial}{\partial \zeta_k}.
\]
Recall that $\delta_{\zeta - z}$ anti-commutes with $\bar{\partial}$. We allow these operators to act on forms of all bidegrees. In particular, the contraction of a function is zero.

A weight with respect to $z$ is a smooth differential form $g = g_{0,0} + g_{1,1} + \cdots + g_{n,n}$ such that $\nabla_{\zeta - z} g = 0$ and $g_{0,0}(z) = 1$. The subscripts denote bidegree.

Let $s$ be any $(1,0)$-form such that $\delta_{\zeta - z} s = 1$ outside of $\{\zeta = z\}$, e.g.,
\[
s = \frac{\partial |\zeta|^2}{2\pi i (|\zeta|^2 - \bar{\zeta} \cdot z)},
\]
where the dot sign denotes the pairing given by $a \cdot b = \sum a_i b_i$. Next we set
\[
u = s + s \wedge \bar{\partial}s + \cdots + s \wedge (\bar{\partial}s)^{n-1},
\]
which is defined whenever $s$ is defined. We note that $\delta_{\zeta - z} \bar{\partial}s = -\bar{\partial}\delta_{\zeta - z} s = -\bar{\partial}1 = 0$. Since $s \wedge (\bar{\partial}s)^n$ must vanish, we have $(\bar{\partial}s)^n = \delta_{\zeta - z} (s \wedge (\bar{\partial}s)^n) = 0.$
The reader may check that $\nabla_{\zeta-z} u = 1$. In fact, this can be seen elegantly by using functional calculus of differential forms; then $u = s/\nabla_{\zeta-z} s = s/(1 - \overline{\partial}s) = s \wedge \sum_{1}^{n-1} (\overline{\partial}s)^{k}$, and $\nabla_{\zeta-z} u = \nabla s/\nabla s = 1$.

One can construct a weight $g_{z}(\zeta)$ with respect to $z$, compactly supported in the ball of radius $r + \varepsilon$, such that $(z, \zeta) \mapsto g_{z}(\zeta)$ is holomorphic in $z$ in the ball of radius $r - \varepsilon$. This is accomplished by setting

$$g_{z}(\zeta) = \chi - \overline{\partial}\chi \wedge u,$$

where $\chi$ is a cut-off function that is 1 whenever $|\zeta| \leq r - \varepsilon$ and 0 whenever $|\zeta| > r + \varepsilon$. Note that $u$ is well-defined on the support of $\partial\chi$. We see that $g_{z}$ is a weight since $\nabla_{\zeta-z}$ is an anti-derivation; $\nabla_{\zeta-z} g_{z} = -\overline{\partial}\chi + \overline{\partial}\delta_{\zeta-z} \chi \wedge u + \overline{\partial}\chi = 0$ (as $\chi$ is a function, we have $\delta_{\zeta-z} \chi = 0$).

**Proposition 3.1.** — If $g$ is a weight with respect to $z$ which has compact support, and if $\phi$ is holomorphic in a neighbourhood of the support of $g$, then

$$\phi(z) = \int \phi(\zeta)g(\zeta). \quad (3.1)$$

**Proof.** — As in the construction of a weight with compact support above, we define forms

$$b = \frac{\partial|\zeta - z|^{2}}{2\pi i|\zeta - z|^{2}}$$

and $u = b \wedge \sum (\overline{\partial}b)^{k}$ such that $\delta_{\zeta-z} b = 1$ and $\nabla_{\zeta-z} u = 1$ hold outside of $\{\zeta = z\}$. The highest degree term of $u$ is the Bochner-Martinelli kernel. We now want to determine the residue $R = 1 - \nabla_{\zeta-z} u$ (where $\nabla_{\zeta-z}$ is taken in the sense of currents) at $\{\zeta = z\}$. The $(k, k-1)$ bidegree component $u_{k,k-1}$ of $u$ is $\mathcal{O}(|\zeta-z|^{-2k+1})$, so only the highest component, $\overline{\partial}u_{n,n-1} = \overline{\partial}(b \wedge (\overline{\partial}b)^{n-1})$ of $\nabla_{\zeta-z} u$ will contribute to the residue. Using Stokes’ theorem, it is easy to check that $R = [z]$, the point evaluation current at $z$. Clearly $\nabla_{\zeta-z}(\phi g) = 0$, so $\nabla_{\zeta-z}(u \wedge \phi g) = \phi g - [z] \wedge \phi g$. Taking highest order terms, we get

$$d(u \wedge \phi g)_{n,n-1} = \overline{\partial}(u \wedge \phi g)_{n,n-1} = [z] \wedge \phi g_{0,0} - \phi g_{n,n} = [z] \wedge \phi - \phi g_{n,n},$$

so by Stokes’s theorem

$$\int \phi(\zeta)g(\zeta) = \int \phi(\zeta)g_{n,n}(\zeta) = [z].\phi = \phi(z).$$

□
4. Finishing the proof of Theorem 1.2

We now begin constructing a weight associated with Berndtsson’s division formula for an ideal \( I \subset \mathcal{O}_n \). Take \( h = (h_i) \) to be an \( m \)-tuple of so called Hefer forms with respect to the generators \( f_i \) of \( I \); these (germs of) \((1, 0)\)-forms are holomorphic in \( 2n \) variables, and satisfy \( \delta_{\zeta - z} h_i = f_i(\zeta) - f_i(z) \). To see that \( h \) exists, write

\[
  f_i(\zeta) - f_i(z) = \int_0^1 \frac{d}{dt} f_i(z + t(\zeta - z)) dt,
\]

and compute the derivative inside the integral. Define \( \sigma_i = \bar{f}_i / |f|^2 \) and let \( \chi_\varepsilon = \chi(|f|/\varepsilon) \) be a smooth cut-off function, where \( \chi \) is approximatively the characteristic function for \([1, \infty)\). Recall that the dot sign refers to the pairing \( a \cdot b = \sum a_i b_i \).

We now set

\[
  \mu = \min(m, n + 1)
\]

and define the weight

\[
  g_B = (1 - \nabla_{\zeta - z} (h \cdot \chi_\varepsilon \sigma))^\mu = (1 - \chi_\varepsilon + f(z) \cdot \chi_\varepsilon \sigma + h \cdot \overline{\partial} (\chi_\varepsilon \sigma))^\mu \quad (4.1)
\]

where

\[
  A_\varepsilon = \sum_{k=0}^{\mu-1} C_k \chi_\varepsilon \sigma [f(z) \cdot \chi_\varepsilon \sigma]^k [1 - \chi_\varepsilon + h \cdot \overline{\partial} (\chi_\varepsilon \sigma)]^{\mu-k-1} \quad (4.2)
\]

and

\[
  B_\varepsilon = (1 - \chi_\varepsilon + h \cdot \overline{\partial} (\chi_\varepsilon \sigma))^\mu. \quad (4.3)
\]

For convenience, we assume that \( l = 0 \) in Theorem 1.2. The proof goes through verbatim for general \( l \) by just replacing \( \mu \) with \( \mu + l \) in the definition of \( g_B \).

Let \( g \) be any weight with respect to \( z \) which has compact support and is holomorphic in \( z \) near 0. Substitution of the last line of (4.1) into (3.1) applied to the weight \( g_B \wedge g \) yields

\[
  \phi(z) = f(z) \cdot \int \phi(\zeta) A_\varepsilon \wedge g + \int \phi(\zeta) B_\varepsilon \wedge g. \quad (4.4)
\]

To obtain the division we will show two claims:
Claim 4.1. — The second term in (4.4),
\[ \int \phi(\zeta) B_\varepsilon \wedge g, \]
converges uniformly to zero for small \(|z|\).

Claim 4.2. — If \( m \leq n \), the tuple of integrals in (4.4),
\[ \int \phi(\zeta) A_\varepsilon \wedge g, \]
converges uniformly as \( \varepsilon \to 0 \).

We give an argument for the case \( m > n \) of Theorem 1.2 at the end of
the paper. Letting \( \varepsilon \) go to zero in (4.4), these claims give that \( \phi \in I \).

To prove Claim 4.1, we will soon find a function \( F(\zeta) \) integrable near
\( \zeta = 0 \), such that \( |\phi(\zeta) B_\varepsilon| \leq F \). Now we note that the integrand of Claim 4.1
has support on the set \( S_\varepsilon = \{|f| \leq 2\varepsilon\} \); outside of \( S_\varepsilon \), we have that \( \chi_\varepsilon = 1 \),
so \( B_\varepsilon = (h \cdot \partial \sigma)^\mu \), which vanishes regardless of whether \( \mu = n+1 \) or \( \mu = m \).
In the latter case apply \( \partial \) to \( f \cdot \sigma = 1 \) to see that \( \partial \sigma \) is linearly dependent.
Thus for small \(|z|\), we get
\[ \lim_{\varepsilon \to 0} \left| \int \phi(\zeta) B_\varepsilon \wedge g \right| \leq C \lim_{\varepsilon \to 0} \int_{S_\varepsilon} F = 0, \]
where we used that \( g \) is smooth.

The existence of \( F \) is a consequence of the main estimate of the previous
chapter and a little bookkeeping that we will now carry out. Straightforward
calculations, based on the fact that \( \chi' \) is bounded, give that
\[ \bar{\partial} \chi_\varepsilon = \mathcal{O}(1) |f|^{-1} \sum \bar{\partial} f_j \quad \text{and} \quad \bar{\partial} \sigma_i = \mathcal{O}(1) |f|^{-2} \sum \bar{\partial} f_j, \]
(4.5)
since \(|f| \sim \varepsilon \) on the support of \( \bar{\partial} \chi_\varepsilon \). Note also that \(|\sigma| = |f|^{-1}\). It is easy to
see that \( \mathcal{O}(1) \) actually represents a function that does not depend on \( \varepsilon \).

Using these facts, as we binomially expand (4.3), we get that \( \phi(\zeta) B_\varepsilon \) is
a linear combination of terms that are given by
\[ \phi(\zeta) (\bar{\partial} \chi_\varepsilon h \cdot \sigma)^a \wedge (\chi_\varepsilon h \cdot \bar{\partial} \sigma)^b (1 - \chi_\varepsilon)^c = \phi(\zeta) |f|^{-2(a+b)} \bar{\partial} f_j \wedge \mathcal{O}(1), \]
(4.6)
where $a + b + c = \mu$, $J \subset \{1, 2 \ldots m\}$, $|J| = a + b$ and $\partial f_J = \bigwedge_{i \in J} \overline{\partial f_i}$. Since $\partial f_J = 0$ whenever $a + b > n$ we can assume that $a + b \leq \min(m, n)$. We now set $F$ to be the sum of the right hand side of (4.6) over all possible $J$, i.e.

$$F = \sum_{|J| \leq \min(m, n)} \phi(\zeta)|f|^{-2|J|} \overline{\partial f_J} \wedge \mathcal{O}(1).$$

(4.7)

Clearly $|\phi(\zeta)B_\varepsilon| \leq F$. Applying Proposition 2.1 with $k = \min(m, n)$ to (4.7), it follows that $F$ is indeed locally integrable. □

Before dealing with Claim 4.2, we note that there is a way around it; clearly, the integrals in the claim are holomorphic for each $\varepsilon > 0$, so the first term (4.4) belongs to $I$ for fixed $\varepsilon > 0$. Thus, due to Claim 4.1, $\phi$ is in the closure of $I$ with respect to uniform convergence. All ideals are however closed under uniform convergence, see [H90] Chapter 6, so $\phi$ belongs to $I$.

The proof of Claim 4.2 is similar to the proof of Claim 4.1. Since we have assumed $m \leq n$, we have $\mu = \min(m, n + 1) = m$. Expanding $\phi(\zeta)A_\varepsilon$, displayed in (4.2), we get a linear combination of terms that are given by

$$\phi(\zeta)(f(z) \cdot \chi_\varepsilon \sigma)^k (\overline{\partial \chi_\varepsilon} h \cdot \sigma)^a \wedge (h \cdot \overline{\partial \sigma})^b = \phi(\zeta)|f|^{-(1+k+2a+2b)} \overline{\partial f_J} \wedge \mathcal{O}(1),$$

where $a + b \leq \mu - k - 1$, $k \leq \mu - 1$ and $|J| = a + b$. The sum $1 + k + 2a + 2b$ is at most $2\mu - 1$, and this happens when $k = 0$ and $a + b = \mu - 1$. By an argument almost identical to the one proving that $F$ was integrable, we get an integrable upper bound for $\phi A_\varepsilon$ independent of $z$ and $\varepsilon$. This is, of course, an upper bound also for the limit

$$A := \lim_{\varepsilon \to 0} A_\varepsilon = \sum_{k=0}^{\mu-1} C_k \sigma[f(z) \cdot \sigma]^k [h \cdot \overline{\partial \sigma}]^{\mu-k-1}.$$

As in the beginning of the proof of Claim 4.1, one sees that $\int \phi(\zeta)A_\varepsilon \wedge g$ converges uniformly to $\int \phi(\zeta)A \wedge g$. □

The case $m > n$ presents an additional difficulty as our upper bound fails to be integrable. Also, $\phi A \wedge g$ will not be integrable. A remedy is to consider a reduction of the ideal $I$, that is, an ideal $\mathfrak{a} \subset I$ generated by $n$ germs such that $\mathfrak{a} = \overline{I}$, see for example Lemma 10.3, Ch. VIII in [D07]. If $a_i$ generate $\mathfrak{a}$ we have that $|a| \sim |f|$, so $\mathfrak{a}^{(k)} = \overline{I}^{(k)}$ for any integer $k \geq 1$. Thus we have reduced to the case $m \leq n$, which has already been proved. □
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Bibliography


