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Ahlfors’ currents in higher dimension


<http://afst.cedram.org/item?id=AFST_2010_6_19_1_121_0>
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RÉSUMÉ. — On considère une application holomorphe non dégénérée $f : V \mapsto X$ où $(X, \omega)$ est une variété Hermitienne compacte de dimension supérieure ou égale à $k$ et $V$ est une variété complexe, connexe, ouverte de dimension $k$. Dans cet article, nous donnons des critères qui permettent de construire des courants d’Ahlfors dans $X$.

ABSTRACT. — We consider a nondegenerate holomorphic map $f : V \mapsto X$ where $(X, \omega)$ is a compact Hermitian manifold of dimension larger than or equal to $k$ and $V$ is an open connected complex manifold of dimension $k$. In this article we give criteria which permit to construct Ahlfors’ currents in $X$.

0. Introduction

Let $f : V \mapsto X$ be a nondegenerate holomorphic map between an open connected complex manifold $V$ (non-compact) of dimension $k$ and a compact Hermitian manifold $(X, \omega)$ of dimension larger than or equal to $k$. We consider an exhaustion function $\tau$ on $V$. This means that (see [14]):

(i) $\tau : V \mapsto [0, +\infty]$ is $C^1$.

(ii) $\tau$ is proper (i.e. $\tau^{-1}(\text{compact}) = \text{compact}$).

(iii) There exists $r_0 > 0$ such that $\tau$ has only isolated critical points in $\tau^{-1}([r_0, +\infty[)$.

In this article we will employ the notation $V(r) = \tau^{-1}([0, r[)$.
The first important example is $V = \mathbb{C}^k$ and $\tau = ||z||^2$. When $k = 1$ we are studying entire curves in $X$. Another example is that of a pseudoconvex domain $V$ in $\mathbb{C}^k$. If $\tau_0$ is its exhaustion function, we can easily transform $\tau_0$ into a function $\tau$ which satisfies the previous hypothesis (see [11] p. 63-65).

The goal of this article is to construct Ahlfors’ currents in $X$ starting from $V$ and $f$. By definition, an Ahlfors’ current is a \textbf{closed} positive current of bidimension $(k, k)$ which is the limit of a sequence $\frac{f_*[V(r_n)]}{\text{volume}(f(V(r_n)))}$ (here $r_n \to +\infty$ and volume$(f(V(r_n))) := \int_{V(r_n)} f_*^*\omega^k$ is the volume of $f(V(r_n))$ counted with multiplicity). When $V = \mathbb{C}$ and $\tau = ||z||^2$, M. McQuillan constructed such currents in [10] (see [1] too). These currents are fundamental tools in the study of the hyperbolicity of $X$ (see for example [6]). When the dimension of $V$ is larger than or equal to 2 it is not always possible to produce Ahlfors’ currents. Indeed, for example, there exist domains $\Omega$ in $\mathbb{C}^2$ which are biholomorphic to $\mathbb{C}^2$ and such that $\Omega \neq \mathbb{C}^2$ (Fatou-Bieberbach domains). As a consequence, to produce Ahlfors’ currents it is necessary to add a hypothesis on $f$.

When the dimension of $X$ is equal to $k$, there exist criteria which imply that $f(V)$ is dense in $X$ (see [3], [13], [14], [8], [7], [2] and [12]). These criteria use the degrees of $f$ (see [3]) or the growth of the function $f$.

Our goal is to give criteria which use these degrees in order to produce Ahlfors’ currents in $X$. Of course, in the case where the dimension of $X$ is equal to $k$, the existence of such currents will automatically imply that $f(V)$ is dense in $X$. Indeed, $[X]$ is the only positive closed current of bidimension $(k, k)$ in $X$ (up to normalization).

In this article, we will use the following degrees ($t_{k-1}$ will be slightly different from Chern’s one):

$$t_k(r) = \int_{V(r)} f_*^*\omega^k,$$

which is the volume of $f(V(r))$ counted with multiplicity, and

$$t_{k-1}(r) = \int_{V(r)} i\partial\tau \wedge \overline{\partial}\tau \wedge f_*^*\omega^{k-1}.$$

Let $C$ be the set of critical values of $\tau$ in $[r_0, +\infty[$. $V$ is connected and non-compact so we can suppose that $[r_0, +\infty[ \subset \tau(V)$.

The criteria that we will give on $t_k$ and $t_{k-1}$ will strongly use the following inequality:
Ahlfors' currents in higher dimension

Theorem 0.1.— The functions $t_k$ and $t_{k-1}$ are $C^1$ on $]r_0, +\infty[\mathcal{C}$ and $C^0$ on $]r_0, +\infty[\mathcal{C}$. If $r \in]r_0, +\infty[\mathcal{C}$ then

$$\|\partial f_*[V(r)]\|^2 \leq K(X) t'_{k-1}(r) t'_k(r).$$

Here $K(X)$ is a constant which depends only on $(X, \omega)$ and

$$\|\partial f_*[V(r)]\| := \sup_{\Psi \in \mathcal{F}(k-1, k)} |\langle \partial f_*[V(r)], \Psi \rangle|$$

where $\mathcal{F}(k-1, k)$ is the set of smooth $(k-1, k)$ forms $\Psi$ with $\|\Psi\| := \max_{x \in X} \|\Psi(x)\| \leq 1$.

By using the previous inequality we can prove some criteria which imply the existence of Ahlfors' currents. Indeed, the difficulty for the construction of Ahlfors’ currents is the closedness of a limit of $f_*(V(r_n))$ and the previous Theorem gives an estimate for $\|\partial f_*[V(r_n)]\|$. Here we give the following two criteria:

**Theorem 0.2.**— We suppose that $f$ is nondegenerate and of finite-type (i.e. there exist $C_1, C_2, r_1 > 0$ such that $\text{volume}(f(V(r))) \leq C_1 r^{C_2}$ for $r \geq r_1$).

If

$$\limsup_{r \to +\infty} \frac{t_{k-1}(r)}{r^2 t_k(r)} = 0$$

then there exists a sequence $r_n$ which goes to infinity such that $\frac{f_*(V(r_n))}{\text{volume}(f(V(r_n)))}$ converges to a closed positive current with bidimension $(k, k)$ and mass equal to 1.

When $V = \mathbb{C}$ and $\tau = \|z\|^2$, the finite-type hypothesis holds modulo a Brody renormalization (see for example [9]).

We now give one criterion which does not use this hypothesis.

**Theorem 0.3.**— If $f$ is nondegenerate and if there exist $\varepsilon > 0$ and $L > 0$ such that:

$$\limsup_{r \notin \mathbb{C}, r \to +\infty} \frac{t'_{k-1}(r)}{rt_k(r)^{1-\varepsilon}} \leq L$$

then there exists a sequence $r_n$ which goes to infinity such that $\frac{f_*(V(r_n))}{\text{volume}(f(V(r_n)))}$ converges to a closed positive current with bidimension $(k, k)$ and mass equal to 1.
The plan of this article is the following: in the first part we prove the inequality (Theorem 0.1), in the second one we give the proof of both criteria (Theorems 0.2 and 0.3). In the third part, we give a new formulation of the criteria in the special case where $V = \mathbb{C}^k$.

1. Proof of the inequality

Let $\mathcal{C}$ be the set of critical values of $\tau$ in $[r_0, +\infty[$. We recall that we can suppose $[r_0, +\infty[ \subset \tau(V)$. Notice that point (iii) in the hypothesis on $\tau$ implies that $\mathcal{C}$ is discrete. When $r \in ]r_0, +\infty[$ and $r \notin \mathcal{C}$ then $\tau : \tau^{-1}(]r - \varepsilon, r + \varepsilon[) \mapsto ]r - \varepsilon, r + \varepsilon[ \text{ is a }$ submersion for $\varepsilon > 0$ small enough. In particular, $\tau^{-1}(r)$ is a submanifold of $V$ and $\partial V(r) = \tau^{-1}(r)$. When $r \in \mathcal{C}$, then $\tau^{-1}(r)$ is a compact set which is a submanifold of $V$ outside a neighbourhood of a finite number of points.

We begin now with the following lemma:

**Lemma 1.1.** — *The functions $t_k$ and $t_{k-1}$ are $C^1$ on $]r_0, +\infty[ \setminus \mathcal{C}$ and $C^0$ on $]r_0, +\infty[$.*

**Proof.** — The form $f^*\omega^k$ is positive and smooth and $i\partial \tau \wedge \partial \tau \wedge f^*\omega^{k-1}$ is positive and continuous ($\tau$ is $C^1$) so it is enough to show that $t(r) = \int_{\tau^{-1}(r)} \Phi$ is $C^1$ on $]r_0, +\infty[ \setminus \mathcal{C}$ and $C^0$ on $]r_0, +\infty[ \setminus \mathcal{C}$ with $\Phi$ a positive continuous form of bidegree $(k, k)$.

We take $r \in ]r_0, +\infty[ \setminus \mathcal{C}$ and $\varepsilon > 0$ such that $\tau : \tau^{-1}(]r - \varepsilon, r + \varepsilon[) \mapsto ]r - \varepsilon, r + \varepsilon[ \text{ is a submersion}$. Now, if $r' \in ]r - \varepsilon, r[$, we have:

$$\frac{t(r) - t(r')}{r - r'} = \frac{1}{r - r'} \int_{\tau^{-1}(]r', r[)} \Phi = \frac{1}{r - r'} \int_{]r', r[} \tau_* \Phi.$$

The form $\tau_* \Phi$ is continuous so it is equal to $\alpha(s)ds$ with $\alpha$ in $C^0(]r - \varepsilon, r + \varepsilon[)$. We obtain:

$$\frac{t(r) - t(r')}{r - r'} = \frac{1}{r - r'} \int_{r'}^r \alpha(s)ds$$

which converges to $\alpha(r)$ when $r' \to r$. The same thing happens when we consider $r' \in ]r, r + \varepsilon[$, so the function $t$ is differentiable at $r$ and $t'(r) = \alpha(r)$. In particular $t$ is $C^1$ on $]r_0, +\infty[$.

**Remark 1.2.** — Notice that here we did not use that $\Phi$ is positive. We will use this remark in the proof of Theorem 0.1.
Ahlfors’ currents in higher dimension

Now, consider $r \in \mathcal{C}$. If we take $\varepsilon > 0$, then we can find two neighbourhoods $W_\varepsilon \subset W_{2\varepsilon}$ of the (finite) number of the critical points in $\{\tau = r\}$ such that $\int_{W_{2\varepsilon}} \Phi \leq \varepsilon$ (because $\Phi$ is continuous). Now, let $\psi$ be a $C^\infty$ function which is equal to 1 in a neighbourhood of $W_\varepsilon$ and to 0 outside $W_{2\varepsilon}$ $(0 \leq \psi \leq 1)$. Then, if $r' < r$,

$$t(r) - t(r') = \int_{V(r) \setminus V(r')} \psi \Phi + \int_{V(r) \setminus V(r')} (1 - \psi) \Phi \leq \varepsilon + \int_{V(r) \setminus V(r')} (1 - \psi) \Phi.$$

If $\alpha > 0$ is small then $\tau$ is a submersion on $\tau^{-1}([r - \alpha, r + \alpha]) \cap (V \setminus W_\varepsilon)$. In particular the function

$$r' \mapsto \int_{V(r) \setminus V(r')} (1 - \psi) \Phi = \int_{r'}^r \tau_*((1 - \psi) \Phi)$$

goes to 0 when $r' \to r$. The same thing happens when we take $r' > r$. As a consequence, there exists $\delta > 0$ such that if $|r - r'| < \delta$ then $|t(r) - t(r')| \leq 2\varepsilon$, i.e. $t$ is continuous at $r$. \hfill $\Box$

We give now the proof of Theorem 0.1.

We take $r \in ]r_0, +\infty[ \setminus \mathcal{C}$. We have:

$$\|\partial f_*[V(r)]\| = \sup_{\Psi \in \mathcal{F}(k-1,k)} |\langle \partial f_*[V(r)], \Psi \rangle|$$

where $\mathcal{F}(k - 1, k)$ is the set of smooth $(k - 1, k)$ forms $\Psi$ with $\|\Psi\| = \max_{x \in X} \|\Psi(x)\| \leq 1$. If $\Psi \in \mathcal{F}(k-1,k)$ then we can write (see for example [5] chapter III Lemma 1.4)

$$\Psi = \sum_{i=1}^{K(X)} \theta_i \wedge \Omega_i$$

where $K(X)$ is a constant which depends only on $X$, the $\theta_i$ are smooth forms of bidegree $(0, 1)$ with $\|\theta_i\| \leq 1$ and the $\Omega_i$ are (strongly) positive smooth forms of bidegree $(k - 1, k - 1)$ with $\|\Omega_i\| \leq K(X)$. So, to prove the inequality it is sufficient to bound from above $|\langle \partial f_*[V(r)], \theta \wedge \Omega \rangle|^2$ by $K'(X) t_{k-1}(r) t_k(r)$ with $\theta$ a smooth form of bidegree $(0, 1)$ with $\|\theta\| \leq 1$, $\Omega$ a positive smooth form of bidegree $(k - 1, k - 1)$ with $\|\Omega\| \leq 1$ and $K'(X)$ a constant which depends only on $(X, \omega)$.

If $\varepsilon > 0$ is small then $\tau : \tau^{-1}([r - \varepsilon, r + \varepsilon[) \mapsto [r - \varepsilon, r + \varepsilon[ is a submersion. Now, if we take $r' \in ]r - \varepsilon, r[$, we have:
\[ A(r', r) := \left| \frac{1}{r - r'} \int_{r'}^{r} \langle \partial f_* [V(s)], \theta \wedge \Omega \rangle ds \right| \]

\[ = \left| \frac{1}{r - r'} \int_{r'}^{r} \langle [\partial V(s)], f^* \theta \wedge f^* \Omega \rangle ds \right|. \]

If we use the Stokes’ Theorem, we have:

\[ A(r', r) = \left| \frac{1}{r - r'} \int_{r}^{r'} \langle [\partial V(s)], f^* \theta \wedge f^* \Omega \rangle ds \right| \]

\[ = \left| \frac{1}{r - r'} \int_{r}^{r'} \langle [\tau = s], f^* \theta \wedge f^* \Omega \rangle ds \right|, \]

because for \( s \in ]r - \varepsilon, r + \varepsilon[ \) the boundary of \( V(s) \) is \( \{ \tau = s \} \).

We obtain:

\[ A(r', r) = \left| \frac{1}{r - r'} \int_{r}^{r'} \left( \int_{\tau = s} f^* \theta \wedge f^* \Omega \right) ds \right|. \]

Now \( \tau : \tau^{-1}(]r - \varepsilon, r + \varepsilon[) \mapsto ]r - \varepsilon, r + \varepsilon[ \) is a submersion, so by using Fubini’s Theorem (see [4] p. 334), we have:

\[ A(r', r) = \left| \frac{1}{r - r'} \int_{V(r) \setminus V(r')} d\tau \wedge f^* \theta \wedge f^* \Omega \right| \]

\[ = \left| \frac{1}{r - r'} \int_{V(r) \setminus V(r')} \partial \tau \wedge f^* \theta \wedge f^* \Omega \right|. \]

Now, if we consider,

\[ \{ \phi, \psi \} := \int_{V(r) \setminus V(r')} i\phi \wedge \overline{\psi} \wedge f^* \Omega \]

where \( \phi \) and \( \psi \) are continuous forms of bidegree \((1, 0)\), then \( \{ \phi, \phi \} \geq 0 \) (because \( \Omega \) is positive) and so by using the proof of the Cauchy-Schwarz’s inequality we obtain that:

\[ |\{ \phi, \psi \}| \leq (\{ \phi, \phi \})^{1/2}(\{ \psi, \psi \})^{1/2}. \]
Ahlfors' currents in higher dimension

In particular,

$$A(r', r)^2 \leq \left| \frac{1}{r - r'} \int_{V(r) \setminus V(r')} i\partial \tau \wedge \overline{\partial} \tau \wedge f^* \Omega \right| \times \left| \frac{1}{r - r'} \int_{V(r) \setminus V(r')} i\overline{f^* \theta} \wedge f^* \theta \wedge f^* \Omega \right|.$$ 

Now $i\overline{f^* \theta} \wedge f^* \theta \wedge f^* \Omega$ is equal to $f^*(i\overline{\theta} \wedge \theta \wedge \Omega)$ and $i\overline{\theta} \wedge \theta \wedge \Omega \leq K'(X)\omega^k$ (which means that $K'(X)\omega^k - i\overline{\theta} \wedge \theta \wedge \Omega$ is a (strongly) positive form). Here $K'(X)$ depends only on $(X, \omega)$ because $\|\theta\| \leq 1$ and $\|\Omega\| \leq 1$.

As a consequence, we have:

$$\left| \frac{1}{r - r'} \int_{V(r) \setminus V(r')} i\overline{f^* \theta} \wedge f^* \theta \wedge f^* \Omega \right| \leq K'(X) \left| \frac{1}{r - r'} \int_{V(r) \setminus V(r')} f^* \omega^k \right| = K'(X) \left( \frac{t_k(r) - t_k(r')}{r - r'} \right).$$

On the other hand, there exists a constant $K''(X)$ with $\Omega \leq K''(X)\omega^{k-1}$ (we use $\|\Omega\| \leq 1$). So, we have

$$\left| \frac{1}{r - r'} \int_{V(r) \setminus V(r')} i\partial \tau \wedge \overline{\partial} \tau \wedge f^* \Omega \right| \leq K''(X) \left( \frac{t_{k-1}(r) - t_{k-1}(r')}{r - r'} \right).$$

We obtain:

$$A(r', r)^2 \leq K(X) \left( \frac{t_{k-1}(r) - t_{k-1}(r')}{r - r'} \right) \left( \frac{t_k(r) - t_k(r')}{r - r'} \right). \quad (1.1)$$

Now, when $r' \to r$

$$A(r', r)^2 \to |\langle \partial f_*[V(r)], \theta \wedge \Omega \rangle|^2$$

because the function $s \mapsto \langle \partial f_*[V(s)], \theta \wedge \Omega \rangle = -\int_{V(s)} \partial f^*(\theta \wedge \Omega)$ is continuous on $[r - \varepsilon, r + \varepsilon]$ (see remark 1.2).

Finally, if we take $r' \to r$ in the inequality (1.1), we have:

$$|\langle \partial f_*[V(r)], \theta \wedge \Omega \rangle|^2 \leq K(X)t'_{k-1}(r)t'_k(r)$$

which gives the desired inequality.
2. Proof of Theorems 0.2 and 0.3

2.1. Proof of the first criterion

We begin with this lemma:

**Lemma 2.1.** — If \( f \) is nondegenerate and of finite-type then there exists a constant \( K > 0 \) such that:

\[
\forall r_2 > 0 \ \exists r \geq r_2 \text{ with } \text{volume}(f(V(2r))) \leq K \text{volume}(f(V(r))).
\]

**Proof.** — The hypothesis implies that there exist \( C_1, C_2, r_1 > 0 \) such that \( \text{volume}(f(V(r))) \leq C_1 r^{C_2} \) for \( r \geq r_1 \).

If the conclusion of the lemma fails then for all \( K > 0 \) there exists \( r_2 > 0 \) such that for all \( r \geq r_2 \) we have \( \text{volume}(f(V(2r))) \geq K \text{volume}(f(V(r))) \).

So, if we take \( K >> 2^{C_2} \) then we obtain (if \( l \) is large enough):

\[
C_1 (2^l r_2)^{C_2} \geq \text{volume}(f(V(2^l r_2))) \geq K^l \text{volume}(f(V(r_2))).
\]

As a consequence we have

\[
\text{volume}(f(V(r_2))) \leq C_1 r_2^{C_2} \left( \frac{2^{C_2}}{K} \right)^l
\]

which implies that \( \text{volume}(f(V(r_2))) = 0 \) when we take \( l \to \infty \). It contradicts the fact that \( f \) is nondegenerate. \( \square \)

By using this lemma, we can find a sequence \( R_n \to +\infty \) which satisfies

\[
\text{volume}(f(V(2R_n))) \leq K \text{volume}(f(V(R_n))).
\]

Theorem 0.1 gives now that:

\[
\int_{R_n}^{2R_n} \| \partial f_s[V(r)] \| dr \leq K(X) \int_{R_n}^{2R_n} \sqrt{t_k^l(r)} \sqrt{t_k^l(r)} dr.
\]

We give the following sense to the integrals: for example, if there is one point \( a_n \) of \( C \) in \( [R_n, 2R_n] \), we consider \( \int_{R_n}^{2R_n} = \lim_{\varepsilon \to 0} \int_{[R_n, a_n-\varepsilon] \cup [a_n+\varepsilon, 2R_n]} \). All the functions that we consider are non negative, so the limit exists in \([0, +\infty]\).
Ahlfors’ currents in higher dimension

Now, by using the Cauchy-Schwarz’s inequality, the last integral is smaller than
\[
K(X) \left( \int_{R_n}^{2R_n} t'_{k-1}(r) dr \right)^{1/2} \left( \int_{R_n}^{2R_n} t'_k(r) dr \right)^{1/2} \leq K(X) \sqrt{t_{k-1}(2R_n)/t_k(2R_n)}.
\]

For the last inequality it is important to use that \( t_{k-1} \) and \( t_k \) are continuous on \( r_0, +\infty \) (see Theorem 0.1).

It implies that there exists a sequence \( r_n \in [R_n, 2R_n] \) such that:
\[
\|\partial f[V(r_n)]\| \leq \frac{K(X)}{R_n} \sqrt{t_{k-1}(2R_n)/t_k(2R_n)},
\]
i.e.
\[
\frac{\|\partial f[V(r_n)]\|}{\text{volume}(f(V(r_n)))} \leq 2K(X) \sqrt{\frac{t_{k-1}(2R_n)}{(2R_n)^2 t_k(2R_n)}} \times \frac{t_k(2R_n)}{t_k(r_n)}
\]
because \( \text{volume}(f(V(r_n))) = t_k(r_n) \).

Now we have
\[
\frac{t_k(2R_n)}{t_k(r_n)} \leq \frac{t_k(2R_n)}{t_k(R_n)} \leq K
\]
and by using the hypothesis,
\[
\sqrt{\frac{t_{k-1}(2R_n)}{(2R_n)^2 t_k(2R_n)}} \to 0.
\]

So, we obtain that
\[
\frac{\|\partial f[V(r_n)]\|}{\text{volume}(f(V(r_n)))} \to 0.
\]

The current \( T_n := \frac{f[V(r_n)]}{\text{volume}(f(V(r_n)))} \) is positive with bidimension \((k, k)\) and mass equal to 1, so there exists a subsequence of \((T_n)\) which converges to a positive current \( T \) with bidimension \((k, k)\) and mass 1. Moreover,
\[
\|\partial T_n\| = \frac{\|\partial f[V(r_n)]\|}{\text{volume}(f(V(r_n)))} \to 0,
\]
so the limit current \( T \) is closed. This proves the first criterion.
2.2. Proof of the second criterion

Take $\varepsilon > 0$ and $L > 0$ such that

$$\limsup_{r \not\in C, \, r \to +\infty} \frac{t_{k-1}'(r)}{rt_k(r)^{1-\varepsilon}} \leq L.$$ 

Let $R_n$ be a sequence of positive reals which goes to $+\infty$. By using Theorem 0.1, we have (see the proof of the last criterion for the definition of the integrals):

$$\int_{r_0+1}^{R_n} \frac{\|\partial f_*[V(r)]\|^2}{t_{k-1}'(r)t_k(r)^{1+\varepsilon}} dr \leq K(X) \int_{r_0+1}^{R_n} \frac{t_k'(r)}{t_k(r)^{1+\varepsilon}} dr.$$ 

This last integral is smaller than $\frac{K(X)}{\varepsilon t_k(r_0+1)^{\varepsilon}} \leq K'(X, f)$ (here we use the fact that $\frac{1}{t_k(r)}$ is continuous on $]r_0, +\infty[)$.

So, we have

$$\int_{r_0+1}^{+\infty} \frac{1}{r} \left( \frac{r\|\partial f_*[V(r)]\|^2}{t_{k-1}'(r)t_k(r)^{1+\varepsilon}} \right) dr \leq K'(X, f),$$

and $\int_{r_0+1}^{+\infty} \frac{1}{r} dr = +\infty$ implies that there exists a sequence $r_n \to +\infty$ such that $r_n \not\in C$ and:

$$\varepsilon(n) := \frac{r_n\|\partial f_*[V(r_n)]\|^2}{t_{k-1}'(r_n)t_k(r_n)^{1+\varepsilon}} \to 0.$$ 

We obtain

$$\left( \frac{\|\partial f_*[V(r_n)]\|}{\text{volume}(f(V(r_n)))} \right)^2 = \frac{\varepsilon(n)}{r_n} \frac{t_{k-1}'(r_n)}{t_k(r_n)^{1-\varepsilon}} \leq (L + 1)\varepsilon(n),$$

by hypothesis (for $n$ large enough).

So,

$$\frac{\|\partial f_*[V(r_n)]\|}{\text{volume}(f(V(r_n)))} \to 0.$$ 

Now, by using exactly the same argument as in the proof of the previous criterion, we obtain that there exists a subsequence of $T_n := \frac{f_*[V(r_n)]}{\text{volume}(f(V(r_n)))}$ which converges to a closed positive current of bidimension $(k, k)$ and with mass equal to 1.
3. The special case $V = \mathbb{C}^k$

In this paragraph we consider the special case where $V = \mathbb{C}^k$.

Let $\beta$ be the standard Kähler form in $\mathbb{C}^k$. We want to transform our previous criteria by using $\beta$ instead of $i\partial \tau \wedge \overline{\partial} \tau$. More precisely, we consider:

$$a_k(r) = \int_{B(0,r)} f^* \omega^k$$

and

$$a_{k-1}(r) = \int_{B(0,r)} \beta \wedge f^* \omega^{k-1}.$$

Then we can prove a new formulation of our three Theorems:

**Theorem 3.1.** — The functions $a_k$ and $a_{k-1}$ are $C^1$ on $]0, +\infty[$ and for $r > 0$ we have

$$\|\partial f_*[B(0,r)]\|_2^2 \leq K(X)a'_{k-1}(r)a'_k(r).$$

Here $\|\cdot\|$ is the norm in the sense of currents and $K(X)$ is a constant which depends only on $(X, \omega)$.

**Proof.** — We apply Theorem 0.1 with $V = \mathbb{C}^k$ and $\tau = \|z\|^2$ (here we have $\mathcal{C} = \{0\}$) and then for $r > 0$:

$$\|\partial f_*[V(r^2)]\|_2^2 \leq K'(X)t'_{k-1}(r^2)t'_k(r^2).$$

Now, $a_k(r) = t_k(r^2)$, so $a_k$ is $C^1$ in $]0, +\infty[$ and

$$t'_k(r^2) = \frac{a'_k(r)}{2r}.$$

The function $a_{k-1}(r) = t(r^2)$ with $t(r) = \int_{V(r)} \beta \wedge f^* \omega^{k-1}$ so $a_{k-1}$ is $C^1$ in $]0, +\infty[$ (see proof of Lemma 1.1).

Moreover,

$$t_{k-1}(r^2) = \int_{V(r^2)} i\partial \tau \wedge \overline{\partial} \tau \wedge f^* \omega^{k-1} = \int_{B(0,r)} i\partial \tau \wedge \overline{\partial} \tau \wedge f^* \omega^{k-1},$$

and $i\partial \tau \wedge \overline{\partial} \tau = i \sum_{i,j} \overline{z}_iz_j dz_i \wedge d\overline{z}_j$.

On $B(0,r)$ this last form is smaller than $K(k)\beta r^2$. 

\[\text{Ahlfors' currents in higher dimension}\]
If we take $0 < r' < r$ then
\[
t_{k-1}(r^2) - t_{k-1}(r'^2) = \int_{B(0,r)\setminus B(0,r')} i\partial\tau \wedge \overline{\partial}\tau \wedge f^*\omega^{k-1}
\]
\[
\leq K(k)r^2 \int_{B(0,r)\setminus B(0,r')} \beta \wedge f^*\omega^{k-1}.
\]

If we divide by $r - r'$ and take the limit $r' \to r$, we obtain:
\[
2 rt'_{k-1}(r^2) \leq K(k)r^2 a'_{k-1}(r).
\]

Finally, we have:
\[
\|\partial f_*[B(0,r)]\|_2 = \|\partial f_*[V(r^2)]\|_2 \leq K'(X)t'_{k-1}(r^2)t'_k(r^2) \leq K(X)a'_{k-1}(r)a'_k(r),
\]
with $K(X) = K(k)K'(X)$ (we recall that the dimension of $X$ is larger than or equal to $k$). This is the inequality that we were looking for. □

Now if we replace in the proof of Theorems 0.2 and 0.3 the function $t_{k-1}$ by $a_{k-1}$, the function $t_k$ by $a_k$ and $V(r)$ by $B(0, r)$ then we obtain the two following criteria:

**Theorem 3.2.** — We suppose that $f$ is nondegenerate and with finite-type (i.e. there exist $C_1, C_2, r_1 > 0$ such that $\text{volume}(f(B(0, r))) \leq C_1 r^{C_2}$ for $r \geq r_1$).

If
\[
\limsup_{r \to +\infty} \frac{a_{k-1}(r)}{r^2 a_k(r)} = 0
\]
then there exists a sequence $r_n$ which goes to infinity such that $\frac{f_*[B(0,r_n)]}{\text{volume}(f(B(0,r_n)))}$ converges to a closed positive current with bidimension $(k,k)$ and mass equal to 1.

**Theorem 3.3.** — If $f$ is nondegenerate and if there exist $\varepsilon > 0$ and $L > 0$ such that:
\[
\limsup_{r \to +\infty} \frac{a'_{k-1}(r)}{r a_k(r)^{1-\varepsilon}} \leq L
\]
then there exists a sequence $r_n$ which goes to infinity such that $\frac{f_*[B(0,r_n)]}{\text{volume}(f(B(0,r_n)))}$ converges to a closed positive current with bidimension $(k,k)$ and mass equal to 1.
Ahlfors’ currents in higher dimension

Notice that when $k = 1$ then $a_{k-1}(r) = \pi r^2$ and therefore, in this context, the hypothesis of this criterion is always fulfilled if $f$ is nondegenerate.

Bibliography