MATS ANDERSSON

Uniqueness and factorization of Coleff-Herrera currents


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1. Introduction

Let $X$ be an $n$-dimensional complex manifold and let $Z$ be an analytic variety of pure codimension $p$. The sheaf of Coleff-Herrera currents $\mathcal{CH}_Z$ consists of all $\bar{\partial}$-closed $(\ast, p)$-currents $\mu$ with support on $Z$ such that...
\( \bar{\psi}\mu = 0 \) for each \( \psi \) vanishing on \( Z \), and which in addition fulfill the so-called standard extension property, SEP, see below. Locally, any \( \mu \in \mathcal{CH}_Z \) can be realized as the result of an application of a meromorphic differential operator on the current of integration \([Z]\) (combined with contractions with holomorphic vector fields), see, e.g., [4] and [5].

The model case of a Coleff-Herrera current is the Coleff-Herrera product associated to a complete intersection \( f = (f_1, \ldots, f_p) \),

\[
\mu^f = [\bar{\partial}\frac{1}{f_p} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_1}],
\]

(1.1)

introduced by Coleff and Herrera in [6]. Equivalent definitions are given in [9] and [10]; see also [12]. It was proved in [7] and [9] that the annihilator of \( \mu^f \) is equal to the ideal \( \mathcal{J}(f) \) generated by \( f \). Notice that formally (1.1) is just the pullback under \( f \) of the product \( \mu^w = \bar{\partial}(1/w_1) \wedge \ldots \wedge \bar{\partial}(1/w_p) \). One can also express \( \mu^w \) as \( \bar{\partial} \) of the Bochner-Martinelli form

\[
B(w) = \sum_j (-1)^j \bar{w}_j d\bar{w}_1 \wedge \ldots \wedge d\bar{w}_{j-1} \wedge d\bar{w}_{j+1} \wedge \ldots \wedge d\bar{w}_p / |w|^{2p}.
\]

In [11], \( f^* B \) is defined as a principal value current, and it is proved that \( \mu^f_{BM} = \bar{\partial} f^* B \) is indeed equal to \( \mu^f \). However the proof is quite involved. An alternative but still quite technical proof appeared in [1]. In this paper we prove a uniqueness result which states that any Coleff-Herrera current that is cohomologous to 1 with respect to the operator \( \delta_f - \bar{\partial} \) (see Section 3 for definitions) must be equal to \( \mu^f \). In particular this implies that \( \mu^f = \mu^f_{BM} \).

It is well-known that any Coleff-Herrera current can be written \( \alpha \wedge \mu^f \), where \( \alpha \) is a holomorphic \((*,0)\)-form and \( \mu^f \) is a Coleff-Herrera product for a complete intersection \( f \). However, unless \( Z \) is a complete intersection itself the support of \( \mu^f \) is larger than \( Z \). Using the uniqueness result we can prove

**Theorem 1.1.** — For any \( \mu \in \mathcal{CH}_Z \) (locally) there are residue currents \( R_I \) with support on \( Z \) and holomorphic \((*,0)\)-forms \( \alpha_I \) such that

\[
\mu = \sum_{|I| = p} R_I \wedge \alpha_I.
\]

(1.2)

Here \( R_I \) are currents of Bochner-Martinelli type from [11] associated with a not necessarily complete intersection. In particular, it follows that the Lelong current \([Z]\) admits a factorization (1.2).
The SEP goes back to Barlet, [3]. We will use the following definition: Given any holomorphic \( h \) that does not vanish identically on any irreducible component of \( Z \), the function \( |h|^{2\lambda} \mu \), a priori defined only for \( \text{Re}\lambda > \epsilon \), has a current-valued analytic extension to \( \text{Re}\lambda > -\epsilon \), and the value at \( \lambda = 0 \) coincides with \( \mu \). The reason for this choice is merely practical; for the equivalence to the classical definition, see Section 5. Now, if \( \mu \in \mathcal{CH}_Z \) has support on \( Z \cap \{h = 0\} \), then \( |h|^{2\lambda} \mu \) must vanish if \( \text{Re}\lambda \) is large enough, and by the uniqueness of analytic continuation thus \( \mu = 0 \). In particular, \( \mu = 0 \) identically if \( \mu = 0 \) on \( Z_{\text{reg}} \).

By the uniqueness result we obtain simple proofs of the equivalence of various definitions of the SEP (Section 5) as well as the equivalence of various conditions for the vanishing of a Coleff-Herrera current (Section 6).

2. The Coleff-Herrera product

Let \( f_1, \ldots, f_p \) define a complete intersection in \( X \), i.e., \( \text{codim} Z^f = p \), where \( Z^f = \{f = 0\} \). Notice that (1.1) is elementarily defined if each \( f_j \) is a power of a coordinate function. The general definition relies on the possibility to resolve singularities: By Hironaka’s theorem we can locally find a resolution \( \pi: \tilde{U} \to U \) such that locally in \( \tilde{U} \), each \( \pi^* f_j \) is a monomial times a non-vanishing factor. It turns out that locally \( \mu^f \) is a sum of terms

\[
\sum_{\ell} \pi_* \tau_\ell \tag{2.1}
\]

where each \( \tau_\ell \) is of the form

\[
\tau_\ell = \bar{\partial} \frac{1}{t_1^{a_1}} \wedge \ldots \wedge \bar{\partial} \frac{1}{t_p^{a_p}} \wedge \frac{\alpha}{t_{p+1}^{a_{p+1}} \ldots t_r^{a_r}},
\]

\( t \) is a suitable local coordinate system in \( \tilde{U} \), and \( \alpha \) is a smooth function with compact support. This representation turns out to be very useful; though not explicitly stated, it follows from the definition in [6] as well as from any other reasonable definition of \( \mu^f \) by taking limits in the resolution manifold; see, e.g., [2] for a further discussion.

It is well-known that \( \mu^f \) is in \( \mathcal{CH}_{Z^f} \) but for further reference we sketch a proof. It follows immediately from the definition that \( \mu^f \) is a \( \bar{\partial} \)-closed \((0,p)\)-current with support on \( Z^f \). Given any holomorphic function \( \psi \) we may choose the resolution so that also \( \pi^* \psi \) is a monomial. Notice that each \( |\pi^* \psi|^{2\lambda} \tau_\ell \) has an analytic continuation to \( \lambda = 0 \) and that the value at 0 is equal to \( \tau_\ell \) if none of \( t_1, \ldots, t_p \) is a factor in \( \pi^* \psi \) and zero otherwise.
According to this let us subdivide the set of $\tau_\ell$ into two groups $\tau'_\ell$ and $\tau''_\ell$. Notice that $|\psi|^{2\lambda} \mu^f = \sum_\ell \pi_* (|\pi^* \psi|^{2\lambda} \tau_\ell)$ admits an analytic continuation and that the value at $\lambda = 0$ is $\sum_\ell \pi_* \tau''_\ell$. If $\psi = 0$ on $Z^f$, then $0 = |\psi|^{2\lambda} \mu^f$, and hence $\mu^f = \sum_\ell \pi_* \tau'_\ell$; it now follows that $\bar{\psi} \mu^f = d\psi \wedge \mu^f = 0$. If $h$ is holomorphic and the zero set of $h$ intersects $Z^f$ properly, then $T = \mu^f - |h|^{2\lambda} \mu^f |_{\lambda=0}$ is a current of the type (2.1) with support on $Y = Z^f \cap \{ h = 0 \}$ that has codimension $p + 1$. For the same reason as above, $d\bar{\psi} \wedge T = 0$ for each holomorphic $\psi$ that vanishes on $Y$ and by a standard argument it now follows that $T = 0$ for degree reasons. Thus $\mu^f$ has the SEP and so $\mu^f \in \mathcal{CH}_{Z^f}$. This proof is inspired by a forthcoming joint paper, [2], with Elizabeth Wulcan.

3. The uniqueness result

Let $f = (f_1, \ldots, f_m)$ be a holomorphic tuple on some complex manifold $X$. It is practical to introduce a (trivial) vector bundle $E \to X$ with global frame $e_1, \ldots, e_m$ and consider $f = \sum f_je_j^*$ as a section of the dual bundle $E^*$, where $e_j^*$ is the dual frame. Then $\mathring{f}$ induces a mapping $\delta_f$, interior multiplication with $f$, on the exterior algebra $\Lambda E$. Let $C_{0,k}(\Lambda^\ell E)$ be the sheaf of $(0, k)$-currents with values in $\Lambda^\ell E$, considered as as sections of the bundle $\Lambda(E \oplus T^*(X))$; thus a section of $C_{0,k}(\Lambda^\ell E)$ is given by an expression $v = \sum'_{|I| = \ell} f_I \wedge e_I$ where $f_I$ are $(0, k)$-currents and $d\bar{z}_j \wedge e_k = -e_k \wedge d\bar{z}_j$ etc. Notice that both $\partial$ and $\delta_f$ act as anti-derivations on these spaces, i.e., $\bar{\partial} (f \wedge g) = \bar{\partial} f \wedge g + (-1)^{\text{deg} f} f \wedge \bar{\partial} g$, if at least one of $f$ and $g$ is smooth, and similarly for $\delta_f$. It is straightforward to check that $\delta_f \bar{\partial} = -\bar{\partial} \delta_f$. Therefore, if $\mathcal{L}^k = \bigoplus_j C_{0,j+k}(\Lambda^j E)$ and $\nabla_f = \delta_f - \bar{\partial}$, then $\nabla_f : \mathcal{L}^k \to \mathcal{L}^{k+1}$, and $\nabla_f^2 = 0$. For example, $v \in \mathcal{L}^{-1}$ is of the form $v = v_1 + \cdots + v_m$, where $v_k$ is a $(0, k - 1)$-current with values in $\Lambda^k E$. Also for a general current the subscript will denote degree in $\Lambda E$.

Example 3.1 (The Coleff-Herrera product). — Let $f = (f_1, \ldots, f_m)$ be a complete intersection in $X$. The current

$$V = \left[ \frac{1}{f_1} \right] e_1 + \left[ \frac{1}{f_2} \bar{\partial} \frac{1}{f_1} \right] \wedge e_1 \wedge e_2 + \left[ \frac{1}{f_3} \bar{\partial} \frac{1}{f_2} \wedge \bar{\partial} \frac{1}{f_1} \right] \wedge e_1 \wedge e_2 \wedge e_3 + \cdots$$

(3.1)

is in $\mathcal{L}^{-1}$ and solves $\nabla_f V = 1 - \mu^f \wedge e$, where $\mu^f$ is the Coleff-Herrera product and $e = e_1 \wedge \cdots \wedge e_m$. For definition of the coefficients of $V$ and the computational rules used here, see [9]; one can obtain a simple proof of these rules by arguing as in Section 2, see [2].
Example 3.2 (Residues of Bochner-Martinelli type). — Introduce a Hermitian metric on $E$ and let $\sigma$ be the section of $E$ over $X \setminus Z^f$ with minimal pointwise norm such that $\delta_f \sigma = f \cdot \sigma = 1$. Then $\bar{\partial} \sigma$ has even total degree (it is in $L^0$) and we let $(\bar{\partial} \sigma)^2 = \bar{\partial} \sigma \wedge \bar{\partial} \sigma$, etc. Now

$$u = \sigma + \sigma \wedge \bar{\partial} \sigma + \sigma \wedge (\bar{\partial} \sigma)^2 + \sigma \wedge (\bar{\partial} \sigma)^3 \cdots$$ (3.2)

is smooth outside $Z^f$ and $\nabla_f u = 1$ there; in fact, since $\delta_f (\bar{\partial} \sigma) = -\bar{\partial} \delta_f \sigma = -\bar{\partial} 1 = 0$ we have that $\delta_f (\sigma \wedge (\bar{\partial} \sigma)^k) = (\bar{\partial} \sigma)^k = \bar{\partial} (\sigma \wedge (\bar{\partial} \sigma)^{k-1})$, so $\nabla_f u = (\delta_f - \bar{\partial}) u$ becomes a telescoping sum. (A more elegant way is to notice that (3.2) is equal to $\sigma / \nabla_f \sigma$; then $\nabla_f u = 1$ follows by Leibniz’ rule since $\nabla^2 f = 0$, cf. [1]).

It turns out, see [1], that $u$ has a natural current extension $U$ across $Z^f$. For instance it can be defined as the value at $\lambda = 0$ of the analytic continuation of $|f|^{2\lambda} u$ from $\text{Re} \lambda >> 0$ (the existence of the analytic continuation is of course nontrivial and requires a resolution of singularities). If $p = \text{codim} Z^f$, then $\nabla_f U = 1 - R^f$, where

$$R^f = R^f_p + \cdots + R^f_m,$$

$R^f$ is the value at $\lambda = 0$ of $\bar{\partial} |f|^{2\lambda} \wedge u$ and $R^f_k = \sigma \wedge (\bar{\partial} \sigma)^{k-1}_{|\lambda=0}$. Moreover, these currents have representations like (2.1) so if $\xi \in \mathcal{O}(\Lambda^{m-p} E)$ and $\xi \wedge R^f_p$ is $\bar{\partial}$-closed, then it is in $\mathcal{CH}^f_Z$ by the arguments given in Section 2. Notice that

$$R^f_k = \sum'_{|I|=k} R^f_{I_1} \wedge e_{I_1} \wedge \ldots \wedge e_{I_k}.$$ (3.3)

If we choose the trivial metric, the coefficients $R^f_I$ are precisely the currents introduced in [11]. In particular, if $f$ is a complete intersection, i.e. $m = p$, then, see [1], $R^f_{1, \ldots, p} = \mu^f_{BM} \wedge e$.

**Theorem 3.3** (Uniqueness for Coleff-Herrera currents). — Assume that $Z^f$ has pure codimension $p$. If $\tau \in \mathcal{CH}_{Z^f}$ and there is a solution $V \in L^{p-m-1}$ to $\nabla_f V = \tau \wedge e$, then $\tau = 0$.

**Remark 3.4.** — If $Z^f$ does not have pure codimension, the theorem still holds (with the same proof) with $\mathcal{CH}_{Z^f}$ replaced by $\mathcal{CH}_{Z'}$, where $Z'$ is the irreducible components of $Z^f$ of maximal dimension.

In view of Examples 3.1 and 3.2 we get

**Corollary 3.5.** — Assume that $f$ is a complete intersection. If $\mu \in \mathcal{CH}_{Z'}$ and there is a current $U \in L^{-1}$ such that $\nabla_f U = 1 - \mu \wedge e$, then $\mu$ is equal to the Coleff-Herrera product $\mu^f$. In particular, $\mu^f_{BM} = \mu^f$. 

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The proof of Theorem 3.3 relies on the following lemma, which is probably known. However, for the reader’s convenience we include a proof.

**Lemma 3.6.** — If \( \mu \) is in \( \mathcal{CH}_Z \) and for each neighborhood \( \omega \) of \( Z \) there is a current \( V \) with support in \( \omega \) such that \( \bar{\partial}V = \mu \), then \( \mu = 0 \).

**Proof.** — Locally on \( Z_{\text{reg}} \) we can choose coordinates \( (z, w) \) such that \( Z = \{ w = 0 \} \). We claim that there is a natural number \( M \) such that

\[
\mu = \sum_{|\alpha| \leq M-p} a_\alpha(z) \frac{1}{w_1^{\alpha_1+1}} \wedge \ldots \wedge \frac{1}{w_p^{\alpha_p+1}},
\]

where \( a_\alpha \) are the push-forwards of \( \mu \wedge w^\alpha dw/(2\pi i)^p \) under the projection \( (z, w) \mapsto z \). In fact, since \( \bar{\partial}w_j \mu = 0 \) and \( \bar{\partial}\mu = 0 \) it follows that \( d\bar{\partial}w_j \wedge \mu = 0 \), \( j = 1, \ldots, p \), and hence \( \mu = \mu_0 \wedge d\bar{\partial}z_1 \wedge \ldots \wedge d\bar{\partial}z_p \). Therefore it is enough to check (3.4) for test forms of the form \( \xi(z, w)dw \wedge d\bar{\partial}z \wedge dz \). Since \( \bar{\partial}w_j \mu = 0 \) we have by a Taylor expansion in \( w \) (the sum is finite since \( \mu \) has finite order) that

\[
\int_{z,w} \mu \wedge \xi dw \wedge d\bar{\partial}z \wedge dz = \sum_\alpha \int_{z,w} \mu \wedge \frac{\partial^\alpha \xi}{\partial w^\alpha}(z,0) \frac{w^\alpha}{\alpha!} dw \wedge d\bar{\partial}z \wedge dz
\]

\[
= \sum_\alpha \int_z a_\alpha(z) \frac{\partial^\alpha \xi}{\partial w^\alpha}(z,0) dw \wedge d\bar{\partial}z \wedge dz (2\pi i)^p \]

\[
= \sum_\alpha \int_z a_\alpha(z) \int_w \frac{1}{w_1^{\beta_1+1}} \wedge \ldots \wedge \frac{1}{w_p^{\beta_p+1}} \wedge \xi(z,w) dw \wedge d\bar{\partial}z \wedge dz.
\]

Since \( \mu \) is \( \bar{\partial} \)-closed it follows that \( a_\alpha \) are holomorphic. Notice that

\[
\bar{\partial} \frac{1}{w_1^{\beta_1}} \wedge \ldots \wedge \bar{\partial} \frac{1}{w_p^{\beta_p}} \wedge dw_1^{\beta_1} \wedge \ldots \wedge dw_p^{\beta_p} / (2\pi i)^p = \beta_1 \cdots \beta_p [w = 0],
\]

where \( [w = 0] \) denote the current of integration over \( Z_{\text{reg}} \) (locally). Now assume that \( \bar{\partial}\gamma = \mu \) and \( \gamma \) has support close to \( Z \). We have, for \( |\beta| = M \),

\[
\bar{\partial}(\gamma \wedge dw^\beta) = (2\pi i)^p a_{\beta-1}(z) \beta_1 \cdots \beta_p [w = 0].
\]

If \( \nu \) is the component of \( \gamma \wedge dw^\beta \) of bidegree \( (p, p-1) \) in \( w \), thus

\[
d_w \nu = \bar{\partial}_w \nu = (2\pi i)^p a_{\beta-1} \beta_1 \cdots \beta_p [w = 0].
\]

Integrating with respect to \( w \) we get that \( a_{\beta-1}(z) = 0 \). By finite induction we can conclude that \( \mu = 0 \) locally on \( Z_{\text{reg}} \). Thus \( \mu \) vanishes on \( Z_{\text{reg}} \) and by the SEP it follows that \( \mu = 0 \). □
Proof. — [Proof of Theorem 3.3] Let $\omega$ be any neighborhood of $Z$ and take a cutoff function $\chi$ that is 1 in a neighborhood of $Z$ and with support in $\omega$. Let $u$ be any smooth solution to $\nabla_f u = 1$ in $X \setminus Z^f$, cf. Example 3.2. Then $g = \chi - \partial\chi \wedge u$ is a smooth form in $L^0(\omega)$ and $\nabla_f g = 0$. Moreover, the scalar term $g_0$ is 1 in a neighborhood of $Z$. Therefore,

$$\nabla_f [g \wedge V] = g \wedge \tau \wedge e = g_0 \tau \wedge e = \tau \wedge e,$$

and hence the current coefficient $W$ of the top degree component of $g \wedge V$ is a solution to $\partial W = \tau$ with support in $\omega$. In view of Lemma 3.6 we have that $\tau = 0$. □

4. The factorization

The double sheaf complex $C_{0,k}(\Lambda^\ell E)$ is exact in the $k$ direction except at $k = 0$, where we have the cohomology $O(\Lambda^\ell E)$. By a standard argument there are natural isomorphisms

$$\text{Ker} \, \delta_f O(\Lambda^\ell E)/\delta_f O(\Lambda^{\ell+1}) \simeq \text{Ker} \, \nabla_f L^{-\ell}/\nabla_f L^{-\ell-1}. \quad (4.1)$$

When $\ell = 0$ the left hand side is $O/J(f)$, where $J(f)$ is the ideal sheaf generated by $f$. We have the following factorization result.

**Theorem 4.1.** — Assume that $Z^f$ has pure codimension $p$ and let $\mu \in CH_{Z^f}$ be $(0,p)$ and such that $J(f)\mu = 0$. Then there is locally $\xi \in O(\Lambda^{m-p} E)$ such that

$$\mu \wedge e = R_p \wedge \xi. \quad (4.2)$$

**Proof.** — Since $\nabla_f (\mu \wedge e) = 0$, by (4.1) there is $\xi \in O(\Lambda^{m-p} E)$ such that $\nabla_f V = \xi - \mu \wedge e$. On the other hand, if $U$ is the current from Example 3.2, then $\nabla_f (U \wedge \xi) = \xi - R_p \wedge \xi = \xi - R_p \wedge \xi$. Now (4.2) follows from Theorem 3.3. □

**Proof.** — [Proof of Theorem 1.1] With no loss of generality we may assume that $\mu$ has bidegree $(0,p)$. Let $g = (g_1, \ldots, g_m)$ be a tuple such that $Z^g = Z$. If $f_j = g_j^M$ and $M$ is large enough, then $J(f)\mu = 0$ and hence by Theorem 4.1 there is a form

$$\xi = \sum_{|J|=m-p} \xi_J \wedge e_J$$

such that (4.2) holds. Then, cf. (3.3), (1.2) holds if $\alpha_I = \pm \xi_{I^c}$, where $I^c = \{1, \ldots, m\} \setminus I$. □
Example 4.2. — Let $[Z]$ be any variety of pure codimension and choose $f$ such that $Z = Z^f$. It is not hard to prove that (each term of) the Lelong current $[Z]$ is in $CH_Z$, and hence there is a holomorphic form $\xi$ such that $R_p^f \wedge \xi = [Z] \wedge e$. (In fact, one can notice that the proof of Lemma 3.6 works for $\mu = [Z]$ just as well, and then one can obtain fakto for $[Z]$ in the same way as for $\mu \in CH_Z$. A posteriori it follows that indeed $[Z]$ is in $CH_Z$.) There are natural ways to regularize the current $R_p^f$, see, e.g. [12], and thus we get natural regularizations of $[Z]$.

Next we recall the duality principle, [7], [8]: If $f$ is a complete intersection, then

$$\text{ann } \mu^f = \mathcal{J}(f).$$

In fact, if $\phi \in \text{ann } \mu$, then $\nabla_f U \phi = \phi - \phi \mu \wedge e = \phi$ and hence $\phi \in \mathcal{J}(f)$ by (4.1). Conversely, if $\phi \in \mathcal{J}(f)$, then there is a holomorphic $\psi$ such that $\phi = \delta_f \psi = \nabla_f \psi$ and hence $\phi \mu = \nabla_f \psi \wedge \mu = \nabla(\psi \wedge \mu) = 0$.

Notice that $\text{Hom}_{\mathcal{O}}(\mathcal{O}/\mathcal{J}(f), CH_{Z^f}(\Lambda^p E))$ is the sheaf of currents $\mu \wedge e$ with $\mu \in CH_{Z^f}$ that are annihilated by $\mathcal{J}(f)$. From (4.3) and Theorem 4.1 we now get

**Theorem 4.3.** — If $f$ is a complete intersection, then the sheaf mapping

$$\mathcal{O}/\mathcal{J}(f) \to \text{Hom}_{\mathcal{O}}(\mathcal{O}/\mathcal{J}(f), CH_Z(\Lambda^p E)), \quad \phi \mapsto \phi \mu^f \wedge e,$$

is an isomorphism.

5. The standard extension property

Given the other conditions in the definition of $CH_Z$ the SEP is automatically fulfilled on $Z_{\text{reg}}$; this is easily seen, e.g. as in the proof of Lemma 3.6 (notice that the SEP is a local property), so the interesting case is when the zero set $Y$ of $h$ contains the singular locus of $Z$. Classically, cf. [3], [4], and [5], the SEP is expressed as

$$\lim_{\epsilon \to 0} \chi(|h|/\epsilon)\mu = \mu,$$

where $Y \supset Z_{\text{sing}}$ and $h$ is not vanishing identically on any irreducible component of $Z$. Here $\chi(t)$ can be either the characteristic function for the interval $[1, \infty)$ or some smooth approximand.

**Proposition 5.1.** — Let $\chi$ be a fixed function as above. The class of $\bar{\partial}$-closed $(0, p)$-currents $\mu$ with support on $Z$ that are annihilated by $I_Z$ and satisfy (5.1) coincides with our class $CH_Z$. 

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If \( \chi \) is not smooth the existence of the currents \( \chi(|h|/\epsilon)\mu \) in a reasonable sense for small \( \epsilon > 0 \) is part of the statement.

**Proof.** — [Sketch of proof] Let \( f \) be a tuple such that \( Z = Z^f \). We first show that \( R^f_p \) satisfies (5.1). From the arguments in Section 2, cf. Example 3.2, we know that \( R^f_p \) has a representation (2.1) such that \( \pi^* h \) is a pure monomial (since the possible nonvanishing factor can be incorporated in one of the coordinates) and none of the factors in \( \pi^* h \) occurs among the residue factors in \( \tau^\ell \). Therefore, the existence of the product in (5.1) and the equality follow from the simple observation that

\[
\int_{s_1, \ldots, s_\mu} \chi(|s_1^{e_1} \cdots s_\mu^{e_\mu}|/\epsilon) \frac{\psi(s)}{s_1^{\gamma_1} \cdots s_\mu^{\gamma_\mu}} \rightarrow \int_{s_1, \ldots, s_\mu} \frac{\psi(s)}{s_1^{\gamma_1} \cdots s_\mu^{\gamma_\mu}}
\]

(5.2)

for test forms \( \psi \), where the right hand side is a tensor product of one-variable principal value integrals acting on \( \psi \). Let temporarily \( \mathcal{CH}^cl_Z \) denote the class of currents defined in the proposition. Since each \( \mu \in \mathcal{CH}_Z \) admits the representation (4.2) it follows that \( \mu \in \mathcal{CH}^cl_Z \). On the other hand, Lemma 3.6 and therefore Theorem 3.3 and (4.2) hold for \( \mathcal{CH}^cl_Z \) as well (with the same proofs), and thus we get the other inclusion. \( \square \)

6. Vanishing of Coleff-Herrera currents

We conclude with some equivalent condition for the vanishing of a Coleff-Herrera current. This result is proved by the ideas above, it should be well-known, but we have not seen it in this way in the literature.

**Theorem 6.1.** — Assume that \( X \) is Stein and that the subvariety \( Z \subset X \) has pure codimension \( p \). If \( \mu \in \mathcal{CH}_Z(X) \) and \( \bar{\partial}V = \mu \) in \( X \), then the following are equivalent:

(i) \( \mu = 0 \).

(ii) For all \( \psi \in \mathcal{D}_{n,n-p}(X) \) such that \( \bar{\partial}\psi = 0 \) in some neighborhood of \( Z \) we have that

\[
\int V \wedge \bar{\partial}\psi = 0.
\]

(iii) There is a solution to \( \bar{\partial}w = V \) in \( X \setminus Z \).

(iv) For each neighborhood \( \omega \) of \( Z \) there is a solution to \( \bar{\partial}w = V \) in \( X \setminus \omega \).

**Proof.** — It is easy to check that (i) implies all the other conditions. Assume that (ii) holds. Locally on \( Z_{reg} = \{w = 0\} \) we have (3.4), and by
choosing $\xi(z, w) = \psi(z) \chi(w) dw^\beta \wedge dz \wedge d\bar{z}$ for a suitable cutoff function $\chi$ and test functions $\psi$, we can conclude from (ii) that $a_\beta = 0$ if $|\beta| = M$. By finite induction it follows that $\mu = 0$ there. Hence $\mu = 0$ globally by the SEP. Clearly (iii) implies (iv). Finally, assume that (iv) holds. Given $\omega \supset Z$ choose $\omega' \subset \omega$ and a solution to $\bar{\partial}w = V$ in $X \setminus \omega'$. If we extend $w$ arbitrarily across $\omega'$ the form $U = V - \bar{\partial}w$ is a solution to $\bar{\partial}U = \mu$ with support in $\omega$. In view of Lemma 3.6 thus $\mu = 0$. □

Notice that $V$ defines a Dolbeault cohomology class $\omega^\mu$ in $X \setminus Z$ that only depends on $\mu$, and that conditions (ii)-(iv) are statements about this class. For an interesting application, fix a current $\mu \in \mathcal{CH}_Z$. Then the theorem gives several equivalent ways to express that a given $\phi \in \mathcal{O}$ belongs to the annihilator ideal of $\mu$. In the case when $\mu = \mu^f$ for a complete intersection $f$, one gets back the equivalent formulations of the duality theorem from [7] and [9].

Remark 6.2. — If $\mu$ is an arbitrary $(0, p)$-current with support on $Z$ and $\bar{\partial}V = \mu$ we get an analogous theorem if condition (i) is replaced by: $\mu = \bar{\partial}\gamma$ for some $\gamma$ with support on $Z$. This follows from the Dickenstein-Sessa decomposition $\mu = \mu_{\mathcal{CH}} + \bar{\partial}\gamma$, where $\mu_{\mathcal{CH}}$ is in $\mathcal{CH}_Z$. See [7] for the case $Z$ is a complete intersection and [4] for the general case.

Bibliography

Uniqueness and factorization of Coleff-Herrera currents

