OULD AHMED IZID BIH ISSELKOU

*The Lane-Emden Function and Nonlinear Eigenvalues Problems*


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The Lane-Emden Function and Nonlinear Eigenvalues Problems

ISSELKOU OULD AHMED IZID BIH\(^{(1)}\)

RéSUMÉ. — Nous considérons un problème aux valeurs propres, semi-linéaire elliptique, sur une boule de \(\mathbb{R}^n\) et montrons que ces valeurs et fonctions propres peuvent s'obtenir à partir de la fonction de Lane-Emden.

ABSTRACT. — We consider a semilinear elliptic eigenvalues problem on a ball of \(\mathbb{R}^n\) and show that all the eigenfunctions and eigenvalues, can be obtained from the Lane-Emden function.

1. Introduction

We consider the problem

\[
(P_\lambda^\alpha) \begin{cases} 
\Delta u + \lambda (1 + u)^\alpha = 0, & \text{in } B_1 \\
u > 0, & \text{in } B_1 \\
u = 0, & \text{on } \partial B_1
\end{cases}
\]

where \(B_1\) is the unit ball of \(\mathbb{R}^n\), \(n \geq 3\), \(\lambda > 0\) and \(\alpha > 1\).

This problem arises in many physical models like the nonlinear heat generation and the theory of gravitational equilibrium of polytropic stars (cf. [2] and [11]). It is well known (cf. [2], [10], [12]) that there exists a critical constant \(\lambda^*(\alpha)\), such that \((P_\lambda^\alpha)\) admits, at least, one solution if \(0 < \lambda < \lambda^*(\alpha)\) and no solution if \(\lambda > \lambda^*(\alpha)\). We deal here with these critical constants and the corresponding eigenfunctions.

\(^{(1)}\) Faculté des Sciences et Techniques, B.P. 5026 Nouakchott, Mauritanie.
isselkou@univ-nkc.mr

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Let \( \phi \) be the Lane-Emden function (cf. [1], [5], [6], [15]) in the \( n \)-dimensional space and \( r_0 \) the first "zero" of \( \phi \), we show that

\[
\lambda^*(\alpha) = \max_{r \in [0, r_0]} r^2 \phi^{\alpha-1}(r).
\]

We use this formula to compute \( \lambda^*(\alpha) \), when \( \alpha \) is the Critical Sobolev Exponent. We also extend, to the subcritical case, an estimate of \( \lambda^*(\alpha) \) given in [10] and show qualitative properties of the eigenfunctions.

In the Appendix, we show how to approximate \( \phi \), so one can use numerical approaches (Maple or Matlab) to get estimates of \( \lambda^*(\alpha) \).

2. Scalings of the Lane-Emden function as solutions

When \( 0 < \lambda \leq \lambda^*(\alpha) \), it is known that any regular solution of \((P_{\lambda}^\alpha)\) is radial and the minimal one is stable and analytical (cf. [8], [12]).

**Proposition 2.1.** — Let \( u \) be a regular solution of \((P_{\lambda}^\alpha)\), then

\[
u(r) = (1 + u(0))\phi \left( \sqrt{\lambda} (1 + u(0))^{\frac{\alpha-1}{2}} r \right) - 1, \quad \forall \ r \in [0, 1]
\]

where \( \phi \) is the Lane-Emden function, in the \( n \)-dimensional space.

**Proof.** — The Lane-Emden function (cf. [1], [5], [6], [15]) is the solution of

\[
\begin{align*}
(L - E) \left\{ & \begin{array}{l}
\phi''(r) + \frac{n-1}{r} \phi'(r) + \phi(r) |\phi(r)|^{\alpha-1} = 0,
\phi(0) = 1,
\phi'(0) = 0.
\end{array}
\end{align*}
\]

The proof of the proposition is quite immediate.

3. The Subcritical Case

Let us consider the problem \((P_{\lambda}^\alpha)\), with \( 1 < \alpha < \frac{n+2}{n-2} \). Let \( \phi \) be the Lane-Emden function.

**Proposition 3.1.** — There exists \( r_0 > 0 \), such that \( \phi(r_0) = 0, \phi(r) > 0 \), \( \forall r \in [0, r_0] \) and

\[
\lambda^*(\alpha) = \max_{\rho \in [0, r_0]} \rho^2 \phi^{\alpha-1}(\rho).
\]

We also have

\[
\lambda^*(\alpha) \geq \frac{2}{(\alpha - 1)^2} \left( \alpha(n - 2) - n \right), \quad \text{if} \quad \frac{n}{n-2} < \alpha < \frac{n+2}{n-2}.
\]
Proof. — As \( \phi(0) > 0 \), we infer that \( \phi > 0 \), on a maximal interval \([0, r_0] \). The problem
\[
\begin{align*}
\left\{ \begin{array}{ll}
\Delta u + u^\alpha = 0, & \text{in } \mathbb{R}^n \\
u > 0, & \text{in } \mathbb{R}^n
\end{array} \right.
\end{align*}
\]
does not admit a solution (cf.\([4]\)), so we infer that \( r_0 < \infty \) and \( \phi(r_0) = 0 \).

Let us put
\[
\psi_\rho(r) = \phi(\rho r) - \phi(\rho), \quad \forall r \in [0, 1],
\]
with \( 0 < \rho < r_0 \), then \( \psi_\rho \) is a solution of \( (P_\lambda^\alpha) \), with \( \lambda = \rho^2 \phi^{\alpha-1}(\rho) \). We infer that
\[
\max_{\rho \in [0, r_0]} \rho^2 \phi^{\alpha-1}(\rho) \leq \lambda^*(\alpha).
\]
Let us suppose that
\[
\max_{\rho \in [0, r_0]} \rho^2 \phi^{\alpha-1}(\rho) < \lambda^*(\alpha),
\]
if \( u_{\lambda^*(\alpha)} \) is the unique solution of \( (P_{\lambda^*(\alpha)}^\alpha) \) (cf.\([10]\)), one can use Proposition 1 to show that
\[
u_{\lambda^*(\alpha)}(r) = \left(1 + u_{\lambda^*(\alpha)}(0)\right) \left(\phi\left(\lambda^*(\alpha)\right)^{\frac{1}{2}} \left(1 + u_{\lambda^*(\alpha)}(0)\right)^{\frac{\alpha-1}{2}} r - \frac{1}{1 + u_{\lambda^*(\alpha)}(0)}\right).
\]
Let us put \( \rho_{\lambda^*(\alpha)} = \left(\lambda^*(\alpha)\right)^{\frac{1}{2}} \left(1 + u_{\lambda^*(\alpha)}(0)\right)^{\frac{\alpha-1}{2}} \). As \( u_{\lambda^*(\alpha)} \geq 0 \), we infer that \( \rho_{\lambda^*(\alpha)} < r_0 \). As \( u_{\lambda^*(\alpha)}(1) = 0 \), we infer that
\[
\frac{1}{1 + u_{\lambda^*(\alpha)}(0)} = \phi\left(\lambda^*(\alpha)\right)^{\frac{1}{2}} \left(1 + u_{\lambda^*(\alpha)}(0)\right)^{\frac{\alpha-1}{2}}.
\]
So we get
\[
u_{\lambda^*(\alpha)}(r) = \frac{\phi(\rho_{\lambda^*(\alpha)} r) - \phi(\rho_{\lambda^*(\alpha)})}{\phi(\rho_{\lambda^*(\alpha)})} \text{ and } \lambda^*(\alpha) = (\rho_{\lambda^*(\alpha)})^2 \phi^{\alpha-1}(\rho_{\lambda^*(\alpha)}).
\]
The last equality leads to a contradiction.

To prove the last statement, we use the fact that the maximum here is achieved at a unique \( r_\alpha \) (see the next lemma). So we get
\[
\phi'(r_\alpha) = -\frac{2}{(\alpha - 1)r_\alpha} \phi(r_\alpha), \text{ and }
\]
\[
\phi^{\alpha-3}(r_\alpha) \left(2\phi^2(r_\alpha) + 4r_\alpha(\alpha - 1)\phi(r_\alpha)\phi'(r_\alpha) + (\alpha - 1)r_\alpha^2 \left((\alpha - 2)\left(\phi'(r_\alpha)\right)^2 + \phi(r_\alpha)\phi''(r_\alpha)\right)\right) \leq 0.
\]
We first replace $\phi''(r_\alpha)$ by its value from $(L - E)$ and then $\phi'(r_\alpha)$, from the previous equality, to get

$$\phi^{\alpha - 1}(r_\alpha) \left(- (\alpha - 1)\lambda^*(\alpha) + 2(n - 4) + 4\frac{\alpha - 2}{\alpha - 1}\right) \leq 0.$$  

Simplifying, one gets the estimate.

**Remark 3.2.** — The last statement in Proposition 2 is also true for $\alpha \geq \frac{n + 2}{n - 2}$, with the same proof, provided that $\sup_{r \in \mathbb{R}_+} r^2 \phi^{\alpha - 1}(r)$ is attained (see the next Proposition 6); this has been proved in [10], using sophisticated arguments.

**Lemma 3.3.** — Let us put $g(r) = r^2 \phi^{\alpha - 1}(r)$, $r \in [0, r_0]$, there exists $\rho_0 \in [0, r_0]$ such that $g$ is increasing on $[0, \rho_0]$ and decreasing on $[\rho_0, r_0]$.

**Proof.** — Let $\rho$ be an arbitrary positive constant with $\rho < r_0$, then, as we have already mentioned $\psi_\rho$ is a solution of $(P_\gamma^\alpha)$, where $\gamma = g(\rho)$. As $g'(r) = r^2 \phi^{\alpha - 2}(r) (2 \phi(r) + (\alpha - 1) \rho \phi'(r))$, we infer that $g$ is increasing on a maximal interval $I_0 \subset [0, r_0]$ with $0 \in I_0$. Using Proposition 2, there exists $\rho_0 \in [0, r_0]$, such that $g(\rho_0) = \max_{r \in [0, r_0]} g(r) = \lambda^*(\alpha)$. This $\rho_0$ is unique, otherwise, if there exists $\lambda \in [0, r_0]$, such that $g(\lambda) = \max_{r \in [0, r_0]} g(r) = \lambda^*(\alpha)$, then $\psi_{\rho_0}$ and $\psi_\lambda$ are both solutions of the problem $(P_{\lambda_\alpha}^\alpha)$. As $\phi$ is decreasing on $[0, r_0]$, we infer that $\psi_{\rho_0}(0) = \frac{1 - \phi(\rho_0)}{\phi(\rho_0)} \neq \frac{1 - \phi(\lambda)}{\phi(\lambda)} = \psi_\lambda(0)$. So we get two different solutions of the problem $(P_{\lambda_\alpha}^\alpha)$. This leads to a contradiction (cf. [10]). As $g(r_0) = 0$, we infer that $I_0 \neq [0, r_0]$. Let us put $\delta = \sup I_0$. The function $g$ can’t be constant on a nontrivial interval $J \subset [\delta, r_0]$, for if $g(r) = c$ in $J$, then for every $\lambda \in J$, $\psi_\lambda$ is a solution of $(P_\gamma^\alpha)$. As $\psi_{\lambda_1}(0) \neq \psi_{\lambda_2}(0)$, if $\lambda_1, \lambda_2 \in J$ and $\lambda_1 \neq \lambda_2$, we infer that the problem $(P_\gamma^\alpha)$ admits an infinity of solutions. This leads again to a contradiction (cf. [10]).

So if $g$ is not decreasing on $[\delta, r_0]$, then there exists $\beta_1$ and $\beta_2$ with $r_0 > \beta_2 > \beta_1 > \delta$, such that $g$ is decreasing on $[\delta, \beta_1]$ and increasing on $[\beta_1, \beta_2]$. Let us put $c_0 = \min(g(\delta), g(\beta_2))$, then $c_0 > g(\beta_1)$. Let us choose $c \in ]g(\beta_1), c_0[$, so the problem $g(t) = c$ admits at least three different solutions $\lambda_i \in [0, \beta_2]$, $1 \leq i \leq 3$. As $\psi_{\lambda_i}(0) \neq \psi_{\lambda_j}(0)$, if $i \neq j$, $1 \leq i, j \leq 3$, we obtain three solutions for the problem $(P_\gamma^\alpha)$. So we get a contradiction.

We conclude that $g$ is increasing on $[0, \delta]$, decreasing on $[\delta, r_0]$ and $\delta = \rho_0$.

**Proposition 3.4.** — If $\lambda = \lambda^*(\alpha)$, there exists a unique $\rho_{\lambda^*(\alpha)} \in [0, r_0]$, such that
\( \lambda^*(\alpha) = (\rho_{\lambda^*(\alpha)})^2 \phi^{\alpha-1}(\rho_{\lambda^*(\alpha)}) \) and the unique solution \( u_{\lambda^*(\alpha)} \) of \( (P_{\lambda^*(\alpha)}^\alpha) \) is
\[
u_{\lambda^*(\alpha)}(r) = \frac{\phi(\rho_{\lambda^*(\alpha)}r) - \phi(\rho_{\lambda^*(\alpha)})}{\phi'(\rho_{\lambda^*(\alpha)})} = \psi_{\rho_{\lambda^*(\alpha)}}(r), \quad \forall \, r \in [0, 1].
\]
When \( 0 < \lambda < \lambda^*(\alpha) \), there exist exactly two constants \( r_\lambda \) and \( \rho_\lambda \), such that
\[
0 < r_\lambda < \rho_{\lambda^*(\alpha)} < \rho_\lambda < r_0, \quad \lambda = r_\lambda^2 \phi^{\alpha-1}(r_\lambda) = \rho_\lambda^2 \phi^{\alpha-1}(\rho_\lambda)
\]
and the only two solutions of \( (P_{\lambda}^\alpha) \) are
\[
u_\lambda = \psi_{r_\lambda}, \quad \nu_\lambda = \psi_{\rho_\lambda};
\]
the minimal one (cf. [2]) is \( \nu_\lambda \), \( \lim_{\lambda \to 0} u_\lambda = 0 \) in \( C^0(B_1^0) \) and
\[
\lim_{\lambda \to 0} \nu_\lambda(r) = \infty, \quad \forall \, r \in [0, 1].
\]

Proof. — Using Proposition 2 and Lemma 1, one infers that the only solution of \( (P_{\lambda^*(\alpha)}^\alpha) \) is \( \psi_{\rho_0} \). We put \( \rho_{\lambda^*(\alpha)} = \rho_0 \). If \( 0 < \lambda < \lambda^*(\alpha) \), using the lemma again, we infer that \( g(t) = \lambda \) admits exactly two solutions \( r_\lambda \) and \( \rho_\lambda \), with \( 0 < r_\lambda < \rho_{\lambda^*(\alpha)} < \rho_\lambda < r_0 \). Let us put \( u_\lambda = \nu_{r_\lambda} \) and \( \nu_\lambda = \psi_{\rho_\lambda} \), \( u_\lambda(0) \neq \nu_\lambda(0) \). These two functions \( u_\lambda \) and \( \nu_\lambda \) are solutions of the the problem \( (P_\alpha^\lambda) \), which admits only two ones (cf. [10]).

As \( \phi \) is decreasing on \([0, r_0]\), one can verify that \( u_\lambda(0) < \nu_\lambda(0) \), so we infer that the minimal solution (cf. [2]) is \( u_\lambda \).

As \( \lambda = r_\lambda^2 \phi^{\alpha-1}(r_\lambda) = \rho_\lambda^2 \phi^{\alpha-1}(\rho_\lambda), \quad 0 < r_\lambda < \rho_{\lambda^*(\alpha)} < \rho_\lambda < r_0 \), we get
\[
\lim_{\lambda \to 0} r_\lambda = 0, \quad \lim_{\lambda \to 0} \rho_\lambda = r_0, \quad \lim_{\lambda \to 0} u_\lambda(r) = \lim_{r_\lambda \to 0} \phi(r_\lambda r) - \frac{\phi(r_\lambda r) - \phi(\rho_\lambda)}{\phi'(\rho_\lambda)} = 1 = 0, \quad \text{and} \quad \lim_{\lambda \to 0} \nu_\lambda(r) = \lim_{\rho_\lambda \to r_0} \frac{\phi(\rho_\lambda r) - \phi(\rho_\lambda)}{\phi'(\rho_\lambda)} = \phi(r_0 r) \left( \lim_{\rho_\lambda \to r_0} \frac{1}{\phi'(\rho_\lambda)} \right) = \infty, \quad \forall \, r \in [0, 1].
\]

4. The Critical Sobolev Exponent Case

In this section, we suppose that \( \alpha = \frac{n+2}{n-2} \) and \( n \geq 3 \).

Let us consider the following problem
\[
(P_\alpha^\lambda) \left\{ \begin{array}{ll}
\Delta u + u^\alpha = 0, & \mathrm{in} \, \mathbb{R}^n \\
u > 0, & \mathrm{in} \, \mathbb{R}^n.
\end{array} \right.
\]

Remark 4.1. — Every radially symmetrical solution of \( (P_\alpha^\alpha) \) verifies \( \lim_{r \to \infty} u(r) = 0 \) (cf. [9]).

Following the method of Pohozaev in [14], the problem
\[
(Q_\alpha^\lambda) \left\{ \begin{array}{ll}
u''(r) + \frac{n-1}{r} u'(r) + u^\alpha(r) = 0, & \forall \, r > 0 \\
u > 0, \ u(0) = 1, \ u'(0) = 0
\end{array} \right.
\]

admits a solution \( \phi \).
**Lemma 4.2.** — Let \( u \) be a radially symmetrical regular solution of \( (P^\alpha) \), then
\[
\psi(r) = \psi(0) \phi \left( \frac{\alpha - 1}{\pi} r \right).
\]

**Proof.** — This proof is immediate.

**Lemma 4.3.** — Let us put \( g(r) = r^2 \phi^{\alpha - 1}(r), \ r \in \mathbb{R}_+ \), then there exists \( r_0 > 0 \), such that \( g \) is increasing on \([0, r_0]\), decreasing on \([r_0, \infty)\], with \( \lim_{r \to \infty} g(r) = 0 \).

**Proof.** — As we have already mentioned, \( g \) is increasing near 0. Let us assume that \( g \) is nondecreasing on \([0, \infty)\), then we have two possibilities
\[
\lim_{r \to \infty} g(r) = \infty \text{ or } \lim_{r \to \infty} g(r) = c, \ 0 < c < \infty.
\]

For every \( \rho > 0 \), \( \psi_\rho \) is a solution of \( (P^\alpha_\gamma) \), with \( \gamma = \rho^2 \phi^{\alpha - 1}(\rho) = g(\rho) \). We infer (cf. [2], [10]) that \( g(r) \leq \lambda^*(\alpha), \ \forall \ r > 0 \), so the first limit becomes impossible.

In the second case, we have two subcases: \( c \) is achieved or not.

If \( c \) is not achieved, then \( \forall \ l \) such that \( 0 < l < c \), there exists \( r_1 > 0 \) such that \( g(r_1) = l \). One can verify that \( \forall 0 < l < c \), the problem \( (P^\alpha_l) \) admits the solution \( \psi_{r_1} \), so we infer that \( c = \lambda^*(\alpha) \). Let \( u \) be a radially symmetrical solution (cf. [2], [10] and [3]) of \( (P^\alpha_\gamma) \). As in the proof of Proposition 2, one can verify that
\[
u = \psi_\rho, \ \rho = \sqrt{c(1 + u(0))^{\alpha - 1} - \frac{1}{1 + u(0)}} = \phi(\rho).
\]

As \( c = \rho^2 \phi^{\alpha - 1}(\rho) = g(\rho) \), we get a contradiction.

Let us suppose that \( c \) is achieved, as \( g \) is assumed to be nondecreasing, there exists \( r_0 \) such that \( g(r) = c, \ \forall \ r \geq r_0 \). Let us choose, an arbitrary constant \( \rho > 0 \) such that \( \rho \geq r_0 \). The function \( \psi_\rho \) is a solution of the problem \( (P^\alpha_\gamma) \), where \( \gamma = \rho^2 \phi^{\alpha - 1}(\rho) = g(\rho) = c, \ \forall \ \rho \geq r_0 \). This means that this problem, with such a \( \gamma \), admits an infinity of solutions \( \psi_\rho \); this leads to a contradiction (cf. [2], [10]). So \( g \) is not nondecreasing on \([0, \infty)\). As \( g \) can’t be constant on a nontrivial interval, we deduce that there exists positive constants \( r_1 \) and \( r_2 \), such that \( r_1 < r_2 \), with \( g \) is increasing on \([0, r_1]\) and decreasing on a maximal interval \([r_1, r_2]\). Let us suppose that \( g \) increases again on \([r_2, r_3]\), with \( r_2 < r_3 \). If \( \gamma \in \{g(r_2), \min(g(r_1), g(r_3))\} \), then \( g(r) = \gamma \) admits, at least, three roots, so the problem \( (P^\alpha_\gamma) \) admits, at least, three solutions; this gives again a contradiction (cf. [10]).
Finally, we get the existence of \( r_0 > 0 \), such that \( g \) is increasing on \([0, r_0]\) and decreasing on \([r_0, \infty] \). As \( g > 0 \), we infer that \( \lim_{r \to \infty} g(r) = c_0 \geq 0 \). If \( c_0 > 0 \), then for every \( c \in ]0, c_0[ \), there exists a unique \( \rho_c \in \mathbb{R}_+ \), verifying \( g(\rho_c) = c \). As \( c < \lambda^*(\alpha) \), the problem \( (P^\alpha_c) \) admits exactly two solutions (cf. [10]). One of these two solutions is \( \psi_{\rho_c} \). Let \( u_c \) be the other one, then, using Proposition 2 again, we get

\[
u_c(r) = \psi_\gamma, \quad \gamma = c^{\frac{1}{2}} (1 + u_c(0))^{\frac{\alpha - 1}{2}} = c^{\frac{1}{2}} \phi^{\frac{1 - \alpha}{2}} \left( c^{\frac{1}{2}} (1 + u_c(0))^{\frac{\alpha - 1}{2}} \right).
\]

So we infer that \( c = g(\gamma) \). As the two solutions are different, \( \rho_c \neq \gamma \) and \( \gamma \) is another root of \( g(r) = c \). This gives a contradiction and proves that necessarily \( c = 0 \). This ends the proof of the lemma.

**Proposition 4.4.** — Let us assume \( \alpha = \frac{n + 2}{n - 2} \), \( n \geq 3 \), then

\[
\lambda^*(\alpha) = \max_{r \in [0, \infty[} g(r).
\]

**Proof.** — Let \( \gamma = g(\rho) = \rho^2 \phi^{\alpha - 1}(\rho), \rho \in \mathbb{R}_+^* \), we have seen that \( \psi_\rho \) is a solution of \( (P^\alpha_\gamma) \). So we infer that \( g(\rho) \leq \lambda^*(\alpha) \), \( \forall \rho \in \mathbb{R}_+^* \).

Let us suppose that

\[
\max_{r \in [0, \infty[} g(r) < \lambda^*(\alpha)
\]

and let \( u \) be the unique solution (cf. [10]) of \( (P^\alpha_{\lambda^*(\alpha)}) \). As in the proof of Proposition 2, we get that \( u = \psi_\rho \) and \( \lambda^*(\alpha) = g(\rho) \). This gives a contradiction.

**Proposition 4.5.** — We have \( \lambda^*(\alpha) = \frac{n(n - 2)}{4} \). There exists a unique \( r_{\lambda^*(\alpha)} = \sqrt{n(n - 2)} \), such that \( \lambda^*(\alpha) = r_{\lambda^*(\alpha)}^2 \phi^{\alpha - 1}(r_{\lambda^*(\alpha)}) \) and a unique solution of \( (P^\alpha_{\lambda^*(\alpha)}) \n\]

\[
u_{\lambda^*(\alpha)} = \psi_{r_{\lambda^*(\alpha)}}.
\]

If \( 0 < \lambda < \lambda^*(\alpha) \), there exist exactly two constants

\[
r_\lambda = \sqrt{1 - \frac{2\lambda}{n(n - 2)}} - \sqrt{1 - \frac{4\lambda}{n(n - 2)}} (n(n - 2))^{-1/2} \quad \text{and} \quad \rho_\lambda = \sqrt{1 - \frac{2\lambda}{n(n - 2)}} + \sqrt{1 - \frac{4\lambda}{n(n - 2)}} (n(n - 2))^{-1/2}
\]

such that \( 0 < r_\lambda < r_{\lambda^*(\alpha)} < \rho_\lambda, \lambda = g(r_\lambda) = g(\rho_\lambda) \) and the only two solutions of \( (P^\alpha_\lambda) \) are

\[
u_\lambda = \psi_{r_\lambda} \quad \text{and} \quad \psi_\lambda = \psi_{\rho_\lambda},
\]

the minimal one (cf. [2]) is \( u_\lambda \); \( \lim_{\lambda \to 0} u_\lambda = 0 \), \( \text{in } C^0(\overline{B_1}) \) and \( \lim_{\lambda \to 0} v_\lambda(r) = r^{2-n} - 1, \forall r \in ]0, 1[. \)
Proof. — One can use Lemma 3 to get the existence (and the uniqueness) of \( r_{\lambda^*(\alpha)} = r_0 \), \( r_{\lambda} \) and \( \rho_{\lambda} \). It is then easy to verify that \( \psi_{r_{\lambda^*(\alpha)}} \) is a solution of \( (P^{\alpha}_{\lambda^*(\alpha)}) \), \( u_{\lambda} = \psi_{r_{\lambda}} \) and \( v_{\lambda} = \psi_{\rho_{\lambda}} \) are solutions of \( (P^{\alpha}_{\lambda}) \). The problem \( (P^{\alpha}_{\lambda}) \) admits only two solutions (cf. [10]), as \( \phi \) is decreasing on \( \mathbb{R}_+^* \), one can verify that \( u_{\lambda}(0) < v_{\lambda}(0) \), so \( u_{\lambda} \neq v_{\lambda} \). We conclude that \( u_{\lambda} \) and \( v_{\lambda} \) are the only solutions of \( (P^{\alpha}_{\lambda}) \) and the minimal one (cf. [2]) is \( u_{\lambda} \).

Let us compute the constants \( r_{\lambda^*(\alpha)}, r_{\lambda} \) and \( \rho_{\lambda} \).

It is well known (cf. [13]) that, if \( \alpha = \frac{n+2}{n-2} - 1 = \frac{4}{n-2} \), the problem \( (Q^{\alpha}) \) admits the continuum of spherically symmetrical "instantons"
\[
\psi(r) = \gamma^{\frac{n+2}{n-2}} (n(n-2))^{\frac{n-2}{n-2}} (\gamma^2 + r^2)^{\frac{2-n}{2}}, \quad \gamma > 0.
\]

Let us fix \( \gamma > 0 \), so \( u_{\gamma}(0) = \gamma^{\frac{2-n}{2}} (n(n-2))^{\frac{n-2}{n-2}} \). Using Lemma 2, we get the expression of the Lane-Emden function
\[
\phi(r) = \frac{1}{u_{\gamma}(0)} u_{\gamma} \left( u_{\gamma}(0) \right)^{\frac{n-2}{n-2}} = \left( 1 + \frac{r^2}{n(n-2)} \right)^{\frac{2-n}{2}}.
\]

As \( \alpha - 1 = \frac{n+2}{n-2} - 1 = \frac{4}{n-2} \), we infer that
\[
g(r) = r^2 \phi^{\alpha-1}(r) = r^2 \left( 1 + \frac{r^2}{n(n-2)} \right)^{-2}.
\]

Using Proposition 4, a direct calculation gives
\[
\lambda^*(\alpha) = \max_{r>0} r^2 \left( 1 + \frac{r^2}{n(n-2)} \right)^{-2}
\]
\[
= r^2 \left( 1 + \frac{r^2}{n(n-2)} \right)^{-2} \bigg|_{r=r_{\lambda^*(\alpha)}=\sqrt{n(n-2)}} = \frac{n(n-2)}{4}.
\]

In [7], the previous constant has been computed, using the Pohozaev Identity. If \( 0 < \lambda < \lambda^*(\alpha) \), the equation \( g(r) = \lambda \) admits two positive roots
\[
r_{\lambda} = \frac{\sqrt{1 - \frac{2\lambda}{n(n-2)}} - \sqrt{1 - \frac{4\lambda}{n(n-2)}}}{(n(n-2))^{-1} \sqrt{2\lambda}} \quad \text{and} \quad \rho_{\lambda} = \frac{\sqrt{1 - \frac{2\lambda}{n(n-2)}} + \sqrt{1 - \frac{4\lambda}{n(n-2)}}}{(n(n-2))^{-1} \sqrt{2\lambda}}.
\]

This gives us \( u_{\lambda} = \psi_{r_{\lambda}} \) and \( v_{\lambda} = \psi_{\rho_{\lambda}} \); as \( r_{\lambda} < \rho_{\lambda} \), we get \( u_{\lambda}(0) < v_{\lambda}(0) \), so \( u_{\lambda} \) is the minimal solution.
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As \( \lambda = r^{2 \alpha - 1}(r_{\lambda}) = r^{2 \alpha - 1}(\rho_{\lambda}), 0 < r_{\lambda} < r_{\lambda^*}(\alpha) < \rho_{\lambda} < \infty \), one can verify that
\[
\lim_{\lambda \to 0} r_{\lambda} = 0, \lim_{\lambda \to 0} \rho_{\lambda} = \infty, \lim_{\lambda \to 0} u_{\lambda} = 0, \text{ in } C^0(\overline{B_1}) \text{ and }
\lim_{\lambda \to 0} \psi_{\lambda}(0) = \lim_{\rho_{\lambda} \to \infty} \phi(\rho_{\lambda}) - 1 = r^{2 - n} - 1, \forall r \in ]0,1].
\]

5. The Supercritical Case

We consider here the case \( \alpha > \frac{n+2}{n-2}, n \geq 3 \). Let us put
\[
f(\alpha) = \frac{4\alpha}{\alpha - 1} + 4 \sqrt{\frac{\alpha}{\alpha - 1}}, \forall \alpha > 1.
\]
Let’s first detail a condition, \( f(\alpha) > n - 2 \), used in [10].

**Lemma 5.1.** — If \( 3 \leq n \leq 10 \) and \( \alpha > \frac{n+2}{n-2} \)
or \( n > 10 \) and \( \frac{n+2}{n-2} < \alpha < \frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}-4} \),
then \( f(\alpha) > n - 2 \). If \( n > 10 \) and \( \frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}-4} \leq \alpha \), then \( f(\alpha) \leq n - 2 \).

**Proof.** — Let us put \( p(t) = 4t^2 + 4t \) and \( u = \sqrt{\frac{\alpha}{\alpha - 1}} \), so we get \( f(\alpha) = p(u) \). The only positive root of \( p(t) = n - 2 \), is \( t_0 = \frac{\sqrt{n-1}}{2} \) and the equation \( u = \frac{\sqrt{n-1}}{2} \) has the only solution \( \alpha_0 = \frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}-4} \). But \( \alpha_0 > 0 \), if and only if \( n > 10 \).

For every \( \alpha > \frac{n+2}{n-2} \), we have \( \alpha > 1 \) so we get \( \sqrt{\frac{\alpha}{\alpha - 1}} > 1 > \frac{\sqrt{n-1}}{2} \), if
\( 3 \leq n \leq 10 \). We infer that \( f(\alpha) > n - 2 \), if \( 3 \leq n \leq 10 \).

If \( n > 10 \), we have \( \alpha_0 > \frac{n+2}{n-2} > 1 \), one can verify that if \( \frac{n+2}{n-2} < \alpha < \alpha_0 \),
then \( f(\alpha) > n - 2 \) and \( f(\alpha) \leq n - 2 \), if \( \alpha \geq \alpha_0 \).

**Proposition 5.2.** — Let us put \( \lambda_s = \frac{2}{(\alpha-1)^2} (\alpha(n-2) - n) \).

If \( 3 \leq n \leq 10 \) and \( \frac{n+2}{n-2} < \alpha \) or \( n > 10 \) and \( \frac{n+2}{n-2} < \alpha < \frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}-4} \)
then
\[
\lambda^*(\alpha) = \max_{x_+} g(r), \lambda^*(\alpha) > \lambda_s \text{ and } \phi(r) \sim \lambda^{\frac{1}{2}}_s r^{\frac{2}{1-\alpha}}, \text{ as } r \to \infty.
\]

If \( \rho_i \) is an increasing sequence of positive reals, such that \( (\psi_{\rho_i}) \) are solutions of \( (P_{\lambda_s}^*) \) and \( \lim_{i \to \infty} \rho_i = \infty \), then \( \lim_{i \to \infty} \psi_{\rho_i} = \lambda^{\frac{1}{2}}_s (r^{\frac{2}{1-\alpha}} - 1), \forall r \in [0,1]. \)
If $n > 10$ and $\frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}-4} \leq \alpha$ then

$$\lambda^*(\alpha) = \sup_{\mathbb{R}_+^*} g(r) = \lambda_s$$

and $\phi(r) \sim \lambda_s^{\frac{1}{\alpha-1}} r^{\frac{2}{1-\alpha}}$, as $r \to \infty$.

If $(\lambda_i)$ is an increasing positive sequence such that $\lim_{i \to \infty} \lambda_i = \lambda_s$ and $\forall i$, $w_i$ is the unique solution of $(P_{\lambda_i}^\alpha)$, then

$$\lim_{i \to \infty} w_i(r) = \lambda_s^{\frac{1}{\alpha-1}} (r^{\frac{2}{1-\alpha}} - 1), \ \forall \ r \in [0, 1].$$

Proof. — As in the proof of Proposition 4, one can verify that $\lambda^*(\alpha) = \sup_{\mathbb{R}_+^*} g(r)$, where $g(r) = r^2 \phi^\alpha - 1(r)$.

If $(3 \leq n \leq 10$ and $\frac{n+2}{n-2} < \alpha)$ or $(n > 10$ and $\frac{n+2}{n-2} < \alpha < \frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}-4})$, using Lemma 4, we get $f(\alpha) > n - 2$. So we can use Theorem 1 in [10] to infer that $\lambda^*(\alpha) > \lambda_s$, $(P_{\lambda^*(\alpha)}^\alpha)$ admits a unique solution and $(P_{\lambda_s}^\alpha)$ admits an infinity of solutions. Using the unique solution $u_{\lambda^*(\alpha)}$ of $(P_{\lambda^*(\alpha)}^\alpha)$, one can deduce from Proposition 1 that $u_{\lambda^*(\alpha)} = \psi_\rho$, where $\rho \in \mathbb{R}_+^*$ and $g(\rho) = \lambda^*(\alpha)$. We conclude that the supremum is achieved and $\lambda^*(\alpha) = \max_{\mathbb{R}_+^*} g(r)$.

Let us suppose that

$$a = \liminf_{r \to \infty} g(r) < A = \limsup_{r \to \infty} g(r).$$

For every $\lambda \in [a, A]$, the equation $g(r) = \lambda$ admits a sequence of roots $(r_i)$, with $\lim_{i \to \infty} r_i = \infty$. As for every $i$, $\psi_{r_i}$ is a solution of $(P_{\lambda_s}^\alpha)$, we get an infinity of solutions for this problem; but an infinity of solutions exists only when $\lambda = \lambda_s$ (cf. [10]). We get a contradiction and infer that

$$a = A = \lambda_s = \lim_{r \to \infty} g(r), \ \text{so} \ \phi(r) \sim \lambda_s^{\frac{1}{\alpha-1}} r^{\frac{2}{1-\alpha}}, \ \text{as} \ r \to \infty.$$ 

If $(\rho_i)$ is an increasing sequence of positive constants, such that $(\psi_{\rho_i})$ are solutions of $(P_{\lambda_s}^\alpha)$ and $\lim_{i \to \infty} \rho_i = \infty$, then one can use the previous asymptotic behavior of $\phi$ to get $\lim_{i \to \infty} \psi_{\rho_i}(r) = \lambda_s^{\frac{1}{\alpha-1}} (r^{\frac{2}{1-\alpha}} - 1), \ \forall \ r \in [0, 1]$.

If $n > 10$ and $\frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}-4} \leq \alpha$, we get from Lemma 4 that $f(\alpha) \leq n - 2$. Using [10] again, we infer that $\lambda^*(\alpha) = \lambda_s$, $(P_{\lambda_s}^\alpha)$ admits a unique solution for every $\lambda \in [0, \lambda^*(\alpha)]$. As the function $g$ is increasing near $r = 0$, we infer that $g$ is increasing on a nontrivial open interval $I \subset \mathbb{R}_+^*$. For, on one hand, if $g$ decreases on a nontrivial open interval $I \subset \mathbb{R}_+^*$, then the equation $g(r) = \lambda$ admits at least two roots $r_1 < r_2$, if $\lambda \in \min_I g(r), \max_I g(r)$. As $\psi_{r_1}$ and $\psi_{r_2}$ are solutions of $(P_{\lambda_s}^\alpha)$,
with \( \psi_{r_1}(0) \neq \psi_{r_2}(0) \), this violates the uniqueness result of [10]. On another hand, the function \( g \) can’t be constant on a nontrivial interval, otherwise we get an infinity of solutions for some \( \lambda \). One can then see that

\[
\lim_{r \to \infty} g(r) = \sup_{\mathbb{R}^*_+} g(r) = \lambda^*(\alpha); \quad \lambda^*(\alpha) = \lambda_s \quad (c.f. \ [10]).
\]

So \( \phi(r) \sim \lambda_s^{\frac{1}{\alpha - 1}} r^{\frac{2}{\alpha}}, \) as \( r \to \infty \).

Using this asymptotic behavior, one can show the last statement of the proposition.

Let us put

\[
(\mathcal{Q}_\lambda^\alpha) \begin{cases} 
\Delta u + \lambda (1 + u)^\alpha = 0, \text{ in } B_{r_0} \\
u > 0, \text{ in } B_{r_0}^c \\
u = 0, \text{ on } \partial B_{r_0}
\end{cases}
\]

where \( B_{r_0} = \{ x \in \mathbb{R}^n, \|x\| < r_0 \} \). For every solution \( u \) of \( (\mathcal{Q}_\lambda^\alpha) \), we put \( v(r) = u(r_0 r) \) for every \( r \in [0, 1] \). Let \( \lambda^*_{r_0}(\alpha) \), be the maximal eigenvalue of \( (\mathcal{Q}_\lambda^\alpha) \).

**Lemma 5.3.** — A function \( u \) is a solution of \( (\mathcal{Q}_\lambda^\alpha) \), if and only if \( v \) is a solution of \( (\mathcal{P}_\lambda^\alpha) \). In particular, we get \( \lambda^*_{r_0}(\alpha) = \lambda^*_{r_0}(\alpha) \).

**Proof.** — The proof is easy.

**Remark 5.4.** — According to the previous lemma, the results obtained here for \( (\mathcal{P}_\lambda^\alpha) \) (on the unit ball \( B_1 \)), can be easily stated for \( (\mathcal{Q}_\lambda^\alpha) \) (on any ball \( B_{r_0} \)).

### 6. Appendix

Let \( S_i^k \) be the set of all the \((k - i)\)-selections of \( \{1, \ldots, i\} \) and \( s(j) \) the multiplicity of the element \( j, 1 \leq j \leq i \). If \( u \) is an analytical solution of \( (\mathcal{P}_\lambda^\alpha) \), with \( u(r) = \Sigma_{k=0}^\infty a_k r^k \) near \( r = 0, r_0 \) the convergence radius of this series, then

**Proposition 6.1.** —

\[
\forall \ k \geq 0, \quad a_{2k+1} = 0, \quad a_2 = \frac{\lambda}{n-2} (1 + a_0)^\alpha \left( \frac{1}{n} - \frac{1}{2} \right)
\]

and \( \forall \ k > 1, \quad a_{2k} = \frac{\lambda}{n-2} \left( \frac{1}{2k+n-2} - \frac{1}{2k} \right) \times \left( \Pi_{i=1}^{k-1} (1 + a_0)^\alpha - i \right) a_2 \Pi_{p=0}^{i-1} (\alpha - p) \Sigma_{s \in S_i^k} \Pi_{j=1}^{i} a_2(1+s(j)) \)
Proof. — Let us choose $0 < r \leq \rho < r_0$, by standard integrations, we get

$$u(r) - u(\rho) = \frac{\lambda}{n - 2} \times$$

$$\left( (r^{2-n} - \rho^{2-n}) \int_0^r t^{n-1} (1 + u(t))^\alpha dt + \int_\rho^r (t - \rho^{2-n} t^{n-1}) (1 + u(t))^\alpha dt \right).$$

Let us point out that

$$(1 + u(r))^{\alpha} = (1 + u(0) - u(0) + u(r))^{\alpha}$$

$$= (1 + u(0))^{\alpha} \left( 1 + \frac{u(r) - u(0)}{1 + u(0)} \right)^{\alpha} = (1 + a_0)^{\alpha} \left( 1 + \Sigma_{i=1}^\infty \frac{a_i}{1 + a_0 r^i} \right)^{\alpha}, u(0) = a_0.$$ 

By the Maximum Principle, we have $\forall r \in [0, 1[, 0 < u(r) < u(0)$, so we get

$$\left| \frac{u(0) - u(r)}{1 + u(0)} \right| < 1, \forall r \in [0, 1],$$

we infer that

$$(1 + u(r))^{\alpha} = (1 + a_0)^{\alpha} \left( 1 + \Sigma_{j=1}^\infty \frac{\alpha(\alpha - 1) \ldots (\alpha - j + 1)}{j!} \left( \Sigma_{i=1}^\infty \frac{a_i}{1 + a_0 r^i} \right)^j \right).$$

All these series are uniformly convergent on $[0, \rho]$. If we put

$$(1 + u(r))^{\alpha} = \Sigma_{j=0}^\infty c_j r^j,$$

we get

$$u(r) = \frac{\lambda}{n - 2} \left( (r^{2-n} - \rho^{2-n}) \int_0^r t^{n-1} \Sigma_{j=0}^\infty c_j t^j dt + \int_\rho^r (t - \rho^{2-n} t^{n-1}) \Sigma_{j=0}^\infty c_j t^j dt \right)$$

$$= \frac{\lambda}{n - 2} \left( \Sigma_{j=0}^\infty c_j \frac{r^{j+2}}{j + n} - \Sigma_{j=0}^\infty c_j \frac{r^{2-n} r^j}{j + n} + \Sigma_{j=0}^\infty c_j \frac{\rho^{2-n} r^j}{j + 2} - \Sigma_{j=0}^\infty c_j \frac{\rho^{j+2}}{j + n} \right)$$

$$+ \frac{\lambda}{n - 2} \left( -\Sigma_{j=0}^\infty c_j \frac{r^j}{j + n} + \Sigma_{j=0}^\infty c_j \frac{\rho^j}{j + n} \right)$$

$$= \frac{\lambda}{n - 2} \left( \Sigma_{j=2}^\infty c_{j-2} \frac{r^j}{j + n - 2} + \Sigma_{j=0}^\infty c_j \frac{\rho^{j+2}}{j + 2} - \Sigma_{j=0}^\infty c_j \frac{\rho^{j+2}}{j + n} - \Sigma_{j=2}^\infty c_{j-2} \frac{r^j}{j} \right).$$

We finally obtain

$$u(r) = \frac{\lambda}{n - 2} \left( \Sigma_{j=2}^\infty c_{j=2} \left( \frac{1}{j + n - 2} - \frac{1}{j} \right) r^j + \Sigma_{j=0}^\infty c_j \rho^{j+2} \left( \frac{1}{j + 2} - \frac{1}{j + n} \right) \right).$$

Using the previous identity, we obtain

$$a_1 = 0, \forall k > 1, a_k = \frac{\lambda}{n - 2} \left( \frac{1}{k + n - 2} - \frac{1}{k} \right) c_{k-2}.$$
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Using (1), we get

$$c_0 = (1 + a_0)^\alpha, \quad c_1 = \alpha (1 + a_0)^{\alpha - 1} a_1 = 0$$

and

$$\forall k > 1, \quad c_k = (1 + a_0)^\alpha \sum_{j=1}^{k} \frac{1}{j!} \prod_{p=0}^{j-1} (\alpha - p) \frac{1}{(1 + a_0)^j} \sum_{s \in S_k} \prod_{i=1}^{j} a_{1+s(i)}$$

$$= \sum_{j=1}^{k} \frac{1}{j!} \prod_{p=0}^{j-1} (\alpha - p) (1 + a_0)^{\alpha - j} \sum_{s \in S_k} \prod_{i=1}^{j} a_{1+s(i)}.$$ 

Using the previous relation and the fact that $a_1 = 0$, one can verify (by induction) that $a_{2k+1} = 0, \forall k > 0$. We then obtain from (2) and the expression of $c_k$

$$a_{2k} = \frac{\lambda}{n - 2} \left( \frac{1}{2k + n - 2} - \frac{1}{2k} \right) c_{2k-2}$$

$$= \frac{\lambda}{n - 2} \left( \frac{1}{2k + n - 2} - \frac{1}{2k} \right) \sum_{j=1}^{k-1} \frac{1}{j!} \prod_{p=0}^{j-1} (\alpha - p) (1 + a_0)^{\alpha - j} \sum_{s \in S_{k-1}} \prod_{i=1}^{j} a_{2(1+s(i))}.\forall j \in [1, k - 1], \text{Card}(S_{k-1}^j) = C_{k-2}^j.$$

Let us put

$$d_2 = \frac{1}{2n} \text{ and } \forall k > 1,$$

$$d_{2k} = \frac{1}{(2k + n - 2)(2k)} \sum_{j=1}^{k-1} \frac{1}{j!} \prod_{p=0}^{j-1} (\alpha - p) \sum_{s \in S_{k-1}} \prod_{i=1}^{j} a_{2(1+s(j))},$$

then

**Lemma 6.2.** $- a_{2k} = (-1)^k \lambda^k (1 + a_0)^{k(\alpha - 1) + 1} d_{2k}, \forall k > 1.$

**Proof.**

$$a_4 = \frac{\alpha \lambda^2}{(n - 2)^2} = (1 + a_0)^{2\alpha - 1} \left( \frac{1}{n + 2} - \frac{1}{4} \right) \left( \frac{1}{n} - \frac{1}{2} \right)$$

$$= \lambda^2 (1 + a_0)^{2\alpha - 1} \frac{1}{4(n + 2)} \frac{\alpha}{2n} = \lambda^2 (1 + a_0)^{2(\alpha - 1) + 1} \frac{1}{4(n + 2)} \frac{\alpha}{2n}.$$ 

$$d_4 = \frac{1}{4(n + 2)} \sum_{i=1}^{1} \frac{1}{i!} \prod_{p=0}^{i-1} (\alpha - p) \sum_{s \in S_{i}} \prod_{j=1}^{i} d_{2(1+s(j))}$$

$$= \frac{\alpha}{4(n + 2)} d_2 = \frac{1}{4(n + 2)} \frac{\alpha}{2n},$$

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so we infer that the formula is true for \( k = 2 \). Let us suppose it true for every \( j \), such that \( 2 \leq j \leq k \). From Proposition 7, we have

\[
a_{2(k+1)} = \frac{\lambda}{n-2} \left( \frac{1}{2k+n} - \frac{1}{2(k+1)} \right) \sum_{j=1}^{k} \frac{1}{j!} \prod_{p=0}^{j-1} (\alpha - p) (1 + a_0)^{\alpha - j} \sum_{s \in S^j_k} \prod_{i=1}^{j} a_{2(1+s(i))}
\]

\[
= \frac{-\lambda}{(2(k+1) + n - 2)(2(k+1))} \sum_{j=1}^{k} \frac{1}{j!} \prod_{p=0}^{j-1} (\alpha - p) (1 + a_0)^{\alpha - j} \sum_{s \in S^j_k} \prod_{i=1}^{j} a_{2(1+s(i))}.
\]

\( \forall j \in [1,k], \forall s \in S^j_k \), if \( i \in [1,j] \), then \( 1 \leq 1 + s(i) \leq k \), so one can use the hypothesis to get \( \forall i \in [1,j] \),

\[
a_{2(1+s(i))} = (-1)^{1+s(i)} \lambda^{1+s(i)} (1 + a_0)^{(s(i)+1)(\alpha-1)+1} d_{2(1+s(i))}.
\]

We then obtain

\[
\Pi_{i=1}^{j} a_{2(1+s(i))} = (-1)^{j} \lambda^{j} (1 + a_0)^{\alpha j + (\alpha - 1)(k-j)} \prod_{i=1}^{j} d_{2(1+s(i))}
\]

\[
= (-1)^{j} \lambda^{j} (1 + a_0)^{\alpha j + (\alpha - 1)(k-j)} \prod_{i=1}^{j} d_{2(1+s(i))}.
\]

But for every \( s \in S^j_k \), we have \( \Sigma_{i=1}^{j} s(i) = k - j \).

We infer that

\[
\Pi_{i=1}^{j} a_{2(1+s(i))} = (-1)^{k} \lambda^{k} (1 + a_0)^{(\alpha - 1)k + j} \prod_{i=1}^{j} d_{2(1+s(i))}
\]

\[
= (-1)^{k} \lambda^{k} (1 + a_0)^{(\alpha - 1)k + j} \prod_{i=1}^{j} d_{2(1+s(i))}.
\]

Substituting in the expression of \( a_{2(k+1)} \), we obtain

\[
a_{2(k+1)} = (-1)^{k+1} \lambda^{k+1} (1 + a_0)^{k(\alpha-1) + \alpha} \frac{1}{(2(k+1) + n - 2)(2(k+1))} \times
\]

\[
\sum_{j=1}^{k} \frac{1}{j!} \prod_{p=0}^{j-1} (\alpha - p) \sum_{s \in S^j_k} \prod_{i=1}^{j} d_{2(1+s(i))}
\]

\[
= (-1)^{k+1} \lambda^{k+1} (1 + a_0)^{(k+1)(\alpha-1)+1} \frac{1}{(2(k+1) + n - 2)(2(k+1))} \times
\]

\[
\sum_{j=1}^{k} \frac{1}{j!} \prod_{p=0}^{j-1} (\alpha - p) \sum_{s \in S^j_k} \prod_{i=1}^{j} d_{2(1+s(i))}.
\]

\[
= (-1)^{k+1} \lambda^{k+1} (1 + a_0)^{(k+1)(\alpha-1)+1} d_{2(k+1)}.
\]

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Let us compute the first terms of the Lane-Emden function, 
\[ \varphi(r) = \sum_{i=0}^{\infty} a_{2i} r^{2i}, \] 
near \( r = 0 \), where \( a_0 = 1 \), and 
\[ a_{2i} = (-1)^i 2^{i(\alpha-1)+1} d_{2i}, \quad \forall i > 1. \]

\[ d_0 = 1; \quad d_2 = \frac{1}{2n}; \quad d_4 = \frac{1}{4(n+2)} \alpha d_2 = \frac{\alpha}{(2n)(4(n+2))}; \]

\[ d_6 = \frac{1}{6(n+4)} \left( \alpha d_4 + \frac{1}{2} \alpha (\alpha - 1) d_2^2 \right) = \frac{1}{6(n+4)} \left\{ \frac{\alpha^2}{(2n)(4(n+2))} + \frac{\alpha(\alpha - 1)}{2(2n)^2} \right\}; \]

\[ d_8 = \frac{1}{8(n+6)} \left( \alpha d_6 + \alpha (\alpha - 1) d_4 d_2 + \frac{\alpha(\alpha - 1)(\alpha - 2)}{6} d_2^3 \right) \]

\[ = \frac{1}{8(n+6)} \left\{ \frac{\alpha^3}{(2n)(4(n+2))(6(n+4))} + \frac{\alpha^2(\alpha - 1)}{2(2n)^2(6(n+4))} + \frac{\alpha^2(\alpha - 1)}{(2n)^2(4(n+2))} \right\}; \]

\[ d_{10} = \frac{1}{10(n+8)} \left\{ \alpha d_8 + \frac{\alpha(\alpha - 1)}{2} (2d_2 d_6 + d_4^2) + \frac{\alpha(\alpha - 1)(\alpha - 2)}{6} d_2^2 d_4 + \frac{\alpha(\alpha - 1)(\alpha - 2)(\alpha - 3)}{24} d_2^4 \right\} \]

\[ = \frac{1}{10(n+8)} \left\{ \frac{\alpha^4}{(2n)(4(n+2))(6(n+4))(8(n+6))} + \frac{\alpha^3(\alpha - 1)}{2(2n)^2(6(n+4))(8(n+6))} \right\}; \]

\[ + \frac{\alpha^3(\alpha - 1)}{(2n)^2(4(n+2))(8(n+6))} + \frac{\alpha^2(\alpha - 1)(\alpha - 2)}{6(2n)^3(8(n+6))} + \frac{\alpha^3(\alpha - 1)}{(2n)^2(4(n+2))(6(n+4))} \]

\[ + \frac{\alpha^2(\alpha - 1)^2}{2(2n)^3(6(n+4))} + \frac{\alpha^3(\alpha - 1)}{2(2n)^2(4(n+2))^2} + \frac{\alpha^2(\alpha - 1)(\alpha - 2)}{2(2n)^3(4(n+2))} \]

\[ + \frac{\alpha(\alpha - 1)(\alpha - 2)(\alpha - 3)}{24(2n)^4} \right\}. \]

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Bibliography


