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Local Peak Sets in Weakly Pseudoconvex Boundaries in $\mathbb{C}^n$ (*)

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**Abstract.** — We give a sufficient condition for a $C^\omega$ (resp. $C^\infty$)-totally real, complex-tangential, $(n-1)$-dimensional submanifold in a weakly pseudoconvex boundary of class $C^\omega$ (resp. $C^\infty$) to be a local peak set for the class $\mathcal{O}$ (resp. $A^\infty$). Moreover, we give a consequence of it for Catlin’s multitype.

**Résumé.** — On donne une condition suffisante pour qu’une sous variété $C^\omega$ (resp. $C^\infty$), totalement réelle, complexe-tangentielle, de dimension $(n-1)$ dans le bord d’un domaine faiblement pseudoconvexe de $\mathbb{C}^n$, soit un ensemble localement pic pour la classe $\mathcal{O}$ (resp. $A^\infty$). De plus, on donne une conséquence de cette condition en terme de multitype de D. Catlin.

1. Introduction and basic definitions

This article is a part of the Ph.D thesis of the author. The $\mathcal{O}$ part was motivated by the paper of Boutet de Monvel and Iordan [B-I] and $A^\infty$ part by the methods of Hakim and Sibony [H-S]. Let $D$ be a domain in $\mathbb{C}^n$ with $C^\omega$ (resp. $C^\infty$)-boundary. We denote for an open set $\mathcal{U}$ by $\mathcal{O}$ (resp. $A^\infty$) the class of holomorphic functions on $\mathcal{U}$ (resp. the class of holomorphic functions in $\mathcal{U}$ which have a $C^\infty$-extension to $\overline{\mathcal{U}}$).

We say that $M \subset bD$ is a local peak set at a point $p \in M$ for the class $\mathcal{O}$ (resp. $A^\infty$), if there exist a neighborhood $\mathcal{U}$ of $p$ in $\mathbb{C}^n$ and a function...
Let $D$ be a pseudoconvex domain with $C^\omega$ (resp. $C^\infty$)-boundary. Let $M$ be an $(n-1)$ dimensional submanifold of $bD$ which is totally real and complex-tangential in a neighborhood of a point $p \in M$. Let $(V, \gamma)$ be a $C^\omega$ (resp. $C^\infty$)-parametrization of $M$ at $p$, where $V$ is a neighborhood of the origin in $\mathbb{R}^{n-1}$ such that $\gamma(0) = p$. Let $X$ be a $C^\omega$ (resp. $C^\infty$)-vector field on $M$ such that $X(p) = 0$. Denote by $\zeta = (\zeta_1, \ldots, \zeta_{n-1})$ the coordinates of

2. Preliminaries
Local Peak Sets in Weakly Pseudoconvex Boundaries in $\mathbb{C}^n$

a point in $V$. Then $X$ can be written as $X = \sum_i d_i(\zeta) \frac{\partial}{\partial \zeta^i}$ where $d_i$ are $C^\omega$ (resp. $C^\infty$)-functions on $V$. We set $D_0$ the Jacobian matrix at the origin: \[
\left\{ \frac{\partial d_i}{\partial \zeta^j}(0) \right\}_{i,j \leq n-1}.
\] Now, we introduce our first hypothesis:

$(H_1)$ The matrix $D_0$ is diagonalizable and has $\tilde{m}_1 \geq \ldots \geq \tilde{m}_{n-1}$ eigenvalues with $\tilde{m}_i \in \mathbb{N}^*$ for all $i$.

We say that $M$ admits a peak-admissible $C^\omega$ (resp. $C^\infty$)-vector field $X$ of weights $(\tilde{m}_1, \ldots, \tilde{m}_{n-1})$ at $p \in M$ for the class $O$ (resp. $A^\infty$). $(H_1)$ is independent of the choice of the parametrization and the $\tilde{m}_i$ and their multiplicities are uniquely determined. Using hypothesis $(H_1)$, one can easily prove that there exists a $C^\omega$ (resp. $C^\infty$)-change of coordinates on $V$ such that $X = \sum_i \tilde{m}_i \zeta_i \frac{\partial}{\partial \zeta_i}$. This representation of $X$ is invariant if we apply a “weight-homogeneous” polynomial transformation of coordinates as below:

**Lemma 2.1.** Let $\Lambda = (\Lambda_1, \ldots, \Lambda_{n-1})$ be a $C^\omega$ (resp. $C^\infty$)-change of coordinates on $V$ such that $\Lambda(0) = 0$ and $d\Lambda(X) = X$. Then $\Lambda$ is a polynomial map. More precisely, if $\zeta = (\zeta_1, \ldots, \zeta_{n-1}) \in V$, $I = (i_1, \ldots, i_{n-1}) \in \mathbb{N}^{n-1}$ and we set $|I|_* = \sum \nu_i \tilde{m}_\nu$ then for every $1 \leq j \leq n-1$, $\Lambda_j(\zeta) = \sum_{|I|_* = \tilde{m}_j} a_j^{i_1} \cdots \zeta_{n-1}^{i_{n-1}}$ with $a_j^i \in \mathbb{R}$. Conversely, any $\Lambda$ of this form preserves $X$.

**Proof.** The integral curves of $X$ are $\kappa_\zeta(\lambda) = (\lambda^{\tilde{m}_1} \zeta_1, \ldots, \lambda^{\tilde{m}_{n-1}} \zeta_{n-1})$, $\lambda \in \mathbb{R}$. Since $d\Lambda(X) = X$, $\Lambda$ transforms an integral curve passing through $\zeta$ to an integral curve passing through $\eta = \Lambda(\zeta)$. So we obtain

\[
(\lambda^{\tilde{m}_1} \Lambda_1(\zeta), \ldots, \lambda^{\tilde{m}_{n-1}} \Lambda_{n-1}(\zeta)) = (\Lambda_1(\kappa_\zeta(\lambda)), \ldots, \Lambda_{n-1}(\kappa_\zeta(\lambda))). \tag{2.1}
\]

Let $1 \leq j \leq n-1$ be fixed. We write $\Lambda_j$ as: $\Lambda_j(\zeta) = \Lambda^*(\zeta) + R(\zeta)$ where $\Lambda^*(\zeta) := \sum_{|I|_* = \tilde{m}} a^{i_1} \cdots i_{n-1} \zeta_1^{i_1} \cdots \zeta_{n-1}^{i_{n-1}}$ is non identically zero for a smallest integer $\tilde{m}$ that satisfies this condition: there exists a constant $C > 0$ such that $|R(\kappa_\zeta(\lambda))| \leq C|\lambda|^{\tilde{m}+1}$. From (2.1), we have

\[
\lambda^{\tilde{m}_j} \Lambda_j(\zeta) = \Lambda_j(\kappa_\zeta(\lambda)) = \lambda^{\tilde{m}} \Lambda^*(\zeta) + R(\kappa_\zeta(\lambda)). \tag{2.2}
\]

Now we divide (2.2) by $\lambda^{\tilde{m}}$. When $\lambda$ tends to 0 we obtain $\tilde{m} = \tilde{m}_j$ and $\Lambda_j(\zeta) = \Lambda^*(\zeta)$ for all $\zeta \in \mathbb{R}^{n-1}$. $\square$
So let the coordinates be chosen such that $X = \sum_i \tilde{m}_i \zeta \frac{\partial}{\partial \zeta_i}$. For $\zeta = (\zeta_1, \ldots, \zeta_{n-1})$, $\eta = (\eta_1, \ldots, \eta_{n-1}) \in \mathbb{R}^{n-1}$ and $\lambda, \mu \in \mathbb{R}$, we set $\sigma := \zeta + i.\eta \in \mathbb{C}^{n-1}$, $\kappa_\zeta(\lambda) := (\lambda_1^m \zeta_1, \ldots, \lambda_{n-1}^m \zeta_{n-1})$ and $\kappa_\sigma(\mu, \lambda) := \kappa_\zeta(\mu) + i.\kappa_\eta(\lambda)$. Let $\rho$ be a local defining function of $D$ at $p \in bD$ and $\overline{\gamma} : \tilde{V} \to \theta(\tilde{V}) := \tilde{M}$ be a holomorphic-extension (resp. almost-holomorphic extension) of the parametrization $\gamma$ of $M$. In the $C^\omega$-case $\tilde{M}$ is a complexification of $M$ and $\tilde{V}$ is an open neighborhood of the origin in $\mathbb{C}^{n-1}$. Let $M, K \in \mathbb{N}^*$ be such that $M \leq K$ and $m_j := M/\tilde{m}_j \in \mathbb{N}^*$, $k_j := K/\tilde{m}_j \in \mathbb{N}^*$. We set $E = \{\zeta \in \mathbb{R}^{n-1}/ \sum_j \zeta_j^{2m_j} = 1\}$. Now, we introduce our second hypothesis:

$(\mathcal{H}_2)$ There exist constants $\varepsilon > 0$, $0 < c \leq C$ such that for every $\sigma = \zeta + i.\eta \in E + i.\mathbf{E}$, $|\lambda| < \varepsilon$, $|\mu| < \varepsilon$, we have: $c|\lambda|^{2M}(|\mu| + |\lambda|)^{2(K-M)} \leq \rho(\overline{\gamma}(\kappa_\sigma(\mu, \lambda))) \leq C|\lambda|^{2M}(|\mu| + |\lambda|)^{2(K-M)}$.

**Definition 2.2.** — If a $C^\infty$ (resp. $C^\infty$)-vector field $X$ on $M$ verifies $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ we say that $X$ is peak-admissible of peak-type $(K, M; \tilde{m}_1, \ldots, \tilde{m}_{n-1})$ at $p \in M$ for the class $\mathcal{O}$ (resp. $A^\infty$).

**Remark 2.3.** —

1) The hypothesis $(\mathcal{H}_2)$ does not depend neither on the choice of the defining function of the boundary $bD$ nor the choice of the almost-holomorphic extension (see Lemma 4.3 in section 4).

2) The geometric meaning of $(\mathcal{H}_2)$ will become clear in inequality $(\mathcal{H})$.

**3. A sufficient condition for the existence of local peak set for the class $\mathcal{O}$**

**Theorem 3.1.** — Let $D$ be a pseudoconvex domain in $\mathbb{C}^n$ with $C^\omega$-boundary. Let $M$ be an $(n-1)$-dimensional $C^\omega$-submanifold in $bD$ that is totally real and complex-tangential at $p \in M$. We suppose that $M$ admits a peak-admissible $C^\omega$-vector field $X$ of peak-type $(K, M; \tilde{m}_1, \ldots, \tilde{m}_{n-1})$ at $p$ for $\mathcal{O}$. Then $M$ is a local peak set at $p$ for the class $\mathcal{O}$.

**Proof.** — The proof is based on Propositions 3.2 and 3.4 below after several holomorphic coordinates changes. Also we allow shrinking of $M$. □

**Proposition 3.2.** — Let $D$ be a domain in $\mathbb{C}^n$ with $C^\omega$ (resp. $C^\infty$)-boundary $bD$. Let $M$ be an $(n-1)$-dimensional $C^\omega$-submanifold in $bD$ which
Local Peak Sets in Weakly Pseudoconvex Boundaries in $\mathbb{C}^n$

is totally real and complex-tangential near $p$. Then there exists a holomorphic change (resp. an almost-holomorphic change) of coordinates $(Z, w)$ with $Z = X + iY \in \mathbb{C}^{n-1}$ and $w = u + iv \in \mathbb{C}$, such that $p$ corresponds to the origin and in an open neighborhood $U$ of the origin, we have:

i) $\mathbf{M} = \{(Z, w) \in U/Y = w = 0\}$. Moreover, $\mathbf{M}$ is contained in an $n$-dimensional totally real submanifold $\mathbf{N} = \{(Z, w) \in U/Y = u = 0\}$ of $bD$.

ii) For every $c \in \mathbb{R}$, $\mathbf{M}_c = \{(Z, w) \in \mathbf{N}/v = c\}$ is complex-tangential or empty.

iii) $D \cap U = \{(Z, w) \in U/\rho(Z, w) < 0\}$ with

$$\rho(Z, w) = u + A(Z) + vB(Z) + v^2R(Z, v).$$

iv) $A$ and $B$ vanish of order $\geq 2$ when $Y = 0$.

Proof. — We give the proof in the $C^\omega$-case. Let $\gamma$ be a $C^\omega$-parametrization of $\mathbf{M}$ defined on a neighborhood of the origin in $\mathbb{R}^{n-1}$. After a translation and a rotation of the coordinates in $\mathbb{C}^n$ we may assume that $p$ is the origin and the real tangent space at 0 to $bD$ is $T_0(bD) = \mathbb{C}^{n-1} \times i\mathbb{R}$. We set $L(Z, w) = i\mathbf{n}(Z, w)$ where $\mathbf{n}$ is the vector field of the outer exterior normal to $bD$. Then, for every $(Z, w) \in bD$, there exists a $C^\omega$-integral curve $l_{(Z, w)}(\lambda) \in bD$ of $L$ satisfying $l_{(Z, w)}(0) = (Z, w)$ and

$$\frac{dl_{(Z, w)}}{d\lambda}(\lambda) = L(l_{(Z, w)}(\lambda)).$$

Now, we consider the map $\theta : (t, \lambda) \mapsto l_{\gamma(t)}(\lambda)$. It is clear that $\theta$ is a $C^\omega$-diffeomorphism from a neighborhood $U$ of the origin in $\mathbb{R}^n$ into an $n$-dimensional submanifold $N' := \theta(U)$ of $bD$ which is totally real. By complexification of $\theta$ in a neighborhood $W$ of the origin in $\mathbb{C}^n$, we obtain in the new holomorphic coordinates $(Z', w')$, $M' = \{(Z', w') \in W/\gamma' = w' = 0\}$ and $N' = \{(Z', w') \in W/\gamma' = w' = 0\}$. We remark that the system

$$\{\Sigma_q = T_q(N') \cap T_q^{\omega}(bD), q \in W\}$$

is $C^\omega$ and involutive. By Frobenius theorem [Bo] the leaves $M'_c = \{(Z', w') \in W/\gamma'/w' = c\}_{c \in \mathbb{R}}$ are complex-tangential to $bD$. Now, we change coordinates again by defining: $Z = Z'$ and $w = iw'$. We obtain in a neighborhood $U$ of the origin $i)$ and $ii)$. Representing $bD$ as a graph over $\mathbb{C}^{n-1} \times i\mathbb{R}$, we obtain iii). Since $\mathbf{M} \subset bD$ is complex-tangential $A$ vanishes of order $\geq 2$ if $Y = 0$. As $\frac{\partial}{\partial v}$ is tangent to $\mathbf{N}$ and the complex gradient $\nabla p = (0, -1)$ is constant along $\mathbf{N}$, we obtain that $B$ vanishes of order $\geq 2$ if $Y = 0$. This achieves iv) and the proposition. □

Let the change of coordinates of Proposition 3.2 for the vector field $\mathbf{X}$ which verifies hypothesis $(H_2)$ be achieved. Now we show the impact of $(H_2)$. We set $\kappa := K/M = k_j/m_j \geq 1$. Since $\kappa$ is independent of $j$,
we define in a sufficiently small neighborhood \( V \) of the origin in \( \mathbb{C}^{n-1} \) the following pseudo-norms of the \( Z = (z_1, \ldots, z_{n-1}) \) coordinates of Proposition 3.2:

\[
||Y|| = \left( \sum_j y_j^{2m_j} \right)^{1/2M} \quad \text{and} \quad ||Z||_* = \left( \sum_j |z_j|^{2k_j} \right)^{1/2K}.
\]

We note that

\[
A(Z) = \rho(\tilde{\gamma}(\kappa_{\sigma}(\mu, \lambda))) \quad \text{where} \quad Z = X + iY = \kappa_{\sigma}(\mu, \lambda).
\]

Therefore, from now on we may assume that \( A \) verifies:

\[
(\mathcal{H}) \quad \text{There exist two constants } 0 < c \leq C \text{ such that, for every } Z = X + iY \in \mathbb{C}^{n-1} \text{ near the origin, we have:}
\]

\[
c||Y||_*^{2M} \cdot ||Z||_*^{2K-2M} \leq A(Z) \leq C||Y||_*^{2M} \cdot ||Z||_*^{2K-2M}.
\]

Remark 3.3. —

1) The proof of Proposition 3.2 remains true in the \( C^\infty \)-case. We indicate the modification in Lemma 4.2 (section 4).

2) If \( Z = (z_1, \ldots, z_{n-1}) \in \mathcal{V} \) where \( \mathcal{V} \) is a small open neighborhood of the origin in \( \mathbb{C}^{n-1} \), then \( \sum z_j|^{2(k_j-m_j)} \approx \left( \sum |z_j|^{2m_j} \right)^{\kappa-1} \).

Moreover, we may replace \( k_j \) by \( m_j \) and \( K \) by \( M \) in the definition of the pseudo-norm \( ||Z||_* \).

3) If \( K = M = \tilde{m}_1 = \ldots = \tilde{m}_{n-1} = 1 \), we find the property on \( A \) for a strongly pseudoconvex boundary.

Proposition 3.4. —

1) If the real hyperplane \( H = \mathbb{C}^{n-1} \times \mathbb{R} = \{(Z, iv)/Z \in \mathbb{C}^{n-1}, v \in \mathbb{R}\} \) lies outside of \( D \) in a neighborhood \( \mathcal{U} \) of the origin, then there exists a constant \( T > 0 \) such that \( B^2 \leq TA \) near the origin.

2) If there exists a constant \( T > 0 \) such that \( B^2 \leq TA \) near the origin, then there exist a sufficiently small neighborhood \( \mathcal{U} \) of the origin and a holomorphic function \( \psi \) on \( \mathcal{U} \) (resp. an almost-holomorphic function with respect to \( \mathcal{N} \cap \mathcal{U} \)) which satisfies: \( \Re \psi < 0 \) on \( \overline{D} \cap \mathcal{U} \) if \( w \neq 0 \) and \( \psi = 0 \) if \( w = 0 \). Here \( \psi = \frac{w}{1 - 2K_1w} \) with a suitable constant \( K_1 > 0 \).

Proof. — The proof is elementary. See also [B-I]. \( \square \)

In order to apply Proposition 3.4 2), we should determine the order of vanishing for certain functions on \( M \) at \( p = 0 \in M \). We begin by defining the \( Z \)-weights and the \( Y \)-weights for polynomial functions.
**Definition 3.5.** — Let $\chi = a_{I,J} z_1^{i_1} \ldots z_n^{i_n}$, with $a_{I,J} \neq 0$, be a monomial. We define the $Z$-weight $P_Z(\chi)$ of $\chi$ as: $P_Z(\chi) = \sum_{\nu} \tilde{m}_\nu (i_\nu + j_\nu)$.

If $g \neq 0$ is a polynomial function in $Z$ and $\bar{Z}$ we define the $Z$-weight of $g$ as the smallest $Z$-weight in the decomposition of $g$ by monomials. If $g$ is a sum of monomials which have the same $Z$-weight $L$, we say that $g$ is homogeneous with respect to the $Z$-weight. Let $X \in \mathbb{R}^{n-1}$ be fixed and $\Xi = \alpha_{I,J} (X) y_1^{i_1} \ldots y_n^{i_n}$, with $\alpha_{I,J} (X) \neq 0$, be a monomial at $Y$. We define the $Y$-weight $P_Y(\Xi)$ of $\chi$ as $\sum_{\nu} \tilde{m}_\nu i_\nu$. If $h \neq 0$ is a polynomial function in $Y$ we define the $Y$-weight of $h$ to be the smallest $Y$-weight in the decomposition of $h$. If $h$ is a sum of monomials which have the same $Y$-weight $L'$, we say that $h$ is homogeneous with respect to the $Y$-weight of order $L'$.

**Lemma 3.6.** — Let $R, S \in \mathbb{N}$, $R \geq S$ and $F(X,Y) = \sum_{I,J} F_{I,J} Y^I X^J$ be a $C^\omega$-function on an open neighborhood of the origin of $\mathbb{C}^{n-1}$ such that, for all multi-indices $I = (i_1, \ldots, i_{n-1})$, $J = (j_1, \ldots, j_{n-1})$ in $\mathbb{N}^{n-1}$, $F_{I,J} = 0$ or $P_Y(F_{I,J} Y^I X^J) \geq S$ and $P_Z(F_{I,J} Y^I X^J) \geq R \geq S$. Then, there exists a constant $C > 0$ such that, $|F(Z)| \leq C ||Y||_N^S ||Z||_N^{R-S}$, $\forall Z = X + i Y$ near the origin.

**Proof.** — This can be seen by Taylor expansion and standard arguments.

**Lemma 3.7.** — With the notations of Lemma 3.6, if $S \geq M$ and $R \geq K = \kappa M$, then $\frac{|F|^2}{A}$ is uniformly bounded on a sufficiently small neighborhood of the origin.

**Proof.** — This follows immediately from Lemma 3.6 and inequality $(\mathcal{H})$.

In order to know the weights of $A$ and $B$ we analyze the restrictions which are imposed on the functions $A$ and $B$ by the pseudoconvexity of $bD$. We assume that $B \neq 0$ and we set $(P_Y(B), P_Z(B)) = (S, R)$. From $(\mathcal{H})$ we have $(P_Y(A), P_Z(A)) = (2M, 2K)$. Next, a simple computation of the Levi form at a point near the origin to $bD$ for $t = \sum_{\nu} \tilde{m}_\nu y_\nu \chi_\nu \in T^C(bD)$, with $\chi_\nu = i \left[ \frac{\partial}{\partial z_\nu} - \frac{i}{\eta} \frac{\partial}{\partial z_\nu} \frac{\partial}{\partial w} \right]$ and $\eta = \frac{1}{2} \left( i + B + 2vR + v^2 \frac{\partial R}{\partial v} \right)$, gives $\text{Lev}_\nu[t] = \mathcal{A}(Z) + v \mathcal{B}(Z) + v^2 \mathcal{R}(v, Z)$, $Z$ varying on $\overline{M}$, the complexification of $M$. By pseudoconvexity of $bD$ and Proposition 3.4.1) there exists a positive constant $T^* > 0$ such that

$$B \geq T^* A.$$  \hspace{1cm} (3.1)
It remains to study the $Z$-weight and $Y$-weight of $A$ and $B$ and their relationship with the weights of $A$ and $B$ and finally to show $S \geq M$ and $R \geq K$. Some necessary auxiliaries results are given in Lemmas 3.8 and 3.9 below. We denote by $\partial^2_{\nu\mu}$ the partial derivative $\frac{\partial^2}{\partial \nu \partial \mu}$ and $O_Y(L)$ (resp. $O_Z(L)$) is the set of functions that admit an $Y$-weight (resp. a $Z$-weight) $\geq L$ ($L \in \mathbb{N}$).

- Suppose that $S < M$.

The expressions of $A$ and $B$ are:

$$A = \sum_{\nu,\mu} \partial^2_{\nu\mu} A \tilde{m}_\nu \tilde{m}_\mu y_\nu y_\mu + O_Y(2M + 1)$$

$$B = \sum_{\nu,\mu} \partial^2_{\nu\mu} B \tilde{m}_\nu \tilde{m}_\mu y_\nu y_\mu + O_Y(2S).$$

By Lemma 3.8 $A = A_{2M} + \bar{A}$ with $\mathcal{P}_Y(A_{2M}) = 2M$ and every term of $\bar{A}$ has an $Y$-weight $> 2M$. We put $A_{2M} := \sum_{\nu,\mu} \partial^2_{\nu\mu} A_{2M} \tilde{m}_\nu \tilde{m}_\mu y_\nu y_\mu$. By Lemma 3.9 we obtain $A_{2M} \neq 0$ and $\mathcal{P}_Y(A_{2M}) = 2M$. Similary, we have $B = B_S + \bar{B}_S$ where every term of $\bar{B}_S$ has an $Y$-weight $> 2M$. We put $B_S := \sum_{\nu,\mu} \partial^2_{\nu\mu} B_S \tilde{m}_\nu \tilde{m}_\mu y_\nu y_\mu$. We obtain $B_S \neq 0$ and $\mathcal{P}_Y(B_S) = S$. Inequality (3.1) becomes:

$$(B_S + O_Y(S + 1)^2 \leq T^*(A_{2M} + O_Y(2M + 1)). \quad (3.2)$$

Since $B_S \neq 0$ there exists $Z_0 = X_0 + iY_0$ with $Y_0 = (y_{0,1}, \ldots, y_{0,n-1}) \neq 0$ such that $B_S(Z_0) \neq 0$. Since every term in the decomposition of $B_S$ has an $Y$-weight $S$, we consider for $\lambda > 0$, $\phi_{Y_0}(\lambda) = (\lambda^m y_{0,1}, \ldots, \lambda^m y_{0,n-1})$. Then $B_S(X_0 + i, \phi_{Y_0}(\lambda))$ becomes an homogeneous polynomial in $\lambda$ of degree $S$ (i.e. $B_S(X_0 + i, \phi_{Y_0}(\lambda)) = \lambda^S B_S(X_0 + i, Y_0)$). Therefore, we obtain

$$\lim_{\lambda \to 0^+} \frac{1}{\lambda^S} B_S(X_0 + i, \phi_{Y_0}(\lambda)) \neq 0.$$ 

Now we replace $Z$ by $X_0 + i, \phi_{Y_0}(\lambda)$ in inequality (3.2) and divide by $\lambda^{2S}$. We obtain $B_S^2(X_0 + i, \phi_{Y_0}) \leq 0$ when $\lambda$ tends to $0^+$. So $B_S(X_0 + i, Y_0) = 0$ which is a contradiction. Thus, $S \geq M$.

- The case $R < K$ can be falsified in an analogous way by using Lemma 3.9.

Now Lemma 3.7 shows that $\frac{|B|^2}{A}$ is uniformly bounded. Then Proposition 3.4 implies the theorem.

**Lemma 3.8.** — Let $X = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ be fixed and $P_X \in \mathbb{R}[y_1, \ldots, y_{n-1}]$ be homogeneous with respect to the $Y$-weight $L$. Then we have the following equations:
Local Peak Sets in Weakly Pseudoconvex Boundaries in $\mathbb{C}^n$

1) \[ \sum_{\nu=1}^{n-1} \frac{\partial P_X}{\partial y_\nu}(y_1, \ldots, y_{n-1})\tilde{m}_\nu y_\nu = LP_X(y_1, \ldots, y_n). \]

2) \[ \sum_{\nu, \mu} \frac{\partial^2 P_X}{\partial y_\nu \partial y_\mu}(y_1, \ldots, y_{n-1})\tilde{m}_\nu \tilde{m}_\mu y_\nu y_\mu + \sum_{\nu=1}^{n-1} \frac{\partial P_X}{\partial y_\nu}(y_1, \ldots, y_{n-1})\tilde{m}_\nu^2 y_\nu = L^2 P_X(y_1, \ldots, y_{n-1}). \]

Proof. — For $1 \leq \nu \leq n - 1$, we set $y_\nu = \tilde{y}_\nu^{m_\nu}$. Now, we consider the polynomial $Q_X$ defined by: $Q_X(\tilde{y}_1, \ldots, \tilde{y}_{n-1}) = P_X(\tilde{y}_1^{m_1}, \ldots, \tilde{y}_{n-1}^{m_{n-1}})$. $Q_X$ is an homogeneous polynomial at $\tilde{Y} = (\tilde{y}_1, \ldots, \tilde{y}_{n-1})$ in the classic sense, of degree $L$. Then the result follows from Euler’s equation. \hfill \Box

**LEMMA 3.9.** — If $P_X \not\equiv 0$ is a polynomial in $\mathbb{R}[y_1, \ldots, y_{n-1}]$ not containing neither constant nor linear terms which is homogeneous with respect to the $Y$-weight $L \geq 2$ then \[ \sum_{\nu, \mu} \frac{\partial^2 P_X}{\partial y_\nu \partial y_\mu}(y_1, \ldots, y_{n-1})\tilde{m}_\nu \tilde{m}_\mu y_\nu y_\mu \not\equiv 0. \]

Proof. — Let $P_X$ be a polynomial which depends exactly on $(n - r - 1)$-variables, where $0 \leq r \leq n - 2$. By a permutation of variables we may assume that $P_X(y_{r+1}, \ldots, y_{n-1}) = \sum_{I=(i_{r+1}, \ldots, i_{n-1})} a_I(X) y_{r+1}^{i_{r+1}} \cdots y_{n-1}^{i_{n-1}}$. We suppose that the assertion of lemma is false. From Lemma 3.8, we have \[ \sum_{\nu=r+1}^{n-1} \frac{\partial P_X}{\partial y_\nu}(y_1, \ldots, y_{n-1})\tilde{m}_\nu^2 y_\nu = L^2 P_X. \]

Since $\sum_{\nu=r+1}^{n-1} \frac{\partial P_X}{\partial y_\nu}(y_1, \ldots, y_{n-1})\tilde{m}_\nu y_\nu = LP_X$ we get, for all $(y_{r+1}, \ldots, y_{n-1})$:

\[ \sum_{\nu=r+1}^{n-1} \tilde{m}_\nu (L - \tilde{m}_\nu) \frac{\partial P_X}{\partial y_\nu}(y_{r+1}, \ldots, y_{n-1})y_\nu = 0 \quad (3.3) \]

Now, for every $r + 1 \leq \nu \leq n - 1$, we set $\tau_\nu = \tilde{m}_\nu (L - \tilde{m}_\nu)$. We have $\tau_\nu > 0$. In fact, let us suppose that $\tau_\mu = 0$ for a $\mu$ with $r + 1 \leq \mu \leq n - 1$.

For every term of $P_X$ we have: $L = \sum_{\nu=r+1}^{n-1} \tilde{m}_\nu i_\nu$. Then, two cases are possible for this term:
• $i_\mu = 1$ and $i_\nu = 0$ for all $\nu \neq \mu$.
• $i_\mu = 0$.

Since there are no linear terms, the first case is impossible. So, $i_\mu = 0$ for this term. But, this is also impossible from the choice of variables.

Now we show that $P_X$ vanishes identically. In fact, let $Y \neq 0$ be fixed. We consider $f(\lambda) = P_X(\lambda^{r+1} y_{r+1}, \ldots, \lambda^{n-1} y_{n-1})$, $\lambda > 0$. So, we have:

$$f'(\lambda) = \sum_{j=r+1}^{n-1} \frac{\partial P_X}{\partial y_j}(\lambda^{r+1} y_{r+1}, \ldots, \lambda^{n-1} y_{n-1}) \tau_j \lambda^{\tau_j-1} y_j.$$  

For $r + 1 \leq j \leq n - 1$, we set $w_j = \lambda^{\tau_j} y_j$. We get by (3.3):

$$f'(\lambda) = \frac{1}{\lambda} \sum_{j=r+1}^{n-1} \tau_j w_j \frac{\partial P_X}{\partial y_j}(w_{r+1}, \ldots, w_{n-1}) = 0.$$  

So, $f$ is constant. As $f(1) = P_X(y_{r+1}, \ldots, y_{n-1}) = \lim_{\lambda \to 0} f(\lambda) = P_X(0) = 0$, $P_X$ vanishes identically. Therefore, we obtain a contradiction.  

\[\square\]

4. A sufficient condition for the existence of a local peak sets for the class $A^\infty$

This part was inspired by the article of Hakim and Sibony [H-S]. The following lemma can be shown by standard methods [Na].

**Lemma 4.1.** — Let $\widetilde{U}_X$ be a neighborhood of the origin in $\mathbb{R}^n$ and $h : (X, Y) \mapsto h(X, Y)$ a $C^\infty$-function on $\widetilde{U}_X \times \mathbb{R}^n$. We suppose that $h$ is $m$-flat where $Y = 0$. Then there exist a neighborhood $V_Y$ of the origin in $\mathbb{R}^n$, a neighborhood $U_X \subset \subset \widetilde{U}_X$ of the origin and a function $g \in C^\infty(U_X \times \mathbb{R}^n)$ which vanishes on $U_X \times V_Y$ and verifies for $\varepsilon > 0$:

$$\|g - h\|_{U_m^X \times \mathbb{R}} < \varepsilon.$$  

**Lemma 4.2.** — Let $\theta : \widetilde{U} \to \mathbb{C}^n$ be a $C^\infty$-parametrization of the submanifold $N$ in a neighborhood of the origin in $\mathbb{R}^n$. Then $\theta$ has an extension $\widetilde{\theta}$ defined on a neighborhood $U$ of the origin in $\mathbb{C}^n$ and which is almostholomorphic with respect to $N \cap U$.

**Proof.** — Let $T_m(X, Y) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D_X^\alpha \theta(X)(iY)^\alpha$ and $U_X \subset \subset \widetilde{U}_X$ be a neighborhood of the origin in $\mathbb{R}^n$. For $k \in \mathbb{N}$ it is clear that $T_{k+1} - T_k$ is $k$-flat at $Y$ when $Y = 0$. Now we apply the preceding Lemma 4.1 to $T_{k+1} - T_k$. 

\[\text{– 586 –}\]
Local Peak Sets in Weakly Pseudoconvex Boundaries in \( \mathbb{C}^n \)

Then there exist a neighborhood \( V_k^\gamma \) of the origin in \( \mathbb{R}^n \) and a \( C^\infty \)-function \( g_k(X,Y) \) which vanishes on \( U_X \times V_k^\gamma \) such that

\[
||T_{k+1} - T_k - g_k||_{U_X \times \mathbb{R}^n}^k < 2^{-k}. \tag{4.1}
\]

For \( m \in \mathbb{N}^* \), we set \( \tilde{T}_m := T_0 + \sum_{k=0}^m (T_{k+1} - T_k - g_k) \in C^\infty(U_X \times \mathbb{R}^n) \). By (4.1) \( \sum_{k} (T_{k+1} - T_k - g_k) \) is a normal series for all norms \( C^l \) on \( U_X \times \mathbb{R}^n \), \( l \in \mathbb{N} \). So, the sequence \( (\tilde{T}_m)_m \) converges uniformly to \( \tilde{\theta} \in C^\infty(U_X \times \mathbb{R}^n) \). It is clear that for \( m \) and \( k \), \( T_m(X,0) = \theta(X) \), \( g_k(X,0) = 0 \). Hence, \( \tilde{\theta}(X,0) = \lim_{m \to +\infty} \tilde{T}_m(X,0) = \theta(X) \). So \( \theta \) is an \( C^\infty \)-extension of \( \theta \) on \( U_X \times \mathbb{R}^n \). That \( \tilde{\theta} \) is almost-holomorphic with respect to \( U_X \times \mathbb{R}^n \) can be seen by similar arguments as in [H-S]. \( \square \)

The following lemma shows that \( (\mathcal{H}_2) \) does not depend on the choice of the almost-holomorphic extension.

**Lemma 4.3.** — Let \( \bar{\gamma} : \bar{V} \longrightarrow \mathbb{C}^{n-1} \) be an almost-holomorphic extension of \( \gamma \) with respect to \( \bar{V} \cap \mathbb{R}^{n-1} \) which satisfies the hypothesis \( (\mathcal{H}_2) \) (here \( \gamma \) is the \( C^\infty \)-parametrization of \( \mathcal{M} \) defined in section 2). Let \( \bar{\phi} : \bar{W} \longrightarrow \mathbb{C}^{n-1} \) be another almost-holomorphic extension of \( \gamma \) with respect to \( \bar{W} \cap \mathbb{R}^{n-1} \). Then, the hypothesis \( (\mathcal{H}_2) \) is satisfied for \( \bar{\phi} \).

**Proof.** — The passage from \( \bar{\gamma} \) to \( \bar{\phi} \) is given by the transformation \( \bar{\psi} : \bar{W} \longrightarrow \bar{V} \) which is almost-holomorphic with respect to \( \bar{W} \cap \mathbb{R}^{n-1} \). So, we have \( \bar{\psi}|_{\bar{W} \cap \mathbb{R}^{n-1}} = Id \) and \( \bar{\phi} = \bar{\gamma} \circ \bar{\psi} \). It is sufficient to prove for every \( \sigma \in \bar{W} \) and for all \( l \in \mathbb{N} \): \( |\bar{\psi}(\sigma) - \sigma| \lesssim |\Im \sigma|^l \).

Let \( \sigma = \zeta + i.\eta \) with \( \zeta \in \bar{W} \cap \mathbb{R}^{n-1} \) and \( l \in \mathbb{N} \) be fixed. Then, we have

\[
\bar{\psi}(\sigma) = \sum_{|I| \leq l} \frac{1}{I!} \frac{\partial^{\vert I \vert} \bar{\psi}}{\partial \eta^I}(\zeta)\eta^I + O(|\eta|^{l+1}).
\]

\[
\bar{\psi}(\sigma) = \zeta + \sum_{1 \leq |I| \leq l} \frac{1}{I!} \frac{\partial^{\vert I \vert} \bar{\psi}}{\partial \eta^I}(\zeta)\eta^I + O(|\eta|^{l+1}).
\]

So we can write \( \bar{\psi} \) as \( \bar{\psi}(\sigma) = \zeta + \sum_{j=1}^l \bar{\psi}^{(j)}(\sigma) + O(|\eta|^{l+1}) \) with \( \bar{\psi}^{(j)}(\sigma) = \sum_{|I|=j} \frac{1}{I!} \frac{\partial^{\vert I \vert} \bar{\psi}}{\partial \eta^I}(\zeta)\eta^I \). In particular,
we have
\[
\tilde{\psi}(\sigma) = \zeta + \tilde{\psi}^{(1)}(\sigma) + O(|\eta|^2) = \sum_{i=1}^{n-1} \frac{\partial \tilde{\psi}}{\partial \eta_i}(\zeta) \eta_i + O(|\eta|^2).
\]

Since \( \partial \tilde{\psi} = O(|\eta|) \), we have \( \delta_{kj} \ldots \tilde{U} \) of \( p, \tilde{\theta} \) preserves the distances. In particular, we have: \( \text{dist}(q',N') \approx \text{dist}(q,N) \) with \( q' = \tilde{\theta}(q) \) and \( q \in \tilde{U} \).

Proof. — We set \( N' = \tilde{\theta}(N) \) and \( M' = \tilde{\theta}(M) \). Since \( \tilde{\theta} \) is a local \( C^\infty \)-diffeomorphism on an open neighborhood \( \tilde{U} \) of \( p, \tilde{\theta} \) preserves the distances. In particular, we have: \( \text{dist}(q',N') \approx \text{dist}(q,N) \) with \( q' = \tilde{\theta}(q) \) and \( q \in \tilde{U} \).
Local Peak Sets in Weakly Pseudoconvex Boundaries in $\mathbb{C}^n$

Set $\Psi = \tilde{\theta}^{-1}$, $w = z_n$ and $w' = z'_n$. Since $\tilde{\theta}$ is an almost-holomorphic change of coordinates, the matrix

$$\left\{ \frac{\partial \Psi_i}{\partial z'_j} \right\}_{1 \leq i, j \leq n}^{1 \leq i \leq n}$$

is nonsingular \hspace{1cm} (4.2)

on a sufficiently small neighborhood of the origin.

For $1 \leq i \leq n$, we have

$$\frac{\partial}{\partial z'_j} = \sum_{j=1}^{n} \frac{\partial \Psi_j}{\partial z'_i} \frac{\partial}{\partial z'_j} + \sum_{j=1}^{n} \frac{\partial \Psi_j}{\partial z'_i} \frac{\partial}{\partial z_j}$$

$$= \sum_{j=1}^{n} \frac{\partial \Psi_j}{\partial z'_i} \frac{\partial}{\partial z'_j} + \sum_{j=1}^{n} O \left( \text{dist}(q, N)^{L+1} \right) \frac{\partial}{\partial z_j}$$

The domain $D'$ is defined by $\rho' = \rho \circ \Psi$. Let $t' = (t'_1, \ldots, t'_n) \in T^C_q(bD')$

Thus $\sum_{j=1}^{n} \frac{\partial \rho'(q^i)}{\partial z'_j} t'_j = 0$. This implies

$$\sum_{i, j=1}^{n} \frac{\partial \rho}{\partial z_i} \frac{\partial \Psi_i}{\partial z'_i} t'_j + O \left( \text{dist}(q, N)^{L+1} \right) = 0.$$ 

For $1 \leq i \leq n$ we set $t_i = \sum_{i, j=1}^{n} \frac{\partial \Psi_i}{\partial z'_i} t'_j$.

From (4.2) we get: $\sum_{i=1}^{n} \frac{\partial \rho}{\partial z_i} t_i = O \left( |t'| \text{dist}(q, N)^{L+1} \right) = O \left( |t| \text{dist}(q, N)^{L+1} \right)$.

Now we decompose $t$ into tangential component $t^H$ and a normal component $t^N$. So, $t = t^H + t^N$ with $t^H \in T^C_q(bD)$, $t^N \perp T^C_q(bD)$ and $|t^H| + |t^N| \leq 2|t|$. Moreover, $t^N = \kappa(q) \mathbf{n}(q)$ with $\kappa(q) \in \mathbb{C}$ and, for all $1 \leq i \leq n$, we have $t^N_i = \kappa(q) \frac{\partial \rho(q)}{\partial z_i}$. This implies

$$\kappa(q) \sum_{i=1}^{n} \left| \frac{\partial \rho(q)}{\partial z_i} \right|^2 = \sum_{i=1}^{n} \frac{\partial \rho(q)}{\partial z_i} \kappa(q) \frac{\partial \rho(q)}{\partial z_i}$$

$$= \sum_{i=1}^{n} \frac{\partial \rho(q)}{\partial z_i} t_i^N = \sum_{i=1}^{n} \frac{\partial \rho(q)}{\partial z_i} t_i$$

$$= O \left( |t| \text{dist}(q, N)^{L+1} \right).$$

Consequently,

$$|t^N| = |\kappa(q)| = O \left( |t| \text{dist}(q, N)^{L+1} \right).$$

(4.3)
Now, we compute the Levi form of $\rho'$. As
\[
\frac{\partial \rho'(q')}{\partial z'_i} = \sum_{i=1}^{n} \frac{\partial \rho(q)}{\partial z'_i} \frac{\partial \Psi_i(q')}{\partial z'_i} + O(\text{dist}(q, N)^{L+1})
\]
and by replacing $L$ by $L + 1$, we get
\[
\frac{\partial^2 \rho'(q')}{\partial z'_i \partial z'_j} = \sum_{k,l=1}^{n} \frac{\partial^2 \rho(q)}{\partial z_k \partial z'_l} \frac{\partial \Psi_i(q')}{\partial z'_i} \frac{\partial \Psi_j(q')}{\partial z'_j} + O(\text{dist}(q, N)^{L+1})
\]
By (4.3) it follows that
\[
\sum_{i,j=1}^{n} \frac{\partial^2 \rho'(q')}{\partial z'_i \partial z'_j} t'_i t'_j = \sum_{k,l=1}^{n} \frac{\partial^2 \rho(q)}{\partial z_k \partial z'_l} \left( \sum_{i=1}^{n} \frac{\partial \Psi_i(q')}{\partial z'_i} t'_i \right) \left( \sum_{j=1}^{n} \frac{\partial \Psi_j(q')}{\partial z'_j} t'_j \right) + O(\text{dist}(q, N)^{L+1})
\]
\[
= \sum_{k,l=1}^{n} \frac{\partial^2 \rho(q)}{\partial z_k \partial z'_l} t'_k t'_l + O(|t|^2 \text{dist}(q, N)^{L+1})
\]
From $(\mathcal{H}_3)$ and (4.3) we get:
\[
\sum_{k,l=1}^{n} \frac{\partial^2 \rho(q)}{\partial z_k \partial z'_l} t'_k t'_l \geq C |t|^2 \text{dist}(q, N)^L
\]
\[
\geq C |t|^2 \text{dist}(q, N)^L + O(|t|^2 \text{dist}(q, N)^{L+1})
\]
Thus there exists a constant $C' > 0$ such that $\text{Lev } \rho'(q'[t']) \geq C'|t|^2 \text{dist}(q, N)^L$. This means that $D'$ is a locally pseudoconvex at the origin. □

**Definition 4.5.** — Let $F$ be a $C^\infty$-function on a neighborhood $\mathcal{V}$ of the origin in $\mathbb{C}^{n-1}$. We say that $F$ has $Y$-weight $\mathcal{P}_Y(F) \geq S$ ($S \in \mathbb{N}$) if there exists a constant $C > 0$ such that $|F(X,Y)| \leq C ||Y||_s^S$, $\forall Z = X + iY \in \mathcal{V}$. Also, we say that $F$ has $Z$-weight $\mathcal{P}_Z(F) \geq R \geq S$ ($R \in \mathbb{N}$) if there exists a constant $c > 0$ such that $|F(X,Y)| \leq c ||Z||_s^R$, $\forall Z = X + iY \in \mathcal{V}$.

In the sequel we have to take into account the following obvious assertions.

**Remark 4.6.** —

1) Let $F$ be a polynomial function with respect to $Y$. Then $\mathcal{P}_Y(F) \geq S$ if and only if
\[
S \iff F(X,Y) = \sum_{I=(i_1,\ldots,i_{n-1})}^{n-1} F_I(X)Y^I \text{ with } \sum_{\nu=1}^{n-1} \bar{m}_\nu i_\nu \geq S.
\]
Local Peak Sets in Weakly Pseudoconvex Boundaries in $\mathbb{C}^n$

2) Let $F$ be a polynomial function with respect to $X$ and $Y$. Then

$$\mathcal{P}_Z(F) \geq R \iff F(X,Y) = \sum_{I=(i_1,\ldots,i_{n-1})}^n \sum_{J=(j_1,\ldots,j_{n-1})} F_{I,J} X^I Y^J$$

with $\sum_{\nu=1}^{n-1} \tilde{m}_\nu (i_\nu + j_\nu) \geq R$.

3) If $||Y|| < 1$ then there exists a constant $a > 0$ such that $||Y|| \leq a ||Y||^*$.

Now, we give a version of Lemma 3.6 in the $C^\infty$-case. Its proof is similar.

**Lemma 4.7.** — Let $R, S \in \mathbb{N}$, $R \geq S$ and $F$ be a $C^\infty$-function on an open sufficiently small neighborhood $\mathcal{V}$ of the origin in $\mathbb{C}^n$. We suppose

\begin{itemize}
  \item There exist two positives constants $C$ and $L$ such that
    \begin{equation}
      \text{Lev} \rho(q)[t] \geq C|t|^2 \text{dist}(q, M)^L, \forall q \in \mathcal{U} \cap bD, \forall t \in T^\mathbb{C}_q(bD).
    \end{equation}
  \item $M$ admits a peak-admissible $C^\infty$-vector field $X$ of peak-type $(K, M; \tilde{m}_1, \ldots, \tilde{m}_{n-1})$ at $p$ for $A^\infty$.
\end{itemize}

Then,

i) $M$ is a local peak set at $p$ for the class $A^\infty$.

ii) $M$ is a local interpolation set at $p$ for the class $A^\infty$.

**Proof.** — i) After an almost-analytic change of coordinates we obtain the following properties: The point $p \in M$ corresponds to the origin and in an open neighborhood of the origin, we have $M' = \tilde{\theta}(M) = \{(Z', w')/Y' = w' = 0\}$, $D' = \tilde{\theta}(D)$ has $\rho'(Z', w') = u' + A(Z') + v'B(Z') + v'^2 R(Z', v')$ as local defining function at the origin. Moreover, $M'$ is locally contained in an $n$-dimensional submanifold $N' = \{(Z', w')/Y' = 0 \text{ and } u' = 0\}$ of $bD'$ which is totally real. By Lemma 4.4, the condition $(\mathcal{H}'_3)$ guarantees that $D'$ is a locally pseudoconvex at the origin. Moreover, the hypothesis on $M$ implies:
There exist two constants $0 < c_1' \leq c'2$ such that, for every $Z' = X' + i.Y' \in \mathbb{C}^{n-1}$ near the origin, we have:

$$c_1'||Y'||^2_{*}||Z'||^{2K-2M}_{*} \leq A(Z') \leq c_2'||Y'||^2_{*}||Z'||^{2K-2M}_{*}.$$ 

From (H) and Lemma 4.7 we get $\frac{|B|^2}{A}$ is uniformly bounded in a sufficiently small neighborhood of the origin in $\mathbb{C}^{n-1}$. By Proposition 3.4, there exists an almost-holomorphic function with respect to $N' \cap \mathcal{U}'$, $\widetilde{\psi}(w') = \frac{w'}{1-2K_1w'}$ defined on an open neighborhood $\mathcal{U}'$ of the origin in $\mathbb{C}^{n}$ such that: $\Re \widetilde{\psi} < 0$ on $\overline{D'} \cap \mathcal{U}'$ if $w' \neq 0$ and $\psi = 0$ if $w' = 0$.

As $|\tilde{\psi}(w')| \lesssim |w'|$, we have for every $(Z', w') \in \overline{D'} \cap \mathcal{U}'$,

$$A(Z') = \rho'(Z', w') - v'B(Z') - v'^2R(Z', v') - u' \leq -v'B(Z') - v'^2R(Z', v') - u' \lesssim |u'| + |v'| \lesssim |w'|.$$

Moreover, if $\mathcal{U}'$ is sufficiently small we get:

$$\text{dist}((Z', w'), M') \lesssim ||Y'|| + |w'|. \quad (4.4)$$

Since $||Y'||^2_{*}||Z'||^{2(K-M)}_{*} \lesssim A(Z') \lesssim |w'|$ and $||Y'||_{*} \leq ||Z'||_{*}$ we have $||Y'||^{2K}_{*} \lesssim |w'|$. By Remark 4.6 inequality (4.4) gives: For every $(Z', w') \in \overline{D'} \cap \mathcal{U}'$: $\text{dist}((Z', w'), M') \lesssim |w'|^{1/2K}$. This has two consequences:

a) $\overline{D}' \left( \frac{1}{\widetilde{\psi}} \right)$ has a $C^\infty$-extension on $\mathcal{U}' \cap \overline{D'}$.

b) If $F \in C^\infty(\mathcal{U}' \cap D')$ is an almost-holomorphic function with respect to $N' \cap \mathcal{U}'$ then $\frac{1}{\widetilde{\psi}} F$ has a $C^\infty$-extension on $\mathcal{U}' \cap \overline{D'}$.

(Here $\overline{\partial'}$ denotes the $\overline{\partial}$-operator on $D'$. Set $\widetilde{\Psi} := \overline{\partial}^{-1}$. If $f' \in C^\infty(\mathcal{U}' \cap D')$ then $\overline{\partial}' f' = \widetilde{\Psi}^* \left( \overline{\partial} (f' \circ \overline{\partial}) \right)$ where $\widetilde{\Psi}^*$ is the pull-back of $\widetilde{\Psi}$).

Proof. —

a) On $\mathcal{U}' \cap D'$ we have $\overline{\partial}' \left( \frac{1}{\psi} \right) = - \left( \frac{1-2K_1w'}{w'} \right)^2 \overline{\partial}' \tilde{\psi}$. As $\tilde{\psi}$ is an almost-holomorphic function with respect to $N' \cap \mathcal{U}'$ we get for all $L \in \mathbb{N}^*$ and $(Z', w') \in \mathcal{U}' \cap \overline{D'}$,

$$|\overline{\partial}' \tilde{\psi}(w')| \lesssim \text{dist}((Z', w'), N')^L \lesssim \text{dist}((Z', w'), M')^L \lesssim |w'|^{L/2K}. \quad (4.5)$$
b) With an analogous reasoning, we have for every \((Z', w') \in \mathcal{U}' \cap \mathcal{D}'\) and for all \(L \in \mathbb{N}^*\), \(|\overline{\partial} F(Z', w')| \lesssim \text{dist}((Z', w'), \mathcal{M})^L \lesssim |w'|^{L/2}\). By (4.5) we see that the \((0,1)\)-form \(\overline{\partial} \left(\frac{1}{\psi}\right)\) has a \(\overline{\partial}\)-closed \(C^\infty\)-extension on \(\mathcal{U}' \cap \mathcal{D}'\). We set \(\psi = \tilde{\psi} \circ \tilde{\theta}\) and get that \(\overline{\partial} \left(\frac{1}{\psi}\right)\) is a \(\overline{\partial}\)-closed \((0,1)\)-form of class \(C^\infty\) on \(\mathcal{U} \cap \mathcal{D}\).

Let \(0 < \varepsilon \ll 1\) be such that \(\overline{B}(0, \varepsilon) \subset \mathcal{U}\) and \(bB(0, \varepsilon) \cap bD\) be a transversal intersection. Due to Corollary 2 in [Mi] there exists a function \(g \in C^\infty(\overline{B}(0, \varepsilon) \cap \overline{D})\) such that \(\overline{\partial} g = \overline{\partial} \left(\frac{1}{\psi}\right)\) on \(\overline{B}(0, \varepsilon) \cap \overline{D}\). Adding a constant, we may assume that \(\Re g > 0\). If \(\varepsilon\) is sufficiently small, we get \(|g\psi| \leq \frac{1}{2}\) on \(\overline{B}(0, \varepsilon) \cap \overline{D}\). Now we consider \(h = \frac{\psi}{1 - g\psi}\). It is clear that \(h \in C^\infty(\overline{B}(0, \varepsilon) \cap \overline{D})\). As \(\overline{\partial} h = -\frac{1}{(\frac{1}{\psi} - g)^2} \overline{\partial} \left(\frac{1}{\psi} - g\right) = 0\) on \(B(0, \varepsilon) \cap \overline{D}\) we obtain \(h \in A^\infty(B(0, \varepsilon) \cap \overline{D})\). Moreover, \(\psi \mid_\mathcal{M} = 0\) so \(h \mid_\mathcal{M} = 0\). For every \((Z, w) \in B(0, \varepsilon) \cap \overline{D} \setminus \mathcal{M}\) we have \(\Re h = \Re \left(\frac{1}{\frac{1}{\psi} - g}\right) = \frac{\Re \psi - \Re g}{|\frac{1}{\psi} - g|^2} < 0\). Thus, \(\mathcal{M}\) is a local peak set at \(p\) for the class \(A^\infty\). \(\square\)

ii) Using the notations as above, let \(F \in C^\infty(\overline{\mathcal{M} \cap B(0, \varepsilon_1)})\) with \(0 < \varepsilon_1 \leq \varepsilon\). Let \(\tilde{F}\) be an almost-holomorphic extension of \(F\) on \(B(0, \varepsilon_2)\) with respect to \(\mathcal{N} \cap B(0, \varepsilon_2)(\varepsilon_2 \leq \varepsilon_1)\). By b) the \((0,1)\)-form \(\frac{1}{\psi} \overline{\partial} \tilde{F}\) has a \(C^\infty\)-extension on \(\overline{B}(0, \varepsilon_2) \cap \overline{D}\). Since \(\frac{1}{h} = (1 - g\psi)^{-1} \frac{\psi}{h}\), \(\frac{1}{h} \overline{\partial} \tilde{F}\) is \(\overline{\partial}\)-closed on \(B(0, \varepsilon_2) \cap \overline{D}\). Moreover, \(\frac{1}{h} \overline{\partial} \tilde{F}\) has a \(C^\infty\)-extension on \(\overline{B}(0, \varepsilon_2) \cap \overline{D}\).

Let \(0 < \varepsilon_3 \leq \varepsilon_2\) be such that \(bB(0, \varepsilon_3) \cap bD\) is a transversal intersection. By Corollary 2 of [Mi] there exists a function \(G \in C^\infty(\overline{B}(0, \varepsilon_3) \cap \overline{D})\) such that \(\overline{\partial} G = \frac{1}{h} \overline{\partial} \tilde{F}\) on \(\overline{B}(0, \varepsilon_3) \cap \overline{D}\). Now we set \(f = \tilde{F} - hG\) on \(\overline{B}(0, \varepsilon_3) \cap \overline{D}\). It is clear that \(f \in C^\infty(\overline{B}(0, \varepsilon_3) \cap \overline{D})\). Moreover, we have \(f \mid_{\mathcal{M} \cap \overline{B}(0, \varepsilon_3)} = \tilde{F} \mid_{\mathcal{M} \cap \overline{B}(0, \varepsilon_3)} = F\) and \(\overline{\partial} f = \overline{\partial} \tilde{F} - h \overline{\partial} G = 0\). The theorem is completely proved. \(\square\)
5. Some implications from the sufficient hypotheses for the multitype

We want to interpret the sufficient hypotheses \((\mathcal{H}_1)\) and \((\mathcal{H}_2)\) in terms of Catlin’s multitype. In this section we first recall various concepts of types and we give the multitype for the points on the submanifold \(M\).

Let \(D\) be a bounded pseudoconvex in \(\mathbb{C}^n\) with \(C^\infty\)-boundary. Let \(\rho\) be a local defining function at a point \(p \in bD\). The variety (1-)type \(\Delta_1(bD, p)\) (or \(\Delta_1(p)\) if no confusion can occur), introduced by D’Angelo [DA], is defined as

\[
\Delta_1(bD, p) := \sup_z \left\{ \frac{\nu(z^* \rho)}{\nu(z - p)} \right\},
\]

where the supremum is taken over all germs of nontrivial one-dimensional complex curves \(z : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, p)\) with \(z(0) = p\). Here, \(\nu(f)\) denotes the vanishing order of the function \(f\) at 0 and \(z^* \rho \equiv \rho \circ z\).

More generally, one can define the \(q\)-type, \(\Delta_q(bD, p)\) [DA], \(1 \leq q \leq n\),

\[
\Delta_q(bD, p) := \inf_z \Delta_1(bD \cap S, p).
\]

Here \(S\) runs over all \((n - q + 1)\)-dimensional complex hyperplanes passing through \(p\), and \(\Delta_1(bD \cap S, p)\) denotes the 1-type of the domain \(D \cap S\) (considered as a domain in \(S\)) at \(p\). Note that the \(q\)-types are biholomorphic invariants [DA], [Ca].

Next we recall the definition of Catlin’s multitype. Let \(\Gamma_n\) denote the set of all \(n\)-tuples of numbers \(\mu = (\mu_1, \ldots, \mu_n)\) with \(1 \leq \mu_i \leq \infty\) such that

(i) \(\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n\);

(ii) For each \(k\), either \(\mu_k = \infty\) or there is a set of nonnegative numbers \(a_1, \ldots, a_k\), with \(a_k > 0\) such that \(\sum_{j=1}^{k} a_j / \mu_j = 1\).

An element of \(\Gamma_n\) will be referred to as a weight. The set of weights can be ordered lexicographically, i.e., if \(\mu' = (\mu'_1, \ldots, \mu'_n)\) and \(\mu'' = (\mu''_1, \ldots, \mu''_n)\), then \(\mu' < \mu''\) if for some \(k\), \(\mu'_j = \mu''_j\) for all \(j < k\), but \(\mu'_k < \mu''_k\). A weight \(\mu \in \Gamma_n\) is said to be distinguished if there exist holomorphic coordinates \((z_1, \ldots, z_n)\) about \(p\), with \(p\) mapped to the origin, such that

\[
\text{If } \sum_i \frac{\alpha_i + \beta_i}{\mu_i} < 1, \text{ then } D^\alpha \overline{D}^\beta \rho(p) = 0. \tag{5.1}
\]
Here $D^\alpha$ and $\overline{D}^\beta$ denote the partial differential operators:

\[
\frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}} \text{ and } \frac{\partial^{|\beta|}}{\partial \overline{z}_1^{\beta_1} \ldots \partial \overline{z}_n^{\beta_n}},
\]

respectively.

**Definition 5.1.** — The multitype $\mathcal{M}(bD, p)$ (or $\mathcal{M}(p)$) is defined to be the least weight $\mathcal{M}$ in $\Gamma_n$ (smallest in the lexicographic sense) such that $\mathcal{M} \geq \mu$ for every distinguished weight $\mu$.

We call a weight $\mu$ linearly distinguished if there exist a complex linear change of coordinates about $p$ with $p$ mapped to the origin and such that in the new coordinates (5.1) holds. The linear multitype $\mathcal{L}(bD, p)$ is defined to be the smallest weight $\mathcal{L} = (l_1, \ldots, l_n)$ such that $\mathcal{L} \geq \mu$ for every linearly distinguished weight $\mu$.

Clearly $\mathcal{L}(bD, p)$ is invariant under linear change of coordinates and we have $\mathcal{L}(bD, p) \leq \mathcal{M}(bD, p)$. It is easy to see that the first component of $\mathcal{M}(p)$ is always 1.

Let us $\Delta(p) := (\Delta_n(p), \ldots, \Delta_1(p))$ where $\Delta_q(p)$ stands for the $q$-type. Let the multitype of $p$ be $\mathcal{M}(p) = (\mu_1, \ldots, \mu_n)$. By the main theorem (property 4) in [Ca] it is always true that $\mathcal{M}(p) \leq \Delta(p)$ in the sense that $\mu_{n-q+1} \leq \Delta_q(p)$, for all $q = 1, \ldots, n$.

**Theorem 5.2.** — Let $D$ be a pseudoconvex domain in $\mathbb{C}^n$ with $C^\omega$-boundary. Let $\mathcal{M}$ be an $(n-1)$-dimensional submanifold of $bD$ which is totally real and complex-tangential in a neighborhood $U$ of $p \in \mathcal{M}$. We suppose that $\mathcal{M}$ admits a peak-admissible $C^\omega$-vector field $X$ of peak-type $(K, \tilde{M}; \tilde{m}_1, \ldots, \tilde{m}_{n-1})$ at $p$ for the class $\mathcal{O}$. Then

(i) $\mathcal{M}(p) = \Delta(p) = (1, 2k_1, \ldots, 2k_{n-1})$.

(ii) $\mathcal{M}(p') = \Delta(p') = (1, 2m_1, \ldots, 2m_{n-1})$ for $p' \in \mathcal{M} \cap U - \{p\}$.

Here, $m_j = M/\tilde{m}_j$, $k_j = K/\tilde{m}_j$ for all $1 \leq j \leq n-1$.

**Remark 5.3.** — An analogous result holds true in the $A^\infty$-case.

**Proof.** — i) From Proposition 3.2 we know that there exists a holomorphic coordinates change (denoted $\theta$) such that the point $p \in \mathcal{M}$ corresponds to the origin and in an open neighborhood of the origin in $\mathbb{C}^n$, the defining function $\rho'$ of the boundary of $D' = \theta(D)$ is $\rho' = u' + A + v'B + v'^2R$. By hypothesis inequality $(\mathcal{H})$ holds in the new coordinates. So, we may identify the complexification $\mathcal{M} = \mathcal{M} + i.\mathcal{M}$ of $\mathcal{M}$ to $\mathbb{C}^{n-1} = T_0^C(bD')$ and we may
assume that $\rho^t|_M \equiv A$ in a sufficiently small neighborhood of the origin in $\mathbb{C}^{n-1}$. Let $Z'_0 = X'_0 + i.Y'_0 \neq 0$ near the origin in $\mathbb{C}^{n-1}$ be fixed. We consider $f(\lambda) = A(\lambda Z'_0)$, $\lambda \in [0, 1]$. We set $m = \max_{1 \leq i \leq n-1} m_i$, $m' = \min_{1 \leq i \leq n-1} m_i$ and $\kappa = K/M \geq 1$. As 

$$f(\lambda) = \left( \sum_{i=1}^{n-1} \lambda^{2m_i} \frac{i_{\nu} \cdot j_{\nu}}{y_{0,i}} \right) \left( \sum_{i=1}^{n-1} \lambda^{2m_i} \left( x_{0,i}^2 + y_{0,i}^2 \right) \right)^{\kappa-1},$$

we have $\lambda^{2m_\kappa} f(1) \leq f(\lambda) \leq \lambda^{2m_\kappa} f(1)$. Therefore, we obtain

$$\frac{f(1)}{2m_\kappa + 1} \leq \int_0^1 f(\lambda) d\lambda \leq \frac{f(1)}{2m_\kappa + 1}.$$ 

By Remark 4 in [B-S], the $1$-type of $bD'$ at 0 is equal to line type in the new system of coordinates. This means that $\Delta_1(bD', 0) = \sup_{v \in \mathbb{C}^{n}, |v| = 1} (\rho^t \circ \ell_v)$, where $\ell_v: \zeta \mapsto \zeta + v$ is a complex line passing through the origin and having $v$ as direction. Inequality (H) implies $\Delta_1(bD', 0) = 2k_{n-1}$. Now we prove that $\Delta(bD', 0) = (1, 2k_1, \ldots, 2k_{n-1})$ is a linearly distinguished weight at 0. Let $F: Z = (z_1, \ldots, z_n) \mapsto (\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_{n-1})$ be a $C$-linear change of coordinates. We set $Z = (\tilde{z}_1, Z') = F(Z)$ with $\tilde{Z}' = (\tilde{z}_2, \ldots, \tilde{z}_n)$ and $\tilde{\rho} = \rho^t \circ F^{-1}$. As $\tilde{\rho}(\tilde{Z}) = \Re(\tilde{\zeta}_1) + A(\tilde{Z}') + (\Im \tilde{\zeta}_1)B(\tilde{Z}') + (\Im \tilde{\zeta}_1)^2R(\tilde{Z}', \Im \tilde{\zeta}_1)$, $\frac{\partial \tilde{\rho}}{\partial \tilde{z}_1}(0) \neq 0$ because $\frac{\partial \rho^t}{\partial z_n}(0) \neq 0$. This implies that $\alpha_1 = \beta_1 = 0$ for the property (5.1). Thus it is sufficient to verify that:

$$\sum_{i=2}^n \frac{\alpha_i + \beta_i}{2k_{i-1}} < 1 \quad \text{implies} \quad D^\alpha D^\beta A(0) = 0.$$ 

In fact, let $\alpha = (\alpha_2, \ldots, \alpha_n)$, $\beta = (\beta_2, \ldots, \beta_n) \in \mathbb{N}^{n-1}$ be such that 

$$\sum_{\nu=2}^n \frac{\alpha_\nu + \beta_\nu}{2k_{\nu-1}} < 1.$$ 

Then, $\sum_{\nu=2}^n \tilde{m}_{\nu-1}(\alpha_\nu + \beta_\nu) < 2k$. Since $A$ is $C^\omega$ on a sufficiently small neighborhood of the origin in $\mathbb{C}^{n-1}$, $A(X, Y) = \sum_{I=(i_2, \ldots, i_n)} A_{I,J} X^J Y^I$ with $X = (x_2, \ldots, x_n)$ and $Y = (y_2, \ldots, y_n)$. We know that the $Z$-weight of $A$ is $\geq 2K$. By Remark 4.6, we have $\sum_{\nu=2}^n \tilde{m}_{\nu}(i_\nu + j_\nu) \geq 2K$. Thus,

$$P_Z(D^\alpha D^\beta A) \geq \sum_{\nu=2}^n \tilde{m}_{\nu-1}(i_\nu + j_\nu) - \sum_{\nu=2}^n \tilde{m}_{\nu-1}(\alpha_\nu + \beta_\nu) > 0.$$ 

– 596 –
We obtain $D^\alpha D^\beta A(0) = 0$. Therefore $\Delta(bD', 0)$ is linearly distinguished and $\Delta(bD', 0) \leq \mathcal{M}(bD', 0)$.

It remains to show that $\mathcal{M}(bD', 0) \leq \Delta(bD', 0)$. Setting $\mathcal{M}(bD', 0) = (\mu_1, \ldots, \mu_n)$, by property 4 of Catlin in [Ca] we have $\mu_{n+1-q} \leq \Delta_q(bD', 0)$ for all $q = 1, \ldots, n$.

It is sufficient to prove that $\Delta_q(bD', 0) = 2k_{n-q}$ for all $1 \leq q \leq n - 1$.

- For $q = 1$, we have already shown that $\Delta_1(bD', 0) = 2k_{n-q}$.

- Let $2 \leq q \leq n - 1$ be fixed. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $C^n$ with $T_0^C(bD') = \text{Span}_C\{e_1, \ldots, e_{n-1}\}$. Consider $V_q = \text{Span}_C\{e_{n-q}, \ldots, e_{n-1}\}$ and $S$ an $(n - q + 1)$-dimensional complex hyperplane in $C^n$.

As

$$\dim (V_q \cap S) = \dim V_q + \dim S - \dim (V_q + S) \geq q + n - q + 1 - n = 1,$$

it follows that there exists a complex line $\ell$ in $S \cap V_q$ that has order of contact $\geq 2k_{n-q}$ with the boundary $bD'$ at 0. Therefore $\Delta_q(bD', 0) = 2k_{n-q}$. Moreover, if we set $\tilde{S} = \text{Span}_C\{e_1, \ldots, e_{n-q}, e_n\}$ then $\tilde{S} \cap V_q = \text{Span}_C\{e_{n-q}\}$. So $\Delta_1(\tilde{S} \cap bD', 0) = 2k_{n-q}$. We therefore obtain $\mathcal{M}(bD', 0) \leq \Delta(bD', 0) = (1, 2k_1, \ldots, 2k_{n-1})$. With $\Delta(bD', 0) = (1, 2k_1, \ldots, 2k_{n-1}) \leq \mathcal{M}(bD', 0)$, we find i).

ii) Let $p' \in \mathcal{M} \cap \mathcal{U} - \{p\}$. We work with the preceding system of coordinates and we set $\theta(p') = \vec{p}' \neq 0$. $\vec{p}'$ is a boundary point of $bD'$ near the origin such that $\Re(\vec{p}') \neq 0$. Let $Z_0' = X_0' + iY_0' \in C^{n-1}$ be fixed such $Y_0' \neq 0$. We consider $f(\lambda) = A(\lambda Z_0' + \vec{p}')$, $\lambda \in [0, 1]$. In this case, there exist two constants $0 < c_1 \leq c_2$ which depend only of $\vec{p}'$ satisfying:

$$c_1 \sum_{i=1}^{n-1} \lambda^{2m_i} y_{0,i}^{2m_i} \lesssim f(\lambda) \lesssim c_2 \sum_{i=1}^{n-1} \lambda^{2m_i} y_{0,i}^{2m_i}.$$

Hence, $\lambda^{2m} f(1) \lesssim f(\lambda) \lesssim f(1) \lambda^{2m'}$. We obtain

$$\frac{f(1)}{2m + 1} \lesssim \int_0^1 f(\lambda) \, d\lambda \lesssim \frac{f(1)}{2m' + 1}.$$

with constants that depend only of $\vec{p}'$. By Remark 4 in [B-S] the 1-type of $\vec{p}'$ is equal to line type. So, $\Delta_1(bD', \vec{p}') = 2m_{n-1}$. In the same way as
before one shows that $\Delta(\bar{p}') = (1, 2m_1, \ldots, 2m_{n-1})$ is linearly distinguished weight. Next, we proceed analogously as i) we obtain the equality and ii) holds. □

**Bibliography**


