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PAUL BROUSSOUS

Representations of PGL(2) of a local field and harmonic cochains on graph


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Representations of PGL(2) of a local field and harmonic cochains on graphs

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Dedicated to Colin Bushnell on his 60th birthday

ABSTRACT. — We give combinatorial models for non-spherical, generic, smooth, complex representations of the group $G = \text{PGL}(2, F)$, where $F$ is a non-Archimedean locally compact field. More precisely we carry on studying the graphs $(\tilde{X}_k)_{k \geq 0}$ defined in a previous work. We show that such representations may be obtained as quotients of the cohomology of a graph $\tilde{X}_k$, for a suitable integer $k$, or equivalently as subspaces of the space of discrete harmonic cochains on such a graph. Moreover, for supercuspidal representations, these models are unique.

RÉSUMÉ. — Nous donnons des modèles combinatoires des représentations lisses, complexes, génériques, non-sphériques du groupe $G = \text{PGL}(2, F)$, où $F$ est un corps localement compact non-archimédien. Plus précisément nous reprenons l'étude des graphes $(\tilde{X}_k)_{k \geq 0}$ inaugurée dans un précédent travail. Nous montrons que de telles représentations se réalisent comme quotients de la cohomologie d'un graphe $\tilde{X}_k$ pour un $k$ bien choisi, ou, de façon équivalente, dans un espace de formes harmoniques discrètes sur un tel graphe. Pour les représentations supercuspidales, ces modèles sont de plus uniques.
Let $F$ be a non-archimedean local field and $G$ be the locally compact group $\text{PGL}(N, F)$, where $N \geq 2$ is an integer. In [1] the author constructed a projective tower of simplicial complexes fibered over the Bruhat-Tits building $X$ of $G$. He addressed the question of understanding the structure of the cohomology spaces of these complexes as $G$-modules. In this article we give a conceptual treatment of the case $N = 2$. In that case $X$ is a homogeneous tree and (a slightly modified version of) the projective tower is formed of graphs $\tilde{X}_n$, $n \geq 0$, acted upon by $G$.

Let $\pi$ be an irreducible generic and non-spherical smooth complex representation of $G$. We show there is a natural $G$-equivariant map
\[
\tilde{\Psi}_\pi : \pi^\vee \longrightarrow \mathcal{H}_\infty(\tilde{X}_{n(\pi)}, \mathbb{C}) ,
\]
where $n(\pi)$ is an integer related to the conductor of $\pi$, $\tilde{\pi}$ denotes the contragredient representation of $\pi$, and $\mathcal{H}_\infty(\tilde{X}_{n(\pi)}, \mathbb{C})$ denotes the space of smooth discrete harmonic cochains on $\tilde{X}_{n(\pi)}$. Our construction is based on the existence of new vectors for irreducible generic representations whose proof is due to Casselman in the case $N = 2$ [5] (see [7] for the general case).

The $G$-space $\mathcal{H}_\infty(\tilde{X}_{n(\pi)}, \mathbb{C})$ is naturally isomorphic to the contragredient representation of the cohomology space $H^1_c(\tilde{X}_{n(\pi)}, \mathbb{C})$ (cohomology space with compact support and complex coefficients). We show that $\tilde{\Psi}_\pi$ corresponds to a non-zero natural $G$-equivariant map:
\[
\Psi_\pi : H^1_c(\tilde{X}_{n(\pi)}, \mathbb{C}) \longrightarrow \pi .
\]
In other words $\pi$ is naturally a quotient of $H^1_c(\tilde{X}_{n(\pi)}, \mathbb{C})$. When $\pi$ is supercuspidal the surjective map $\Psi$ splits and $\pi$ embeds in $H^1_c(\tilde{X}_{n(\pi)}, \mathbb{C})$. We show that this model is unique:
\[
\dim_{\mathbb{C}} \text{Hom}_G(\pi, H^1_c(\tilde{X}_{n(\pi)}, \mathbb{C})) = 1 .
\]
The proof of that fact roughly goes as follows. Using a geodesic Radon transform on the space of 1-cochains with finite support on $\tilde{X}_k$, $k \geq 0$, we construct an intertwining operator:
\[
j_k : H^1_c(\tilde{X}_k, \mathbb{C}) \longrightarrow c\text{-}\text{ind}_T^G 1_T
\]
(the compactly induced representation of the trivial character of the diagonal torus $T$ of $G$). The point is that this map is injective. Moreover we have:
\[
\bigcup_{k \geq 0} \text{Im}(j_k) = c\text{-}\text{ind}_T^G 1_T .
\]
We show that \( c\text{-ind}_T^G 1_T \) naturally embeds in a space of Whittaker functions on \( G \) and we may then rely on the uniqueness of Whittaker model for \( \text{PGL}(2,F) \).

That such a combinatorial realization of the generic non-spherical irreducible representations of \( \text{PGL}(2,F) \) is feasible was actually conjectured by Pierre Cartier more than thirty years ago [4] (but he did not introduce any simplicial structure). Later on Cartier’s student Ahumada Bustamante [3] studied the action of the full automorphism group \( \Gamma = \text{Aut}(X) \) of the tree \( X \) on pairs of vertices at distance \( k+1 \) (i.e. on edges of \( \tilde{X}_k \)). Using an equivalent language, he proved that, under the action of \( \Gamma \), the space \( \mathcal{H}_2(\tilde{X}_k, \mathbb{C}) \) of \( L^2 \) harmonic cochains splits into two irreducible components \( \mathcal{H}^\pm_2(\tilde{X}_k, \mathbb{C}) \), formed of even and odd harmonic cochains respectively.

The present article is not a completion of the previous work [1] where we computed the supercuspidal part of the cohomology space \( H^1_c(\tilde{X}_2, \mathbb{C}) \). (Be aware that the notation slightly differs. In particular, \( \tilde{X}_2 \) is the space \( \tilde{X}_1 \) of [1])). Even though the results are compatible, here we do not determine the structure of the spaces \( H^1_c(\tilde{X}_k, \mathbb{C}), k \geqslant 0 \), as \( G \)-modules. In a forthcoming work [2] we shall work out this structure and its links with types theory.

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The notes are organized as follows. In §1 we shall discuss the link between cohomology and harmonic cochains on graphs. §§2-4 are about the construction of the maps \( \Psi_\pi, \tilde{\Psi}_\pi \) and some of their properties. The Radon transform is defined and studied in §5 in order to prove our uniqueness result (theorem (5.3.2)).

In the sequel we shall use the following notation:

- \( \mathfrak{o} = \mathfrak{o}_F \) is the ring of integers of \( F \),
- \( \mathfrak{p} = \mathfrak{p}_F \) is the maximal ideal of \( \mathfrak{o} \),
- \( v = v_F \) is the normalized additive valuation on \( F \),
- \( k = k_F \) is the residue class field \( \mathfrak{o}/\mathfrak{p} \),
- \( q = |k_F| \) is the cardinal of \( k \),
- \( || \) = \( ||_F \) is the multiplicative valuation on \( F^\times \) normalized in such a way that \( |\varpi|_F = 1/q \) for any generator \( \varpi \) of the ideal \( \mathfrak{p} \).

The contragredient of a representation \( \mathcal{V} \) is denoted by \( \hat{\mathcal{V}} \) or \( \mathcal{V}^\vee \).
1. Proper G-graphs and harmonic cochains

1.1. In this section we let $G$ be any locally profinite group and $Y$ be a locally finite directed graph (each vertex belongs to a finite number of edges). We write $Y^0$ (resp. $Y^1$) for the set of vertices (resp. edges) of $Y$. We have the map $Y^1 \rightarrow Y^0, a \mapsto a^+$ (resp. $a \mapsto a^-$), where for any edge $a$ we denote by $a^+$ and $a^-$ its head and tail respectively. We assume that $G$ acts on $Y$ and preserves the structure of directed graph. For all $s \in Y^0, a \in Y^1$, we have incidence numbers $[a : s] \in \{-1, 1, 0\}$ satisfying $[g.a : g.s] = [a : s]$, for all $g \in G$; these are defined by $[a : a^+] = +1, [a : a^-] = -1$, and $[a : s] = 0$ if $s \notin \{a^+, a^\}$. Finally we assume that the action of $G$ on $Y$ is proper: for all $s \in Y^0$, the stabilizer $G_s := \{g \in G ; g.s = s\}$ is open and compact.

1.2. We let $H^1_c(Y, \mathbb{C})$ denote the cohomology space of the CW-complex $Y$ with compact support and complex coefficients. Recall that it may be calculated as follows. Let $C_0(Y, \mathbb{C})$ (resp. $C_1(Y, \mathbb{C})$) be the $\mathbb{C}$-vector space with basis $Y^0$ (resp. $Y^1$). Let $C^i_c(Y, \mathbb{C})$, $i = 0, 1$, be the $\mathbb{C}$-vector space of $i$-cochains with finite support : $C^i_c(Y, \mathbb{C})$ is the subspace of the algebraic dual of $C_i(Y, \mathbb{C})$ formed of those linear forms whose restrictions to the basis $Y^i$ have finite support. The coboundary map

$$d : C^0_c(Y, \mathbb{C}) \rightarrow C^1_c(Y, \mathbb{C})$$

is given by $df(a) = f(a^+) - f(a^-)$. Then

$$(1.2.1) \quad H^1_c(Y, \mathbb{C}) \simeq C^1_c(Y, \mathbb{C})/dC^0_c(Y, \mathbb{C}) .$$

The group $G$ acts on $C_i(Y, \mathbb{C})$ and $C^i_c(Y, \mathbb{C})$. Since the action of $G$ on $Y$ is proper, these spaces are smooth $G$-modules. The coboundary map is $G$-equivariant and the isomorphism (1.2.1) is $G$-equivariant. So $H^1_c(Y, \mathbb{C})$ is smooth as a $G$-module; it is not admissible in general.

1.3. For $i = 0, 1$, we have a natural pairing:

$$\langle -, - \rangle : C^i(Y, \mathbb{C}) \times C^i_c(Y, \mathbb{C}) \rightarrow \mathbb{C} ,$$

where $C^i(Y, \mathbb{C})$ is the space of $i$-cochains with arbitrary support. The pairings are given by:

$$\langle f, g \rangle = \sum_{x \in Y^i} f(x)g(x) , \ i = 0, 1 .$$

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Via these pairings we may identify the algebraic dual $C^i_c(Y, \mathbb{C})^*$ of $C^i_c(Y, \mathbb{C})$ with $C^i(Y, \mathbb{C})$, $i = 0, 1$. The contragredient representation $C^i_c(Y, \mathbb{C})^\vee$ identifies with the space of smooth linear forms in $C^i(Y, \mathbb{C})$. A straightforward computation gives:

\[(1.3.1) \quad \langle f, dg \rangle = \langle d^*f, g \rangle, \quad f \in C^1(Y, \mathbb{C}), \ g \in C^0_c(Y, \mathbb{C}),\]

where $d^*: C^1(Y, \mathbb{C}) \to C^0(Y, \mathbb{C})$ is defined by

\[d^* f(s) = \sum_{a \in Y^1, \ s \in a} [a : s] f(a), \ s \in Y^0.\]

Of course this latter sum has a finite number of terms. An element of the kernel of $d^*$ is called a harmonic cochain on $Y$. We denote by $\mathcal{H}(Y, \mathbb{C}) = \ker(d^*)$ the space of harmonic cochains. It is naturally acted upon by $G$. The smooth part of $\mathcal{H}(Y, \mathbb{C})$ under the action of $G$, i.e. the space of smooth harmonic cochains is denoted by $\mathcal{H}_\infty(Y, \mathbb{C})$. The following lemma follows from equality (1.3.1).

\[(1.3.2) \quad \text{Lemma.} \quad \text{The algebraic dual of } H^1_c(Y, \mathbb{C}) \text{ naturally identifies with } \mathcal{H}(Y, \mathbb{C}). \text{ Under this isomorphism, the contragredient representation of } H^1_c(Y, \mathbb{C}) \text{ corresponds to } \mathcal{H}_\infty(Y, \mathbb{C}).\]

2. The projective tower of graphs

2.1 In this section, we recall the construction of [1]. The notation is slightly modified. We denote by $X$ the Bruhat-Tits building of $G$ (cf. [8] chap. II, §1). This is a 1-dimensional simplicial complex (a $(q+1)$-homogeneous tree). Let $k \geq 0$ be an integer. An (oriented) $k$-path in $X$ is an injective sequence $(s_0, \ldots, s_k)$ of vertices in $X$ such that, for $i = 0, \ldots, k-1$, $\{s_i, s_{i+1}\}$ is an edge of $X$. We define an oriented graph $\tilde{X}_k$ as follows. Its vertex set (resp. edge set) is the set of $k$-paths (resp. $(k+1)$-paths) in $X$. The structure of oriented graph is given by:

\[a^+ = (t_1, \ldots, t_{k+1}), \ a^- = (t_0, \ldots, t_k), \ \text{if } a = (t_0, \ldots, t_{k+1}).\]

The group $G$ acts on $\tilde{X}_k$. If $k \geq 1$, $\tilde{X}_k$ is a simplicial complex and the $G$-action is simplicial. For $k = 0$ the action preserves the graph structure. For all $k$, it preserves the orientation of $\tilde{X}_k$. Recall ([1] Lemma 4.1) that the simplicial complexes $\tilde{X}_k$, $k \geq 1$ are connected. The directed graph $\tilde{X}_0$
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is obtained from $X$ by doubling the edges (with the same vertex set); it is
obviously connected.

2.2. For any integer $n \geq 1$, we write $\Gamma_0(\mathfrak{p}^n)$ for the image in $G$ of the
following subgroup of $\text{GL}(2, F)$:

$$\tilde{\Gamma}_0(\mathfrak{p}^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, F); \ a, d \in \mathfrak{o}^\times, \ b \in \mathfrak{o}, \ c \in \mathfrak{p}^n \right\}$$

We let $\Gamma_o(\mathfrak{p}^0)$ be the image in $\text{PGL}(2, F)$ of the standard maximal compact
subgroup of $\text{GL}(2, F)$:

$$\tilde{\Gamma}_0(\mathfrak{p}^0) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathfrak{o}); \ ad - bc \in \mathfrak{o}^\times \right\}$$

For all $n \geq 0$, $\Gamma_o(\mathfrak{p}^{n+1})$ is the stabilizer in $G$ of some edges of $\tilde{X}_n$. We fix
such an edge $a_o$.

3. The construction

3.1. We start by recalling Casselman’s result. We fix an irreducible com-
plex smooth representation $(\pi, \mathcal{V})$ of $G$. We assume that:

$$(3.1.1) \quad \pi \text{ has no non-zero vector fixed by } \Gamma_0(\mathfrak{p}^0),$$

$$(3.1.2) \quad \pi \text{ is generic, i.e. it is not of the form } \chi \circ \det, \text{ where } \chi \text{ is a char-
acter of } F^\times/(F^\times)^2 \text{ and } \det : G \longrightarrow F^\times/(F^\times)^2 \text{ is the map induced by the}
determinant map: } \text{GL}(2, F) \longrightarrow F^\times.$$

We have the following result ([5] Theorem 1):

$$(3.1.3) \text{Theorem (Casselman).} \quad \text{— i) For } k \text{ large enough, the space of fixed}
\text{vectors } \mathcal{V}_{\Gamma_o(\mathfrak{p}^{k+1})} \text{ is non-zero.}

\text{ ii) Let } n(\pi) \geq 0 \text{ be such that } \mathcal{V}_{\Gamma_o(\mathfrak{p}^{n(\pi)+1})} \neq \{0\} \text{ and } \mathcal{V}_{\Gamma_o(\mathfrak{p}^{n(\pi)})} = \{0\}. \text{ Then for all } k \geq n(\pi), \text{ we have:}

$$
\dim_\mathbb{C} \mathcal{V}_{\Gamma_o(\mathfrak{p}^{k+1})} = k - n(\pi) + 1.$$
3.2. For all $a \in \tilde{X}_{n(\pi)}^1$ (resp. $s \in \tilde{X}_{n(\pi)}^0$), we write $\Gamma_a$ (resp. $\Gamma_s$) for the stabilizer of $a$ (resp. $s$) in $G$. In particular we have $\Gamma_{a_o} = \Gamma_0(p^{n(\pi)+1})$.

(3.2.1) Lemma. — i) For all $a \in \tilde{X}_{n(\pi)}^1$, we have $\dim\mathcal{V}^{\Gamma_a} = 1$.

ii) Let $a \in \tilde{X}_{n(\pi)}^1$ and $s \in \tilde{X}_{n(\pi)}^0$ with $s \in a$. Then for all $v \in \mathcal{V}^{\Gamma_a}$, we have

$$\sum_{k \in \Gamma_s/\Gamma_a} kv = 0.$$ 

Point i) is obvious. In ii), the vector $\sum_{k \in \Gamma_s/\Gamma_a} kv$ is fixed by $\Gamma_s$. So it must be zero since $\Gamma_s$ is conjugate to $\Gamma_0(p^{n(\pi)})$.

Let us fix a non-zero vector $v_o \in \mathcal{V}^{\Gamma_{a_o}}$; $v_o$ is unique up to a scalar in $\mathbb{C}^\times$. If $a$ is any edge of $X_{n(\pi)}$, we put

(3.2.2) $v_a = gv_o$, where $a = ga_o$.

This is indeed possible since $G$ acts transitively on $\tilde{X}_{n(\pi)}^1$. Moreover, since $v_o$ is fixed by $\Gamma_{a_o}$, $v_a$ does not depend on the choice of $g \in G$ such that $a = ga_o$. Let $\tilde{\mathcal{V}}$ be the contragredient representation of $\mathcal{V}$. We define a map:

$$\tilde{\Psi}_\pi : \tilde{\mathcal{V}} \longrightarrow C^1(X_{n(\pi)}, \mathbb{C})$$

by $\tilde{\Psi}_\pi(\varphi)(a) = \varphi(v_a)$. From (3.2.2) we have that $\tilde{\Psi}_\pi$ is $G$-equivariant.

(3.2.3) Lemma. — i) The image of $\tilde{\Psi}_\pi$ lies in $\mathcal{H}_\infty(X_{n(\pi)}, \mathbb{C})$.

ii) The map $\tilde{\Psi}_\pi$ is injective.

For i), it suffices to prove that $\text{Im}(\tilde{\Psi}_\pi) \subset \mathcal{H}(\tilde{X}_{n(\pi)}, \mathbb{C})$. So we must prove that for all $\varphi \in \tilde{\mathcal{V}}$, $(\varphi(v_a))_{a \in \tilde{X}_{n(\pi)}^1}$ is a harmonic cochain on $\tilde{X}_{n(\pi)}$, that is:

$$\sum_{s \in a} [a : s] \varphi(v_a) = 0, \text{ for all } s \in X_{n(\pi)}^0.$$ 

Let $s$ be any vertex of $\tilde{X}_{n(\pi)}$. Write $\bar{s}$ for the convex hull in $X$ of the set of vertices of $X$ occurring in the path $s$ (this is a segment lying in some
We know that the pointwise stabilizer of \( \bar{s} \) in \( G \) (that is the stabilizer \( \Gamma_s \) of \( s \) in \( G \)) acts transitively on the set of apartments of \( X \) containing \( \bar{s} \). It follows that \( \Gamma_s \) acts transitively on

\[
A^+_s = \{ a \in \tilde{X}^1_{n(\pi)}; \ a^+ = s \} \quad \text{and} \quad A^-_s = \{ a \in \tilde{X}^1_{n(\pi)}; \ a^- = s \}.
\]

Fix some \( a^+_s \in A^+_s \) and \( a^-_s \in A^-_s \). Then

\[
\sum_{s \in a} [a : s] \varphi(v_a) = \varphi\left( \sum_{a \in A^+_s} v_a - \sum_{a \in A^-_s} v_a \right) = \varphi\left( \sum_{k \in \Gamma_s / \Gamma_{a^+_s}} kv_{a^+_s} - \sum_{k \in \Gamma_s / \Gamma_{a^-_s}} kv_{a^-_s} \right) = 0,
\]

thanks to lemma (3.2.1).

The \( G \)-equivariant map \( \tilde{\Psi}_\pi \) is necessarily injective since it is non-zero and since the representation \( \pi \) is irreducible.

Passing to contragredient representations, we get an intertwining operator:

\[
\tilde{\tilde{\Psi}}_\pi : \tilde{H}^1_c(\tilde{X}_{n(\pi)}, \mathbb{C}) \to \tilde{\tilde{V}}.
\]

Recall that for any smooth \( G \)-module \( W \), we have a canonical injection \( W \to \tilde{W} \). It is surjective if and only if \( W \) is admissible. In particular \( V \) and \( \tilde{\tilde{V}} \) are canonically isomorphic since \( \pi \) is irreducible, whence admissible.

(3.2.4) THEOREM. — The map \( \tilde{\tilde{\Psi}}_\pi \) restricts to a non-zero intertwining operator \( \Psi_\pi : H^1_c(\tilde{X}_{n(\pi)}, \mathbb{C}) \to V \simeq \tilde{\tilde{V}} \), given by:

\[
\Psi_\pi(\tilde{\omega}) = \sum_{a \in \tilde{X}^1_{n(\pi)}} \omega(a)v_a.
\]

where for \( \omega \in C^1_c(\tilde{X}_{n(\pi)}, \mathbb{C}) \), \( \tilde{\omega} \) denotes the image of \( \omega \) in \( H^1_c(\tilde{X}_{n(\pi)}, \mathbb{C}) \).

In particular, the representation \( (\pi, V) \) is naturally a quotient of the cohomology space \( H^1_c(\tilde{X}_{n(\pi)}, \mathbb{C}) \).

The theorem follows from a straightforward computation based on Lemma (3.2.1)(ii) and is left to the reader.
Remark. — Assume that $\pi$ is supercuspidal. Then it is projective in the category of smooth complex representations of $G$. So as a corollary of Theorem (3.2.4) we get an injective map $(\pi, V_\pi) \rightarrow H^1_c(X_{n(\pi)}, \mathbb{C})$.

4. Some properties of the map $\tilde{\Psi}_\pi$.

4.1. We keep the notation as in the last section. If $Y$ is any directed graph, we write $\mathcal{H}_c(Y, \mathbb{C})$ for the subspace of $\mathcal{H}(Y, \mathbb{C})$ of harmonic cochains with finite support and $\mathcal{H}_2(Y, \mathbb{C})$ for the subspace of $L^2$-harmonic cochains, that is cochains $f \in \mathcal{H}(Y, \mathbb{C})$ satisfying

$$\sum_{a \in Y^1} |f(a)|^2 < \infty.$$ 

Note that any element of $\mathcal{H}_c(X_{n(\pi)}, \mathbb{C})$ is smooth.

(4.1.1) Proposition. — i) If $\pi$ is a supercuspidal representation then $\text{Im} \tilde{\Psi}_\pi$ is contained in $\mathcal{H}_c(X_{n(\pi)}, \mathbb{C})$.

ii) If $\pi$ is a square-integrable representation then $\text{Im} \tilde{\Psi}_\pi$ lies in $\mathcal{H}_2(X_{n(\pi)}, \mathbb{C})$.

Assume $\pi$ supercuspidal. Let $\lambda \in \tilde{V}$. Then $\tilde{\Psi}_\pi(\lambda)(a) = \lambda(gv_{a_o})$ for all $a = ga_o \in X^1_{n(\pi)}$. Since $\pi$ is supercuspidal, the coefficient $g \in G \mapsto \lambda(gv_{a_o})$ has compact support $C$. Choose a finite number of compact open subgroups $K_i$, $i \in I$, of $G$ and elements $g_i \in G$, $i \in I$, such that $C$ lies in the union of the $g_iK_i$, $i \in I$. Then the support of the harmonic cochain $\tilde{\Psi}_\pi(\lambda)$ lies in

$$\bigcup_{i \in I} g_iK_ia_o = \bigcup_{i \in I} g_iK_i/(K_i \cap \Gamma_{a_o})a_o,$$

a finite set.

Now assume that $\pi$ is square-integrable. With the notation as above, the coefficient $g \mapsto \lambda(gv_{a_o})$ is square-integrable. Consider the Haar measure on $G$ such that $\Gamma_{a_o}$ has volume 1. Then

$$\int_G |\lambda(gv_{a_o})|^2dg = \sum_{a \in X^1_{n(\pi)}} |\lambda(v_a)|^2 < \infty,$$

as required.
(4.1.2) **Corollary.**— If \( \pi \) is supercuspidal, the map \( \tilde{\Psi}_\pi : (\tilde{\mathcal{V}}, \pi^\vee) \rightarrow \mathcal{H}_c(X_n(\pi), \mathbb{C}) \) induces a non-zero (whence injective) map \( \bar{\Psi}_\pi : (\tilde{\mathcal{V}}, \pi^\vee) \rightarrow H^1_c(X_n(\pi), \mathbb{C}) \).

The map \( \tilde{\Psi}_\pi \) is
\[
\lambda \mapsto \tilde{\Psi}_\pi(\lambda) \mod dC^0_c(X_n(\pi), \mathbb{C}) .
\]

It suffices to prove that
\[
\mathcal{H}_c(X_n(\pi), \mathbb{C}) \cap dC^0_c(X_n(\pi), \mathbb{C}) = \{0\} .
\]

If \( f \) lies in the intersection then \( d^*f = 0 \) and \( f = dg \) for some \( g \in C^0_c(X_n(\pi), \mathbb{C}) \). We then have \( d^*\tilde{f} = 0 \) (where \( \tilde{f}(a) \) is the complex conjugate of \( f(a) \)), and
\[
\sum_{a \in X^1_n(\pi)} |f(a)|^2 = \langle \tilde{f}, f \rangle = \langle \tilde{f}, dg \rangle = \langle d^*\tilde{f}, g \rangle = 0 .
\]

Hence \( f = 0 \).

**5. The geodesic Radon transform**

5.1. In corollary (4.1.2) we saw that each supercuspidal irreducible representation of \( G \) may be realized as a \( G \)-invariant subspace of \( H^1_c(X_k, \mathbb{C}) \) for a certain \( k \). In order to prove that this model is unique, we need to embed the space \( H^1_c(X_k, \mathbb{C}) \) in a standard \( G \)-module which contains supercuspidal irreducible representations with multiplicity 1. This standard \( G \)-module is the space of locally constant functions with compact support on the set of all oriented apartments of \( X_k \) endowed with a certain topology (see below). This will be done via a Radon transform that “integrates” 1-cochains over each apartment of \( X_k \).

Recall that an apartment of \( X \) is a doubly infinite geodesic of \( X \), that is the image in \( X \) of an injective sequence of vertices \((s_k)_{k \in \mathbb{Z}}\) such that for all \( k \in \mathbb{Z} \), \( \{s_k, s_{k+1}\} \) is an edge of \( X \).

An oriented apartment \( \tilde{A} \) of \( X \) is by definition a pair \((A, \epsilon)\), where \( A \) is an apartment of \( X \) and \( \epsilon \) is an orientation of \( A \) as a simplicial complex. Our group \( G \) acts on oriented apartments via \( g.(A, \epsilon) = (gA, g\epsilon) \), where \( g\epsilon \) is the unique orientation on \( A \) satisfying \( [ga : gs]_{g\epsilon} = [a : s]_\epsilon \), for all \( a \in A^1 \), \( s \in A^0 \). The group \( G \) acts transitively on the set of oriented apartments.
Let $T$ be the diagonal torus of $G$ (the image of the diagonal torus of $\text{GL}(2, F)$ in $G$). The $G$-set $\tilde{\mathcal{X}}$ of oriented apartments in $\mathcal{X}$ is isomorphic to $G/T$. Indeed the stabilizer of an oriented apartment $(A, c)$, that is the set of $g \in G$ such that $gA = A$ and $gc = c$, is conjugate to $T$. We endow $\tilde{\mathcal{X}}$ with the topology corresponding to the quotient topology of $G/T$. In particular for any $\tilde{\mathcal{A}}$ in $\tilde{\mathcal{X}}$ and any open (compact) subgroup $K$ of $G$, $K\tilde{\mathcal{A}} = \{k\tilde{A}; k \in K\}$ is an open (compact) neighbourhood of $\tilde{A}$ in $\tilde{\mathcal{X}}$. Let $\tilde{A} \in \tilde{\mathcal{X}}$ and $k$ be a non-negative integer. By definition the (oriented) apartment of $\tilde{\mathcal{X}}_k$ corresponding to $\tilde{A}$ is the 1-dimensional subsimplicial complex $\tilde{\mathcal{A}}_k$ of $\tilde{\mathcal{X}}$ whose edges (resp. vertices) are the $(k + 1)$-paths (resp. $k$-paths) in $\mathcal{X}$ contained in $\tilde{\mathcal{A}}$ and such that the orientations of $\tilde{a}$ (resp. $\tilde{s}$) and $\tilde{A}$ are compatible. We denote by $\tilde{\mathcal{A}}_k$ the set of apartments of $\tilde{\mathcal{X}}_k$. Then $\tilde{A} \mapsto \tilde{A}_k$ is a $G$-equivariant bijection between $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{X}}_k$ which allows us to identify both $G$-sets.

Let $k$ be a non-negative integer and $a$ be an edge of $\tilde{\mathcal{X}}_k$. Define a subset $\tilde{\mathcal{A}}_a$ of $\tilde{\mathcal{A}}$ by

$$\tilde{\mathcal{A}}_a = \{\tilde{A} \in \tilde{\mathcal{A}}; \ a \in \tilde{\mathcal{A}}_1\}.$$ 

(5.1.1) Lemma. — i) With the notation as above, we have $\tilde{\mathcal{A}}_a = \Gamma_a \tilde{\mathcal{A}}_0$, for any $\tilde{\mathcal{A}}_0$ in $\tilde{\mathcal{A}}$. In particular $\tilde{\mathcal{A}}_a$ is a compact open subset of $\tilde{\mathcal{A}}$.

ii) The set $\{\tilde{\mathcal{A}}_a; k \geq 0, \ a \in \tilde{\mathcal{X}}_k^1\}$ is a basis of the topology of $\tilde{\mathcal{A}}$ formed of compact open subsets.

The equality $\tilde{\mathcal{A}}_a = \Gamma_a \tilde{\mathcal{A}}_0$ follows from the fact that $\Gamma_a$ acts transitively on the apartments of $\mathcal{X}$ containing the path $a$. For ii), we must prove that any open subset $\Omega$ of $\tilde{\mathcal{A}}$ contains $\tilde{\mathcal{A}}_a$ for some $a \geq 0$ and $a \in \tilde{\mathcal{X}}^1_k$. Replacing $\Omega$ by $g\Omega$ for some $g \in G$ we may assume that it contains the oriented apartment $\tilde{A}_a$ corresponding to the coset $1.T \in G/T$. For $r \geq 1$, let $K_r$ be the image in $G$ of the following congruence subgroup of $\text{GL}(2, F)$:

$$\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}; \ a, \ d \in 1 + p^r, \ b, \ c \in p^r\}.$$ 

Take $r$ large enough so that $K_r \tilde{A}_a \subset \Omega$. Let $T^0$ be the maximal compact open subgroup of $T$. It stabilizes $A_a$ pointwise, whence it fixes $A_a$. So $K_r T^0 \tilde{A}_a \subset \Omega$. The subgroup $K_r T^0$ is $\Gamma_a$ for some $a \in \tilde{A}_1^a$ and we are done.

5.2. Fix $k \geq 0$. We define a (geodesic) Radon transform

$$R = R_k : C^1_c(\tilde{\mathcal{X}}_k, \mathbb{C}) \longrightarrow F(\tilde{A}),$$

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where \( F(\tilde{A}) \) is the space of functions on \( \tilde{A} \), by

\[
R(\omega)(\tilde{A}) = \sum_{a \in \tilde{A}^1} \omega(a) \,.
\]

Note that the image of a 1-cochain \( \omega \) whose support is reduced to a single edge \( a_o \) is \( \omega(a_o)\text{Char}_{\tilde{A}}a_o \), where \( \text{Char} \) denotes a characteristic function. So the image of \( R \) actually lies in the space \( C^0_c(\tilde{A}) \) of locally constant functions with compact support on \( \tilde{A} \). Clearly \( R \) is \( G \)-equivariant.

\textbf{(5.2.1) Lemma.} — For all \( f \in C^0_c(\tilde{X}_k, \mathbb{C}) \), we have \( R(df) = 0 \).

Indeed if \( \tilde{A} \in \tilde{A} \), we have

\[
R(df)(\tilde{A}) = \sum_{a \in \tilde{A}^1} f(a^+) - f(a^-) = \sum_{a \in \tilde{A}^1} f(a^+) - \sum_{a \in \tilde{A}^1} f(a^-) = \sum_{s \in \tilde{A}^0} f(s) - \sum_{s \in \tilde{A}^0} f(s) = 0 ,
\]

since the map \( \tilde{A}^1 \longrightarrow \tilde{A}^0, a \mapsto a^+ \) (resp. \( a \mapsto a^- \)) is a bijection.

\textbf{(5.2.2) Proposition.} — i) The following sequence of \( G \)-modules is exact:

\[
C^0_c(\tilde{X}_k, \mathbb{C}) \xrightarrow{d} C^1_c(\tilde{X}_k, \mathbb{C}) \xrightarrow{R} C^0_c(\tilde{A}) .
\]

In other words \( R \) induces an injective map:

\[
\bar{R} : H^1_c(\tilde{X}_k, \mathbb{C}) \longrightarrow C^0_c(\tilde{A}) .
\]

ii) Moreover we have:

\[
\bigcup_{k \geq 0} R_k(H^1_c(\tilde{X}_k, \mathbb{C})) = C^0_c(\tilde{A}) .
\]

Point ii) follows from the fact that \( C^0_c(\tilde{A}) \) is generated as a \( \mathbb{C} \)-vector space by the functions \( \text{Char}_{\tilde{A}_a} \), where \( a \in \tilde{X}_k^1 \) and \( k \geq 0 \).

To prove i) we need to introduce some more concepts. A path \( p \) in \( \tilde{X}_k \) is a sequence of edges \( a_u, u = 0, \ldots, l - 1 \), such that for \( u = 0, \ldots, l - 2, a_u \) and
$a_{u+1}$ share a vertex. Abusing the notation we shall write $p = (x_0, x_1, \ldots, x_l)$, where for $u = 0, \ldots, l-1$, $\{a_u^+, a_u^-\} = \{x_u, x_{u+1}\}$, keeping in mind that when $k = 0$ two neighbour vertices do not determine a unique edge. We define “incidence coefficients” $[p : a_u] \in \{\pm 1\}$, $u = 0, \ldots, l - 1$, by $[p : a_u] = 1$ if and only if $a_u^- = x_u$ and $a_u^+ = x_{u+1}$.

Let $\omega \in C^1_c(\tilde{X}_k, \mathbb{C})$ be in the kernel of $\mathcal{R}$. The “integral” of $\omega$ along $p$ is by definition

$$\int_p \omega = \sum_{u=0, \ldots, l-1} [p : a_u] \omega(a_u).$$

(5.2.3) Lemma. — With the notation as above, if $p$ is a loop, i.e. if $a_0$ and $a_{l-1}$ share the vertex $x_l = x_0$, then

$$\int_p \omega = 0.$$

We first show that the lemma implies proposition (5.2.2). Fix $s_o \in \tilde{X}_0^0$ and $\alpha_o \in \mathbb{C}$. For any $x \in \tilde{X}_k^0$, we set

$$f(x) = \alpha_o + \int_p \omega,$$

where $p = (x_0, \ldots, x_l)$ is any path satisfying $x_0 = s_o$ and $x_l = x$. We claim that $f(x)$ does not depend on the choice of $p$. Indeed let $q = (y_0, \ldots, y_m)$ be another path satisfying the same assumptions and set $q - p = (z_0, \ldots, z_{m+l+1})$, where $z_u = x_u$, for $u = 0, \ldots, l$, and $z_u = y_{m+l+1-u}$, for $u = l + 1, \ldots, l + 1 + m$. Then one easily checks that

$$(\alpha_o + \int_q \omega) - (\alpha_o + \int_p \omega) = \int_{q - p} \omega = 0,$$

since $q - p$ is a loop.

Let $a \in \tilde{X}_1^1$ and $p = (x_0, \ldots, x_l)$ be a path such that $x_0 = s_o$ and $x_l = a^-$. Set $q = (x_o, \ldots, x_l, a^+)$. Then

$$f(a^+) - f(a^-) = \int_q \omega - \int_p \omega = [q : a] \omega(a) = \omega(a).$$

So we must now prove that one can choose $s_o$ and $\alpha_o$ so that $f \in C^0(\tilde{X}_k, \mathbb{C})$ has finite support. Let $S_k$ be the support of $\omega$ in $\tilde{X}_k^1$ and set

$$S := \bigcup_{a \in S_k} \text{cvx}(a) \subset X,$$

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where \( \text{cvx}(a) \) denotes the convex hull of \( a \) in (the geometric realization of) \( X \). Then \( S \) is a bounded subset of \( X \). Let \( t \) be a vertex in \( S \) and \( \delta \) be an integer large enough so that \( S \subset X(t, \delta) \), where \( X(t, \delta) \) is the subtree of \( X \) whose vertices are at combinatorial distance from \( t \) less than or equal to \( \delta \). Then the complementary set \( ^cX(t, \delta) \) of \( X(t, \delta) \) in \( X \) has the following property; for any \( k \)-path \( a \subset ^cX(t, \delta) \), there exists a half-apartment \( A^+ \) such that \( a \subset A^+ \subset ^cX(t, \delta) \). Set

\[
S'_k = \{ s \in \bar{X}_k^0 \mid \text{cvx}(s) \cap X(t, \delta) \neq \emptyset \}.
\]

Choose \( s_o \) outside \( S'_k \) and set \( \alpha_o = 0 \). We are going to prove that the support of \( f \) is contained in \( S'_k \). Let \( x \) be in \( \bar{X}_k^0 \setminus S'_k \). Choose half-apartments \( A^+_o \) and \( A^+ \) so that:

\[
A^+, A^+_o \subset ^cS \text{ and } \text{cvx}(s_o) \subset A^+_o, \text{ cvx}(x) \subset A^+.
\]

Consider vertices \( s'_o \) and \( x' \) of \( \bar{X}_k \) whose convex hulls in \( X \) lie in \( A^+_o \) and \( A^+ \) respectively. One may choose \( s'_o \) (resp. \( x' \)) away enough from the origin of \( A^+_o \) (resp. of \( A^+ \)) so that there exists an apartment \( B \) of \( X \) containing both \( \text{cvx}(s'_o) \) and \( \text{cvx}(x') \).

**First case:** the vertices \( s'_o \) and \( x' \) induce the same orientation on \( B \) (see figure 1). Let \( \bar{B} \) be the corresponding oriented apartment. Let \( \bar{A}^+_o \) be the oriented half-apartment whose orientation is induced by \( s_o \) and \( \bar{A}^+ \) be the oriented half-apartment whose orientation is induced by \( x \). Let \( p(s_o, s'_o) \) (resp. \( p(x', x) \) and \( p(s'_o, x') \)) be the unique injective path in \( \bar{X}_k \) joining \( s_o \) to \( s'_o \) (resp. \( x' \) to \( x \), \( s'_o \) to \( x' \)) and such that \( p(s_o, s'_o) \subset \bar{A}^+_o \) (resp. \( p(x', x) \subset \bar{A}^+, p(s'_o, x') \subset \bar{B} \)).

By concatenation, we get a path

\[
p(s_o, x) = p(s_o, s'_o) + p(s'_o, x') + p(x', x)
\]

joining \( s_o \) and \( x \). Since those vertices of \( \bar{X}_k \) occurring in \( p(s_o, s'_o) \) or \( p(x', x) \) do not lie in \( S \), we have

\[
\int_{p(s_o, s'_o)} \omega = \int_{p(x, x')} \omega = 0.
\]

Moreover

\[
\int_{p(s'_o, x')} \omega = \pm R(\omega)(\bar{B}) = 0,
\]

by assumption. So

\[
\int_{p(s_o, x)} \omega = 0 \quad \text{and} \quad f(x) = \alpha_o + \int_{p(s_o, x)} \omega = 0,
\]

as required.
Second case: the vertices \( s'_o \) and \( x' \) induce different orientations on \( B \) (see figure 2). Then one can choose a third vertex \( x'' \in S'_k \) such that \( s'_o \) and \( x'' \) (resp. \( x' \) and \( x'' \)) lie in some common apartment \( B_1 \) (resp. \( B_2 \)) and induce the same orientation on that apartment. We denote by \( \tilde{B}_1 \) and \( \tilde{B}_2 \) the corresponding oriented apartments. Then, with the notation as in the first case, one easily shows that

\[
 f(x) = f(s_o) + \int_{p(s_o, s'_o)} \omega + \pm \mathcal{R}(\omega)(\tilde{B}_1) + \pm \mathcal{R}(\omega)(\tilde{B}_2) + \int_{p(x', x)} \omega = 0 .
\]

**Proof of lemma (5.2.3).** — It is based on the following easy lemma whose proof is left to the reader.

(5.2.4) **Lemma.** — Let \( U \) be either \( \mathbb{Z}, \mathbb{N} \) or a finite interval of integers. Let \( p = (x_u)_{u \in U} \) be a path in \( \tilde{X}_k \). Assume that \( p \) satisfies one of the following properties:

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(P1) For all $u \in U$ such that $u + 1 \in U$, we have $a^+_u = a^-_{u+1}$;

(P2) For all $u \in U$ such that $u + 1 \in U$, we have $a^-_u = a^+_{u+1}$.

Then $p$ is injective and there is an apartment $\tilde{A}$ containing $p$. In particular $p$ cannot be a loop.

Remark. — A path $p$ satisfies (P1) or (P2) if and only if the sequence of incidence numbers $(p : a_u)_u$ is constant.

Let $p = (x_0, \ldots, x_l)$ be a loop. We consider the index $u$ as an element of $\mathbb{Z}/l \mathbb{Z}$. According to the previous lemma, the set $V$ of indices $u \in \mathbb{Z}/l \mathbb{Z}$ such that we have neither $a^+_u = a^-_{u+1}$ nor $a^-_u = a^+_{u+1}$ is non-empty. Moreover it has cardinal at least 2. Let us first consider the case $|V| = 2$ (this case indeed occurs when $k = 0$). We may for instance assume that

$$a^-_0 = a^-_{l-1} = x_0 \text{ and } a^+_0 = a^+_u = x_u,$$

for some $u_o \in \mathbb{Z}/l \mathbb{Z}\setminus\{0\}$. Hence we must have

$$a^+_{u+1} = a^-_u, \quad u = 0, \ldots, u - 2 \quad a^-_u = a^+_u, \quad u = u_o, \ldots, l - 2.$$

Choose a half-apartment $A^+_0$, whose vertex set is $\{s_0, s_1, \ldots, s_u, \ldots\}$, satisfying:

- for all $u \geq 0$, there is an edge $b_u$ of $A^+_0$, such that $b^+_u = s_u$, $b^-_u = s_{u+1}$;
- $s_0 = x_0 = b^+_0$.

Similarly choose a half-apartment $A^+_u$, whose vertex set is $\{t_0, t_1, \ldots, t_u, \ldots\}$, satisfying:

- for all $u \geq 0$, there is an edge $c_u$ of $A^+_u$, such that $c^+_u = s_{u+1}$, $c^-_u = s_u$;
- $t_0 = x_u = c^-_{u_o}$. 

Figure 3
Consider the two infinite paths:

\[ p_1 = (\ldots, s_u, \ldots, s_1, s_0 = x_0, x_1, x_2, \ldots, x_{u_o-1}, x_{u_o} = t_0, t_1, t_2, \ldots, t_v, \ldots) \]

\[ p_2 = (\ldots, s_u, \ldots, s_1, s_0 = x_0, x_{l-1}, x_{l-2}, \ldots, x_{u_o+1}, x_{u_o} = t_0, t_1, t_2, \ldots, t_v, \ldots) \]

By lemma (5.2.4) we can find an apartment \( \tilde{A}_1 \) (resp. \( \tilde{A}_2 \)) whose vertices are those of \( p_1 \) (resp. those of \( p_2 \)). Since \( \omega \) has finite support, we can give an obvious meaning to the integrals:

\[ \int_{p_1} \omega \] and \[ \int_{p_2} \omega \]

Moreover we have

\[ \int_{p_1} \omega = \mathcal{R}(\omega)(\tilde{A}_1) = 0 \] and \[ \int_{p_2} \omega = \mathcal{R}(\omega)(\tilde{A}_2) = 0 \]

Finally we have:

\[ \int_{p_1} \omega - \int_{p_2} \omega = \sum_{u=0,\ldots,u_o-1} \omega(a_u) - \sum_{u=l-1,l-2,\ldots,u_o} \omega(a_u) \]

\[ = \sum_{u=0,\ldots,u_o-1} [p : a_u] \omega(a_u) + \sum_{u=l-1,l-2,\ldots,u_o} [p : a_u] \omega(a_u) = \int_p \omega = 0 \]

□.

This proof extends to the case \( \#V > 2 \) by introducing for all \( u \in V \) a half-apartment starting at the vertex \( a_u \cap a_{u+1} \). The details are left to the reader (see figure 4).
5.3 Uniqueness of the model in $H^1_c(\tilde{X}_k, \mathbb{C})$ for supercuspidals

Because of the homeomorphism $\tilde{A} \simeq G/T$, the $G$-module $C^0_c(\tilde{A})$ is isomorphic to $C^0_c(G/T)$, the space of locally constant complex functions with compact support on $G/T$. The $G$-module $C^0_c(G/T)$ by definition is the compactly induced representation $c\text{-}\text{ind}^G_T 1_T$ of the trivial character $1_T$ of $T$ to $G$. So we may rephrase proposition (5.2.2) in the following way.

(5.3.1) PROPOSITION. — For all $k \geq 0$, the Radon transform induces an injective $G$-equivariant map:

$$H^1_c(\tilde{X}_k, \mathbb{C}) \rightarrow c\text{-}\text{ind}^G_T 1_T,$$

where $c\text{-}\text{ind}$ denotes a compactly induced representation and $1_T$ denotes the trivial character of $T$.

(5.3.2) THEOREM. — Let $(\pi, V_\pi)$ be an irreducible supercuspidal representation of $G$. Then we have

$$\dim \mathbb{C}\text{Hom}_G(V_\pi, H^1_c(\tilde{X}_n(\pi), \mathbb{C})) = 1.$$

This theorem will from the following result due to Waldspurger ([9] Prop. 9', p. 31):

(5.3.3) THEOREM (J.-L. Waldspurger). — Let $(\pi, V_\pi)$ be an irreducible unitary smooth representation of $G$. Then we have

$$\dim \mathbb{C}\text{Hom}_G(V_\pi, 1_T) = 1.$$

Proof of (5.3.2). — Waldspurger’s result together with Frobenius reciprocity imply that

$$\dim \mathbb{C}\text{Hom}_G(V_\pi, \text{Ind}^G_T 1_T) = 1,$$

where $\text{Ind}^G_T 1_T$ is the representation of $G$ smoothly induced from the trivial character of $T$. This representation is the smooth part of the $G$-module $C(G/T)$ formed of those locally constant functions on $G$ that are stabilized by an open subgroup of $G$. Since $c\text{-}\text{ind}^G_T 1_T$ naturally embeds as a sub-$G$-module of $\text{Ind}^G_T 1_T$, we get:

$$\dim \mathbb{C}\text{Hom}_G(V_\pi, c\text{-}\text{ind}^G_T 1_T) \leq \dim \mathbb{C}\text{Hom}_G(V_\pi, \text{Ind}^G_T 1_T),$$

and the theorem follows.
Bibliography


[2] Broussous (P.). — Type theory and the symmetric space $PGL(2, F)/T$, where $F$ is local non-archimedean and $T$ is the diagonal torus, in preparation.


