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Renormalized solution for nonlinear degenerate problems in the whole space


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Renormalized solution for nonlinear degenerate problems in the whole space

Mohamed Maliki\(^{(1)}\), Adama Ouedraogo\(^{(2)}\)

\textbf{ABSTRACT.} — We consider the general degenerate parabolic equation:

$u_t - \Delta b(u) + \text{div} \tilde{F}(u) = f$ \quad in $Q = [0, T] \times \mathbb{R}^N$, \quad $T > 0$.

We suppose that the flux $\tilde{F}$ is continuous, $b$ is nondecreasing continuous and both functions are not necessarily Lipschitz. We prove the existence of the renormalized solution of the associated Cauchy problem for $L^1$ initial data and source term. We establish the uniqueness of this type of solution under a structure condition $\tilde{F}(r) = F(b(r))$ and an assumption on the modulus of continuity of $b$. The novelty of this work is that $\Omega = \mathbb{R}^N$, $u_0$, $f \in L^1$, $b$, $\tilde{F}$ are not Lipschitz functions and the techniques are different from those developed in the previous works.

\textbf{RéSUMÉ.} — Nous considérons l’équation parabolique dégénérée général :

$u_t - \Delta b(u) + \text{div} \tilde{F}(u) = f$ \quad dans $Q = [0, T] \times \mathbb{R}^N$, \quad $T > 0$.

Nous supposons que le flux $\tilde{F}$ est continu, $b$ est continue et croissante au sens large et les deux fonctions ne sont pas nécessairement lipschitiennes. Nous prouvons l’existence de solution renormalisée du problème de Cauchy associé à cette équation avec des données (terme source et condition initiale) dans $L^1$. Nous établissons l’unicité de cette solution sous une condition dite de structure du type $\tilde{F}(r) = F(b(r))$ et sous une hypothèse sur le module de continuité de $b$. La nouveauté dans le travail vient du fait que $\Omega = \mathbb{R}^N$, $u_0$, $f \in L^1$, $b$, $\tilde{F}$ ne sont pas des fonctions nécessairement lipschitziennes et les techniques sont différentes de celles développées dans les travaux antérieurs.

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1. Introduction

Let $Q = [0, T] \times \mathbb{R}^N$ with $T > 0$.

We consider the Cauchy problem $(CP) = (CP)(b, F, f, u_0)$:

$$(CP) \begin{cases} 
    u_t - \Delta b(u) + \text{div} F(b(u)) = f & \text{in } Q \\
    u(0, \cdot) = u_0 & \text{on } \mathbb{R}^N,
\end{cases}$$

where $F$ is a Lipschitz continuous function on $\mathbb{R}$ and $b$ is a nondecreasing continuous function. For normalization, we set $F(0) = 0$ and $b(0) = 0$.

$(CP)(b, F, f, u_0)$ is a model of degenerate second order diffusion-convection motions of fluid; it has important applications in two phase flows in porous media (cf. [CJ]) and sedimentation-consolidation processes (cf. [BCBT]). It is well known that there is no classical solution and the weak solution is not unique. There exists a vast literature on the degenerate parabolic equation we consider. In this literature (cf. [AL], [BG], [BT1], [BW], [BT2], [DT], [YJ], [BM], [BR], [O], [BBGPV]), many results are proved about the existence of a weak solution and the uniqueness under various conditions.

In [MT3], we prove the existence and the uniqueness of the entropy solution of $(CP)(b, F, f, u_0)$ when $u_0, f$ are bounded and $\Omega = \mathbb{R}^N$.

In [IW], authors prove that $(CP)(b, F, f, u_0)$ is well posed in the sense of a renormalized solution when $\Omega$ is a bounded domain and $u_0, f \in L^1$.

In [MK], the existence and uniqueness of the renormalized solution of $(CP)$ are given when $\Omega = \mathbb{R}^N$, and $\tilde{F}, b$ are Lipschitz functions.

Here we extend these previous works to the case of $u_0, f$ in $L^1$, $\Omega = \mathbb{R}^N$ and $b$, $\tilde{F}$ not necessarily Lipschitz functions. More precisely, under the structure condition $\tilde{F}(r) = F(b(r))$ and the assumption $(H_2)$ on the modulus of continuity of $b$, we prove the existence and uniqueness of the renormalized solution of $(CP)(b, F, f, u_0)$ in the whole space $\mathbb{R}^N$ with $u_0, f$ in $L^1$. This condition on the modulus of continuity of $b$ appears at the first time in [BK] for conservation laws where the authors give a counterexample to set the optimality of the condition.

Note that if $b(r) = r^m$, this condition is $m > \frac{N - 1}{N}$.

The techniques, ideas and estimations developed in the present paper are based essentially on [C1], [C2], [IU1], [IU2], [IW], [MT3] and [M].

The present work falls into three sections. Section 1 is the introduction.
Section 2 is intended to prove the existence and uniqueness of the weak (entropy) solution when the data $u_0$ and $f$ are bounded. Section 3, on the other hand, using the results of the previous section, aims at proving the existence and uniqueness of the renormalized solution of $(CP)$ in the case of $L^1$ initial data and source term.

2. Existence and uniqueness of the weak solution

In this section we suppose that the initial data and source term satisfy the following hypothesis :

\[ (H1) \begin{cases} 
1) & u_0 \in L^\infty(\mathbb{R}^N); \\
2) & f \in L^1_{\text{Loc}}(Q) \text{ and for a.e } t \in [0,T], \ f(t) \in L^\infty(\mathbb{R}^N); \\
3) & \int_0^T \|f(t)\|_{L^\infty(\mathbb{R}^N)} \, dt < \infty.
\end{cases} \]

We define the operators $H_\epsilon$, $H$, $H_0$ and the truncation function at the level $k$ by:

\[ H_\epsilon(s) = \min(\frac{s+\epsilon}{\epsilon}, 1), \quad H(s) = \begin{cases} 
1 & \text{if } s > 0 \\
[0,1] & \text{if } s = 0 \\
0 & \text{if } s < 0,
\end{cases} \quad H_0(s) = \begin{cases} 
1 & \text{if } s > 0 \\
0 & \text{if } s \leq 0,
\end{cases} \]

\[ T_k(s) = \begin{cases} 
k & \text{if } s > k \\
s & \text{if } |s| \leq k \\
-k & \text{if } s < -k.
\end{cases} \]

**Definition 2.1.** — (Weak solution of $(CP) (b,F,f,u_0)$).

Let $u_0$ and $f$ be such that $(H1)$ is fulfilled. A weak solution of $(CP)$ is a function $u \in L^\infty(Q)$, such that :

\[ u_t \in L^2((0,T);H^{-1}_{\text{loc}}(\mathbb{R}^N)) + L^1((0,T);L^\infty(\mathbb{R}^N)), \quad (2.1) \]

\[ b(u) \in L^2((0,T);H^1_{\text{loc}}(\mathbb{R}^N)), \quad (2.2) \]

\[ u_t - \Delta b(u) + \text{div}F(b(u)) = f \quad \text{in} \quad \mathcal{D}'(Q) \]

and $u(0,x) = u_0$ on $\mathbb{R}^N$.

The last condition must be understood in the sense that

\[ \int_0^T <u_t, \xi> \, dt = -\int_Q u \xi_t \, dx \, dt - \int_{\mathbb{R}^N} u_0 \xi(0) \, dx \quad (2.3) \]

for any $\xi \in L^2((0,T);\mathcal{D}(\mathbb{R}^N)) \cap W^{1,1}((0,T);L^\infty(\mathbb{R}^N))$ so that $\xi(T) = 0$ and $<,>$ represents the duality product between $H^{-1}(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$. 

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Definition 2.2 (Entropy solution of \((CP) (b,F,f,u_0)\)). — Let \(u_0\) and \(f\) verify \((H1)\). An entropy solution \(u\) of \((CP) (b,F,f,u_0)\) is a weak solution of the same problem such that:

\[
\begin{align*}
&\int_Q H_0(u-s)\{\nabla b(u)\nabla \xi - (F(b(u)) - F(b(s)))\nabla \xi - (u-s)\xi_t - f\xi\} \, dx \, dt \\
&\quad - \int_{\mathbb{R}^N}(u_0-s)^+\xi(0) \, dx \leq 0 \\
&\text{and} \\
&\int_Q H_0(s-u)\{\nabla b(u)\nabla \xi - (F(b(u)) - F(b(s)))\nabla \xi - (u-s)\xi_t - f\xi\} \, dx \, dt \\
&\quad + \int_{\mathbb{R}^N}(s-u_0)^+\xi(0) \, dx \geq 0
\end{align*}
\]
for any \(s \in \mathbb{R}\) and \(\xi \in \mathcal{D}(Q), \xi \geq 0\).

Theorem 2.3. — We suppose that \(u_0, f\) verify \((H1)\) and that \(b\) is not a Lipschitz function.

Let \(\omega\) be the modulus of continuity of \(b\); we suppose that \(\omega\) satisfies

\[
\begin{align*}
&\liminf_{\varepsilon \to 0} \frac{\omega(\varepsilon)^N}{\varepsilon^{N-1}} < +\infty \text{ if } N > 2 \text{ and } \liminf_{\varepsilon \to 0} \frac{\omega(\varepsilon)^2}{\varepsilon} = 0 \text{ if } N = 2; \\
&\text{for } N = 1 \text{ there is no condition on } \omega.
\end{align*}
\]

\((H2)\)

Then the Cauchy problem \((CP)(b,F,u_0,f)\) has a unique weak solution.

Remark 2.4. — If \(b = r^m\), we can see by an elementary calculus that \((H2)\) means that \(m > \frac{N-1}{N}\).

Proof of Theorem 2.3. —

1. Given that an entropy solution of \((CP)\) is a weak solution of the same problem, the existence of the weak solution is a consequence of Theorem 3.7 of [MT3].

2. For the uniqueness we use Theorem 3.11 in [MT3]. However, since we are working in the whole space, we cannot take a test function identically equal to one in Kato’s inequality (2.8) below to get the comparison principle. An additional condition on \(b\) is required (hypothesis \((H2)\)).
Let $u$ be a weak solution of $(CP)(b,F,u_0,f)$, $s \in \mathbb{R}$ such that $b(s) \notin E$ where

$$E = \{ r \in \text{Im}(b)/; (b^{-1})_0 \text{ is discontinuous at } r \}.$$\n
Since $(b^{-1})_0$ is a monotone function, $E$ is a countable subset of $\mathbb{R}$.

Let $\xi \in \mathcal{D}(\mathbb{R}^N)$, $\xi \geq 0$. Consider that $R$ is big enough such that $\text{supp} \xi \subset B(0,R) =: \Omega$; in particular we have $\xi \equiv 0$ on $\partial \Omega$. It is known that $u$ is a weak solution of $(CP)$ on $\Omega \times (0,T)$; by Lemma 4 in [IU1], Lemma 5 and Theorem 6 in [C2] we have:

$$\int_Q H_0(u-s)\{(\nabla b(u) + F(b(s)) - F(b(u)))\nabla \xi + (s-u)\xi_t - f\xi\} \, dx \, dt$$

$$- \int_{\mathbb{R}^N} (u_0 - s)^+ \xi(0) \, dx = \lim_{\epsilon \to 0} \int_Q |\nabla b(u)|^2 H'(b(s) - b(u)) \xi \, dx \, dt$$

\hspace{1cm} (2.6)

and

$$\int_Q H_0(s-u)\{(\nabla b(u) + F(b(s)) - F(b(u)))\nabla \xi + (s-u)\xi_t - f\xi\} \, dx \, dt$$

$$- \int_{\mathbb{R}^N} (s-u_0)^+ \xi(0) \, dx = \lim_{\epsilon \to 0} \int_Q |\nabla b(u)|^2 H'(b(s) - b(u)) \xi \, dx \, dt$$

\hspace{1cm} (2.7)

for all $s \in \mathbb{R}$ such that $b(s) \notin E$ and $\xi \in \mathcal{D}(Q)$, $\xi \geq 0$.

By the method of doubling variables introduced by S.N. Kružkhov for the conservation laws and adapted by J. Carrillo for second order parabolic equations, we obtain the so called Kato’s inequality:

For $(u_{01}, f_1)$ and $(u_{02}, f_2)$ satisfying $(H1)$, let $u_1$, $u_2$ be weak solutions of $(CP)(b,F,f_1,u_{01})$, $(CP)(b,F,f_2,u_{02})$ respectively. Then

$$\int_Q \{((\nabla b(u_1) - b(u_2))^+$$

$$+ H_0(u_1 - u_2)(F(b(u_2)) - F(b(u_1)))\nabla \xi - (u_1 - u_2)^+ \xi_t\} \, dx \, dt$$

$$- \int_{\mathbb{R}^N} (u_{01} - u_{02})^+ \xi(0) \, dx \leq \int_Q \nu(f_1 - f_2) \xi \, dx \, dt$$

\hspace{1cm} (2.8)

for any $\nu \in H(u_1 - u_2)$ a.e. and $\xi \in \mathcal{D}([0,T]\times\mathbb{R}^N)$, $\xi \geq 0$.

Let now $W = (u_1 - u_2)^+$, $W_0 = (u_{01} - u_{02})^+$ and $h = (f_1 - f_2)^+$. By using Theorem 3.11 in [MT3], $(H2)$ and the fact that

$$|b(u_1) - b(u_2)|\chi_{\{u_1 > u_2\}} \leq \omega(W), \quad |F(b(u_1)) - F(b(u_2))|\chi_{\{u_1 > u_2\}} \leq C\omega(W),$$

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one has
\[ \int_{\mathbb{R}^N} (u_1(t) - u_2(t))^+ \, dx \leq \int_{\mathbb{R}^N} (u_{01} - u_{02})^+ \, dx + \int_Q \nu(f_1 - f_2) \, dx \, ds \quad (2.9) \]
for \( \nu \in H(u_1 - u_2) \) a.e..

The uniqueness and comparison principle of weak solutions are the consequences of this last inequality (2.9). □

3. Existence and uniqueness of the renormalized solution

In this section we suppose that the initial data and source term satisfy the following hypothesis:

\((H3)\) \quad \begin{align*}
& u_0 \in L^1(\mathbb{R}^N), \quad f \in L^1(Q), \\
& b \text{ satisfies the condition } (H2) \text{ and } F \text{ is a Lipschitz continuous function.}
\end{align*}

It is important to mention that \( b \) and \( \tilde{F} \) are not necessarily Lipschitz functions as in \([MK]\).

**Definition 3.1 (Renormalized solution of \((CP)\) \((b,F,f,u_0)\)).** — A renormalized solution of \((CP)\) \((b,F,f,u_0)\) is a function \( u \) such that:

1. \( u \in L^1(Q) \);
2. \( T_k b(u) \in L^2(0,T,H^{1}_{loc}(\mathbb{R}^N)) \) for any \( k > 0 \);
3. For all \( \xi \in \mathcal{D}([0,T) \times \mathbb{R}^N) \) and \( h \in C(\mathbb{R}) \),

\[
\begin{cases}
- \int \int_Q \xi_t \left( \int_{u_0}^u h(b(r)) \, dr \right) \, dx \, dt + \int \int_Q (\nabla b(u) \cdot \nabla h(b(u))) \xi \, dx \, dt = \int \int_Q fh(b(u))\xi \, dx \, dt.
\end{cases}
\]  

\quad (3.10)

In addition

\[
\int \int_{Q \cap [n\leq|b(u)|\leq n+1]} \{|\nabla b(u)|^2 - F(b(u)).\nabla b(u)\} \, dx \, dt \to 0 \quad \text{as} \quad n \to +\infty.
\]  

\quad (3.11)

**Proposition 3.2.** — Let \( f_1, f_2 \in L^1(Q) \) and \( u_{01}, u_{02} \in L^1(\mathbb{R}^N) \). If \( u_i \) is a renormalized solution of \((CP)\) \((b,F,f_i,u_{0i})\) for \( i = 1,2 \) then

\[
\int_{\mathbb{R}^N} (u_1(t) - u_2(t))^+ \, dx \leq \int_{\mathbb{R}^N} (u_{01} - u_{02})^+ \, dx + \int_0^t \int_{\mathbb{R}^N} \nu(f_1 - f_2) \, dx \, ds
\]  

\quad (3.12)

where \( \nu \in \text{Sign}^+(u_1 - u_2) \).
Proof of Proposition 3.2. — The main idea of the proof is to state that a renormalized solution of \((CP)\) \((b,F,f,u_0)\) satisfies an entropy inequality for an auxiliary problem and use Theorem 2.3 to get the uniqueness.

To be more precise, let \(h \in C_c(\mathbb{R})\) with \(h \geq 0\), \(h(0) > 0\) and

\[
\begin{align*}
\dot{j}_h(r) &= \int_0^r h(b(s)) ds, \quad b_h(r) = \int_0^{b(r)} h(s) ds, \\
F_h(r) &= h(b(r))F(b(r)), \quad f_h(r) = h(b(r))f(r), \\
G_h(r) &= (\nabla b(r) + F(b(r)) ) \cdot \nabla h(b(r)).
\end{align*}
\]

We consider the Cauchy problem below:

\[
(CP)_h \begin{cases}
\frac{\partial}{\partial t} j_h(u) - \nabla (\nabla b_h(u) - F_h(u)) = f_h(u) + G_h(u) & \text{in } Q \\
j_h(u(0)) = j_h(u_0) & \text{on } \mathbb{R}^N.
\end{cases}
\]

The first step is to ensure that if \(u\) is a renormalized solution of \((CP)\) \((b,F,f,u_0)\) then \(u\) satisfies the entropy inequality below of the auxiliary problem \((CP)_h\):

\[
\int \int_Q H_0(u-k) \{(j_h(u) - j_h(k)) \xi_t - (\nabla b_h(u) - (F_h(u) - F_h(k))) \cdot \nabla \xi \} dx \, dt \\
\geq - \int_{\mathbb{R}^N} (j_h(u_0) - j_h(k))^+ \xi(0) dx - \int \int_Q H_0(u-k)(f_h(u) + G_h(u)) \xi dx \, dt.
\]

Let us take \(R\) large enough such that \(supp(\xi) \subset B(0,R) \times (0,T) = Q_1\) and \(\Omega = B(0,R)\).

The last inequality (3.13) is equivalent to

\[
\int \int_{Q_1} H_0(u-k) \{(j_h(u) - j_h(k)) \xi_t - (\nabla b_h(u) - (F_h(u) - F_h(k))) \cdot \nabla \xi \} dx \, dt \\
\geq - \int_{\Omega} (j_h(u_0) - j_h(k))^+ \xi(0) dx - \int \int_{Q_2} H_0(u-k)(f_h(u) + G_h(u)) \xi dx \, dt
\]

which is the same as inequality (2.4) in [IW] ; therefore using the same method, we get the result (3.13). However we do not have to take the same
function $k$ because we are working in $\mathbb{R}^N$ and we do not have any boundary condition.

The second step to prove Proposition 3.2 is to get Kato’s inequality for the auxiliary problem $(CP)_h$ i.e., if $f_1, f_2 \in L^1(Q)$, $u_{01}, u_{02} \in L^1(\mathbb{R}^N)$ and $u_i$ is a renormalized solution of $(CP)$ $(b,F,f_i,u_{0i})$ for $i = 1,2$ then

$$
\begin{aligned}
& \int \int_{Q_1} H_0(u_1 - u_2) \{ (j_h(u_1) - j_h(u_2)) \xi_t - (\nabla (b_h(u_1) - b_h(u_2)))^+ \\
& + F_h(u_1) - F_h(u_2)) \cdot \nabla \xi \} \, dx \, dt \\
\geq - \int_{Q_1} (j_h(u_{01}) - j_h(u_{02}))^+ \xi(0) \, dx - \int \int_{Q_1} \nu((f_h(u_1) + G_h(u_1)) \\
& - (f_h(u_2) + G_h(u_2))) \xi \, dx \, dt \\
\end{aligned}
$$

for any $\nu \in H(u_1 - u_2)$ a.e. and $\xi \in D([0,T] \times \mathbb{R}^N)$, $\xi \geq 0$.

The technique of Kružkov’s doubling variables enables us to deduce (3.15) from entropy inequalities (2.4)-(2.5) as in ([C1], [C2], [KR], ...).

Now given that we are working in the whole space, we cannot take the same test function $\xi \equiv 1$ as in [IW]. We have to use the technique of [MT3] to get the inequality (3.12) from Kato’s inequality (3.15). To this end, we have to verify that the new functions $j_h$, $b_h$ and $F_h$ satisfy the hypothesis $(H_2)$. As long as this is not true for all $h$, we have to make the appropriate choice of $h$ to get the hypothesis $(H_2)$ . Set $h_n(r) = \inf((n + 1 - |r|)^+,1)$ in inequality (3.15) and let $n$ large enough as in [IU2] ; the expressions of $j_{h_n}$, $b_{h_n}$ take the following form :

$$
\begin{cases}
\frac{b^{-1}(-n)}{b^{-1}(-n-1)} (n + 1 + b(s)) \, ds \quad \text{if} \quad r \leq b^{-1}(-n-1) \\
\frac{b^{-1}(-n)}{b^{-1}(n)} (n + 1 + b(s)) \, ds \quad \text{if} \quad b^{-1}(-n-1) \leq r \leq b^{-1}(-n) \\
\frac{r}{b^{-1}(n)} \quad \text{if} \quad b^{-1}(-n) \leq r \leq b^{-1}(n) \\
\frac{b^{-1}(n)}{b^{-1}(n+1)} (n + 1 - b(s)) \, ds \quad \text{if} \quad b^{-1}(n) \leq r \leq b^{-1}(n+1) \\
\frac{b^{-1}(n)}{b^{-1}(n)} (n + 1 + b(s)) \, ds \quad \text{if} \quad r \geq b^{-1}(n+1),
\end{cases}
$$

(3.16)
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\[ b_{h_n}(r) = \begin{cases} 
- n - \frac{1}{2} & \text{if } r \leq b^{-1}(-n - 1) \\
- n + \int_{-n}^{b(r)} (n + 1 + s) ds & \text{if } b^{-1}(-n - 1) \leq r \leq b^{-1}(-n) \\
b(r) & \text{if } b^{-1}(-n) \leq r \leq b^{-1}(n) \\
n + \int_{n}^{b(r)} (n + 1 - s) ds & \text{if } b^{-1}(n) \leq r \leq b^{-1}(n + 1) \\
n + \frac{1}{2} & \text{if } r \geq b^{-1}(n + 1). 
\end{cases} \]

For \( n \) large enough, only the set \([-n + \frac{1}{2}, n + \frac{1}{2}]\) is important and the graph \( b_{h_n}((j_{h_n})^{-1}) \) can be extended by any regular increasing profile (for instance see [IU2]). Let us see \( b_{h_n} \) and \( j_{h_n} \) in this set.

\[ j_{h_n}(r) = \begin{cases} 
b^{-1}(-n) + \int_{b^{-1}(-n)}^{r} (n + 1 + b(s)) ds & \text{if } b^{-1}(-n - \frac{1}{2}) \leq r \leq b^{-1}(-n) \\
r & \text{if } b^{-1}(-n) \leq r \leq b^{-1}(n) \\
b^{-1}(n) + \int_{b^{-1}(n)}^{r} (n + 1 - b(s)) ds & \text{if } b^{-1}(n) \leq r \leq b^{-1}(n + \frac{1}{2}) 
\end{cases} \]

\[ j'_{h_n}(r) = \begin{cases} 
n + 1 + b(r) & \text{if } b^{-1}(-n - \frac{1}{2}) \leq r \leq b^{-1}(-n) \\
1 & \text{if } b^{-1}(-n) \leq r \leq b^{-1}(n) \\
n + 1 - b(r) & \text{if } b^{-1}(n) \leq r \leq b^{-1}(n + \frac{1}{2}). 
\end{cases} \]

On the set \([-n + \frac{1}{2}, n + \frac{1}{2}]\), \( j_{h_n} \) is an increasing Lipschitz function and \( \frac{1}{2} \leq j'_{h_n} \leq 1 \).

Consequently \( j_{h_n} \) is one to one, \( j_{h_n}^{-1} \) exists and \( |(j_{h_n}^{-1})'| \leq 2 \). Then on the set \([-n + \frac{1}{2}, n + \frac{1}{2}]\),

\[ |b_{h_n}(r) - b_{h_n}(r')| = |b(j_{h_n}^{-1}(r) - b(j_{h_n}^{-1})(r')| \leq 2\omega(r - r') \]

with \( \omega \) the modulus of continuity of \( b \) and

\[ |F_{h_n}(r) - F_{h_n}(r')| = |F(j_{h_n}^{-1}(r) - F(j_{h_n}^{-1})(r')| \leq 2C\omega(r - r'). \]

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It is now clear that \( b_{hn} \) and \( F_{hn} \) satisfy (H2). We can use Theorem 2.3 to claim that for \( h = h_n \) and \( n \) large enough we have the following result: if \( u_i \) is a renormalized solution of \((CP) (b, F, f_i, u_{0_i})\) for \( i = 1, 2 \) then:

\[
\begin{cases}
\int_{\mathbb{R}^N} \left( \int_{u_2(t)}^{u_1(t)} h(b(s)) \, ds \right)^+ \, dx 
\leq \int_{\mathbb{R}^N} \left( \int_{u_{01}}^{u_{02}} h(b(s)) \, ds \right)^+ \, dx \\
+ \int_0^t \int_{\mathbb{R}^N} \nu(f_1 h(b(u_1)) - f_2 h(b(u_2))) \, dx \, ds \\
+ \int_0^t \int_{\mathbb{R}^N} \nu[(\nabla b(u_1) + F(b(u_1))) \cdot \nabla (h(b(u_1))) - (\nabla b(u_2)) \cdot \nabla (h(b(u_2)))] \, dx \, ds.
\end{cases}
\] (3.20)

In the inequality (3.20), we take the limit when \( n \) goes to the infinity to get the comparison principle (3.12). In fact,

\[
\begin{align*}
\int_{\mathbb{R}^N} \left( \int_{u_2(t)}^{u_1(t)} h(b(s)) \, ds \right)^+ \, dx &\to \int_{\mathbb{R}^N} (u_1(t) - u_2(t))^+ \, dx, \\
\int_{\mathbb{R}^N} \left( \int_{u_{01}}^{u_{02}} h(b(s)) \, ds \right)^+ \, dx &\to \int_{\mathbb{R}^N} (u_{01} - u_{02})^+ \, dx, \\
\int_0^t \int_{\mathbb{R}^N} \nu(f_1 h(b(u_1)) - f_2 h(b(u_2))) \, dx \, ds &\to \int_0^t \int_{\mathbb{R}^N} \nu(f_1 - f_2) \, dx \, ds.
\end{align*}
\]

The last term in (3.20), on the other hand, goes to 0 when \( n \to \infty \) by definition of the renormalized solution (inequality (3.11)). □

**THEOREM 3.3.** — For all \( u_0 \) and \( f \) verifying (H3), there exists a renormalized solution of the problem \((CP)(b, F, u_0, f)\).

**Proof of Theorem 3.3.** — The proof of the existence of a renormalized solution is based on ideas and techniques developed in [IW], [MT2], [MT3]. By \( B_n \), we denote the ball centered at 0 of radius \( n \). We consider the following Cauchy problem in a bounded domain \((CP)(b, F, u_0, f)_n\):

\[
(CP)_n \begin{cases}
\frac{du}{dt} - \Delta b(u) + \text{div} \, F(b(u)) = f & \text{in } (0,T) \times B_n \\
b(u) = 0 & \text{on } (0,T) \times \partial B_n \\
u(0, x) = u_0 & \text{in } B_n.
\end{cases}
\]

Based on the results of [AI] and [IW], the Cauchy problem \((CP)_n\) has a unique renormalized solution \( u_n \) such that \( u_n \in C([0,T], L^1(B_n)) \) and satisfies the comparison principle. By \( u_n \), we also denote the extension of \( u_n \) to \( \mathbb{R}^N \) by 0 outside of \( B_n \). Let \( \Omega \subset \mathbb{R}^N \) a bounded open set with
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smooth boundary. We consider that \( n \) is large enough such that \( \Omega \subset B_n \).
Thus \( u_n \in C([0,T), L^1(\Omega)) \) and by comparison principle \( u_n \) is a Cauchy
sequence. So when \( n \) goes to infinity,
\[
u_n \to u \quad \text{in} \quad C([0,T), L^1(\Omega)).\tag{3.21}
\]
We now prove that \( u \) is a renormalized solution of \((CP)\).

Set \( Q_1 = (0,T) \times \Omega \) and \( Q_n = (0,T) \times B_n \). Let us take \( h(b(u_n))\xi \) as a test
function in the definition of \( u_n \) with \( h \in w^{1,\infty}(\mathbb{R}) \), \( h \geq 0 \) and \( \xi \in C^1(B_n) \)
such that \( h(b(u_n))\xi \in L^2(0,T,H^1_0(B_n)) \).

With Lemma 4 in [C1] one has
\[
\begin{cases}
- \int_{Q_n} \xi_t j_h(u_n) \, dx \, dt + \int_{Q_n} (\nabla b(u_n) - F(b(u_n))).\nabla (h(b(u_n))\xi) \, dx \, dt \\
+ \int_{B_n} \xi(T) j_h(u_n(T)) \, dx = \int_{B_n} \xi(0) j_h(u_0) \, dx + \int_{Q_n} f h(b(u_n))\xi \, dx \, dt.
\end{cases}
\tag{3.22}
\]
As first step, we prove that if \( n \) goes to infinity then
\[
T_k b(u_n) \rightharpoonup T_k b(u) \quad \text{in} \quad L^2(0,T;H^1_{ loc}(\mathbb{R}^N)) \quad \text{for any} \quad k > 0. \tag{3.23}
\]
If we choose \( h(r) = T_k(r) \) on \( \mathbb{R} \) and \( \xi \equiv 1 \) in (3.22), we get
\[
\begin{cases}
\int_{B_n} j_h(u_n(T)) \, dx + \int_{Q_n} |\nabla T_k b(u_n)|^2 \, dx \, dt \\
= \int_{Q_n} F(b(u_n)).\nabla T_k b(u_n) \, dx \, dt \\
+ \int_{B_n} j_h(u_0) \, dx + \int_{Q_n} f T_k b(u_n)\xi \, dx \, dt.
\end{cases}
\tag{3.24}
\]
One knows that \( \int_{Q_n} F(b(u_n)).\nabla T_k b(u_n) = 0 \) and \( j_h \geq 0 \). Thus we obtain
\[
\int_{Q_1} |\nabla T_k b(u_n)|^2 dx \, dt \leq \int_{Q_n} |\nabla T_k b(u_n)|^2 dx \, dt \\
\leq k(\int_{Q} |f| \, dx \, dt + \int_{\mathbb{R}^N} |u_0| \, dx).
\]
By Poincaré’s inequality we deduce that \( T_k b(u_n) \) is a bounded sequence in
\( L^2(0,T;H^1(\Omega)) \).

We consider a subsequence denoted again by \( n \) such that
\[
T_k b(u_n) \rightharpoonup G \quad \text{in} \quad L^2(0,T;H^1(\Omega)) \quad \text{for any} \quad k > 0,
\]

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Therefore one can easily get (3.23).

Let us now prove that \( u \) satisfies (3.11). We take \( h(r) = T_{k+1}(r) - T_k(r) \), \( \xi \equiv 1 \) in (3.22) and by Lemma 4 in [C1] one has:

\[
\begin{align*}
&\int_{B_n} j_h(u_n(T)) dx + \int \left\{ \int_{[k \leq |b(u_n)| \leq k+1]} (|\nabla b(u_n)|^2 - F(b(u_n)) \nabla b(u_n)) dx \right\} dt \\
&= \int_{B_n} j_h(u_0) dx + \int \int_{Q_n} f h(b(u_n)) dx dt.
\end{align*}
\]

Since \( j_h \geq 0 \) then

\[
\begin{align*}
&\int \int_{[k \leq |b(u_n)| \leq k+1]} (|\nabla b(u_n)|^2 dx dt - \int \int_{[k \leq |b(u_n)| \leq k+1]} F(b(u_n)) \nabla b(u_n)) dx dt \\
&\leq \int \int_{[|u_0| \geq k]} j_h(u_0) dx + \int \int_{[|b(u)| \geq k]} f h(b(u_n)) dx dt.
\end{align*}
\]

As \( \int \int_{[k \leq |b(u_n)| \leq k+1]} F(b(u_n)) \nabla b(u_n) dx dt = 0 \), when \( n \) goes to \( \infty \), one has:

\[
\limsup_{n \to +\infty} \int \int_{[k \leq |b(u_n)| \leq k+1]} |\nabla b(u_n)|^2 dx dt \leq \int \int_{[b(u) \geq k]} |f| dx dt + \int \int_{[|u_0| \geq k]} |u_0| dx.
\]

With the fact that

\[
T_{k+1}b(u_n) - T_kb(u_n) \to T_{k+1}b(u) - T_kb(u) \quad \text{in} \quad L^2(0, T; H^1(\Omega)), \quad (3.25)
\]

we get

\[
\int \int_{[k \leq |b(u)| \leq k+1]} |\nabla b(u)|^2 dx dt \leq \int \int_{[b(u) \geq k]} |f| dx dt + \int \int_{[|u_0| \geq k]} |u_0| dx.
\]

Letting \( k \) goes to \( +\infty \), we obtain (3.11).

The final step of the proof is to have (3.10). To this end, we have to verify that:

\[
|\nabla T_kb(u_n)|^2 \to |\nabla T_kb(u)|^2 \quad \text{in} \quad L^1(Q_1) \quad \text{when} \quad n \to +\infty. \quad (3.26)
\]

We will apply arguments developed in [AW] and [IW] to our case. We consider the regularization in time of \( T_kb(u) \) defined by Landes’s method in [L] :

\[
v_m := m \int_{-\infty}^{t} e^{m(s-t)} T_kb(u(s, x)) ds
\]
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for a.e \((t, x)\); for \(s < 0\) we extend \(b(u)\) by \(0\). Then we have:

\[ v_m \in L^2(0, T; H^1(B_n)) \cap L^\infty(Q_n), \quad v_m(0) = 0; \]

\( v_m \) is differentiable for a.e. \( t \in (0, T) \) and \( \frac{\partial v_m}{\partial t} = m(T_k b(u) - v_m) \in L^2(0, T; H^1(B_n)) \cap L^\infty(Q_n). \)

Let \( \sigma \in D^+(0, T) \) and \( h(l) = h_1(l) = \inf((l + 1 - |r|)^+, 1) \). One can see that \( h \in w^{1, \infty}(\mathbb{R}), h \geq 0 \) and \( \text{supp}(h) \) is compact. So by inequality (3.9) in [IW], we deduce that

\[
\liminf_{l \to \infty} \liminf_{m \to \infty} \lim_{n \to \infty} \int_{Q_n} \sigma(\nabla b(u_n) - F(b(u_n))) \cdot \nabla(h_1(b(u_n))(T_k(b(u_n)) - v_m)) \, dx \, dt \leq 0. \tag{3.27}
\]

Since \( Q_1 \subset Q_n \) for \( n \) large enough, this last inequality implies that:

\[
\liminf_{l \to \infty} \liminf_{m \to \infty} \lim_{n \to \infty} \int_{Q_1} \sigma(\nabla b(u_n) - F(b(u_n))) \cdot \nabla(h_1(b(u_n))(T_k(b(u_n)) - v_m)) \, dx \, dt \leq 0 \tag{3.28}
\]

and by using the same arguments as in [AW] and [IW], we deduce that:

\[
\liminf_{m \to \infty} \lim_{n \to \infty} \int_{Q_1} \sigma|\nabla T_k b(u_n) - \nabla v_m|^2 \, dx \, dt = 0, \tag{3.29}
\]

and then

\[
\lim_{n \to \infty} \int_{Q_1} \sigma|\nabla T_k b(u_n) - \nabla b(u)|^2 \, dx \, dt = 0.
\]

By using (3.23) the last equality implies (3.26) and therefore (3.10).

We then deduce that \( u \) satisfies the definition of renormalized solution which ends the proof of the existence of renormalized solution. \( \Box \)

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Bibliography


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[IU2] Igbida (N.), Urbano (J.M.). — Continuity results for certain nonlinear parabolic PDEs, Preprint LAMFA, Université de Picadie Jules Vernes.


