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Long memory and self-similar processes[(*)]

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Abstract. — This paper is a survey of both classical and new results and ideas on long memory, scaling and self-similarity, both in the light-tailed and heavy-tailed cases.

Résumé. — Cet article est une synthèse de résultats et idées classiques ou nouveaux sur la longue mémoire, les changements d'échelles et l'autosimilarité, à la fois dans le cas de queues de distributions lourdes ou légères.

1. Introduction

The notion of *long memory* (or *long range dependence*) has intrigued many at least since B. Mandelbrot brought it to the attention of the scientific community in the 1960s in a series of papers (Mandelbrot (1965); Mandelbrot and Van Ness (1968) and Mandelbrot and Wallis (1968, 1969)) that, among other things, explained the so-called *Hurst phenomenon*, having to do with unusual behavior of the water levels in the Nile river.

Today this notion has become especially important as potentially crucial applications arise in new areas such as communication networks and finance. It is, perhaps, surprising that what the notion of long memory really is, has never been completely agreed upon. In this survey we attempt to describe the important ways in which one can think about long memory and connections between long memory and other notions of interest, most importantly scaling and self-similarity.

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In the next section we discuss some of the classical approaches to long memory, and point out some of the pitfalls along the way. Section 3 discusses self-similar processes and the connection between the exponent of self-similarity and the length of memory of the increment processes. Section 4 describes recent results on memory of stationary stable processes, and the last section present brief conclusions.

2. In what ways can on think about long memory?

There is an agreement in probability that the notion of long memory should be considered in application to stationary processes only, *i.e.* only in the context of phenomena "in steady state". The point is, however, delicate. First, in various applications of stochastic modeling this term is applied to non-stationary processes. Thus, for example, the usual Brownian motion is sometimes viewed as having long memory because it never really forgets where started from (this is, however, very unreasonable to a probabilist who immediately thinks about independent increments of the Brownian motion.) Second, stationary processes with long memory (in whatever sense) sometimes resemble their non-stationary counterparts, as we will see in the sequel. It is, therefore, possible to think of long memory processes as being that layer among the stationary processes that is "near the boundary" with non-stationarity, or as the layer separating the non-stationary processes from the "well behaved, usual" stationary processes.

What is, then, the difference between the "usual" stationary processes and long memory ones?

The first thought that comes to mind is, obviously, about correlations. Suppose that $(X_n,\, n = 0, 1, 2, \dots)$ is a stationary stochastic process with mean $\mu = EX_0$ and $0 < EX_0^2 < \infty$ (we will consider discrete time processes, but the corresponding formulations for stationary processes with finite variance in continuous time are obvious.) Let $\rho_n = \mathrm{Corr}(X_0, X_n)$, $n = 0, 1, \dots$ be its correlation function. How does the correlation function of the "usual" stationary processes behave? It requires skill and knowledge to construct an example where the correlation function decays to zero (as lag increases) at a slower than exponentially fast rate. For example, the common linear (ARMA) processes, GARCH processes, finite state Markov chains all lead to exponentially fast decaying correlations. A process with correlations that are decaying slower than exponentially fast is, then, "unusual". If the correlations are not even absolutely summable, then the term "long memory" is often used. See Beran (1994).

It is instructive to look at the simplest case, the so called AR(1) model.

Let Z_j, $j = 1, 2, \ldots$ be i.i.d. random variables with zero mean and non-zero finite variance. Choose a number $-1 < \rho < 1$, and an arbitrary (potentially random, but independent of the sequence Z_j, $j = 1, 2, \ldots$) initial state X_0. The AR(1) process X_n, $n = 0, 1, 2, \ldots$ is defined by

$$X_n = \rho X_{n-1} + Z_n, \quad n = 1, 2, \ldots. \tag{2.1}$$

It is elementary to check that the distribution of X_n converges to a limiting distribution, which is then automatically a stationary distribution of this simple Markov process. Choose the initial state X_0 according to this stationary distribution, which can be written in the form

$$X_0 = \sum_{j=0}^{\infty} \rho^j Z_{-j},$$

where $\ldots, Z_{-1}, Z_0$ are i.i.d. random variables independent of $Z_1, Z_2, \ldots$, and with the same distribution. Then the entire AR(1) process is already stationary, and

$$X_n = \sum_{j=0}^{\infty} \rho^j Z_{n-j}, \quad n = 0, 1, \ldots. \tag{2.2}$$

Notice that the correlation function of the stationary AR(1) process given by (2.2) is given by $\rho_n = \rho^n$, $n = 0, 1, 2, \ldots$, and it is exponentially fast decaying. In this sense the stationary AR(1) process is “usual”. Notice also that the exponential rate of decay of correlations becomes slower as ρ approaches ± 1.

Of course, if ρ is exactly equal to -1 or 1, then the AR(1) process in (2.1) cannot be made stationary; in fact, if $\rho = 1$, then the AR(1) process is a random walk, which a is discrete-time equivalent of the Brownian motion. Therefore, in this case there is no real boundary layer between the “usual” stationary processes with exponentially fast decaying correlations and non-stationary ones. Nonetheless, even here, when ρ is close to -1 or 1, we may observe some features in the realizations of a stationary AR(1) process that remind us of non-stationary processes, such as a random walk. See the plots below.

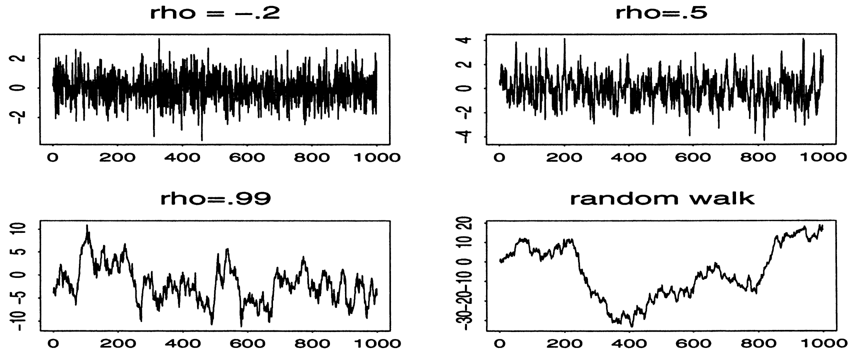

Figure 1. — Different AR(1) models and a random walk.

Overall, however, the class of AR(1) processes is too narrow to observe "unusual" stationary models that are close to being non-stationary.

In order to construct stationary processes with non-summable correlations we have to go to wider classes of processes. A natural way of doing that is via spectral domain. The book of Beran (1994) can be consulted for details. Briefly, we will consider stationary processes with a finite positive variance σ^2, that possess a spectral density f, which is a nonnegative function on $(0,\pi)$ such that for $n = 0, 1, 2, \ldots$,

$$\rho_n = \frac{1}{\sigma^2}\int_0^\pi \cos(nx) f(x)dx. \tag{2.3}$$

Intuitively, fast rate of decay of correlations is associated with a nice spectral density, especially as far as the behavior of the spectral density around zero is concerned. More specifically (but still informally), it is sometimes the case that if a spectral density diverges to infinity at certain rate, then the covariance function converges to zero at an appropriate slow rate (and vice versa). The association is real, even though imprecise statements abound in the literature.

Below is one precise result; see Samorodnitsky (2002).

Theorem 2.1. — *(i) Assume that*

$$\rho_n = n^{-d}L(n),\ n = 0, 1, 2, \ldots, \tag{2.4}$$

where $0 < d < 1$ *and* L *is slowly varying at infinity, satisfying the following assumption:*

for every $\delta > 0$ both functions

$$g_1(x) = x^{\delta} L(x) \text{ and } g_2(x) = x^{-\delta} L(x) \tag{2.5}$$

are eventually monotone.

Then the process has a spectral density, say, f, satisfying

$$f(x) \sim x^{-(1-d)} L(x^{-1}) \frac{2}{\pi} \Gamma(1-d) \sin \frac{1}{2} \pi d \tag{2.6}$$

as $x \to 0$.

(ii) Conversely, assume that the process has a spectral density f satisfying

$$f(x) = x^{-d} L(x^{-1}), \ 0 < x < \pi, \tag{2.7}$$

where $0 < d < 1$, and L is slowly varying at infinity, satisfying assumption (2.5) above. Suppose, further, that f is of bounded variation on the interval (ϵ, π) for any $0 < \epsilon < \pi$. Then the covariances of the process satisfy

$$R_n \sim n^{-(1-d)} L(n) \Gamma(1-d) \sin \frac{1}{2} \pi d \tag{2.8}$$

as $n \to \infty$.

What is the conclusion? If the spectral density of a stationary process satisfies the assumptions of the second part of the theorem, then the correlations of the process will decay to zero at, roughly speaking, the rate of $\rho_n \sim n^{-\theta}$ for some $0 < \theta < 1$, which makes the correlations non-summable. The classical example of a process with this property is the *Fractional Gaussian noise* to be discussed in the sequel. Such models are commonly viewed as long range dependent and, in fact, a not unusual requirement on a long memory process is to have correlations that are regularly varying at infinity with exponent less than 1.

Sample paths of such processes often show features that make one to suspect presence of non-stationarity, as will be seen on the simulations of the Fractional Gaussian noise in the sequel. Observe also that this class of stationary processes is close to non-stationarity models in another respect: once the exponent d describing how fast the spectral density "blows up" near the origin, crosses the level $d = 1$, it stops being the spectral density of a stationary process. The general phenomenon of spectra with densities having a hyperbolic-type of a pole at the origin is often referred to as $1/f$ noise, especially in the physics literature. See Mandelbrot (1983). The distinction between stationarity and non-stationarity is often missed in the engineering and physics literature.

One of the reasons to concentrate on the lack of summability of correlations is the obvious interest in understanding the order of magnitude of the partial sums of the observations. Let $(X_n,\, n = 0, 1, 2, \ldots)$ be a stationary stochastic process with *zero* mean, finite variance σ^2 and correlations (ρ_n). Let $S_0 = 0$, $S_n = X_0 + \ldots + X_{n-1}$, $n \geqslant 1$ be the partial sums of the process. Note that

$$\mathrm{Var} S_n = \sigma^2 \left(n + 2 \sum_{i=1}^{n-1} (n-i)\rho_i \right).$$

If the correlations are summable, then the dominated convergence theorem yields

$$\lim_{n\to\infty} \frac{\mathrm{Var} S_n}{n} = \sigma^2 \left(1 + 2 \sum_{i=1}^{\infty} \rho_i \right),$$

so that the variance of the partial sums grows at most linearly fast (it can grow strictly slower than that if $1 + 2\sum_{i=1}^{\infty} \rho_i = 0$, which does happen in certain common models.) Since the variance is often used as a measure of the size of a random variable, this fact is taken to mean that for the "usual" zero mean stationary models partial sums do not grow faster than the square root of the sample size. Much of the classical statistics is based on that.

Once the correlations stop being summable, the variance of the partial sums can grow faster than linearly fast, and the rate of increase of the the variance is related to the actual rate of decay of correlations. For example, if the correlations (ρ_n) satisfy (2.4) with some $0 < d < 1$, then an easy application of Karamata's theorem (see e.g. Theorem 0.6 in Resnick (1987)) shows that

$$\mathrm{Var} S_n \sim \frac{1}{(1-d)(2-d)} L(n) n^{2-d} \quad \text{as } n \to \infty. \tag{2.9}$$

One can see that a phase transition in the behaviour of the variance of the partial sums occurs when the correlations stop being summable. In particular, the dependence on d in the rate of increase of the variance in (2.9) is the reason why specifically regular variation of the correlations (and the concomitant, if not equivalent, pole of the spectral density at the origin) are often viewed as *the definition* of long range dependence.

However, it is clear that the correlations and the variances give us substantial information about the process only if the process has a nearly Gaussian structure. Furthermore, these notions are entirely undefined if the process does not have a finite variance. Therefore, it is of great interest to develop an approach to long range dependence broader than that based on correlations. Phase transitions in the properties of the process, an example of which was discussed above, can occur regardless of the finiteness of variances. Presence of such phase transitions can itself be taken as a definition of long memory. Such an approach is advocated in Samorodnitsky (2002), especially when the phase transitions are related to large deviations.

The change in the order of magnitude in the variance of the partial sums in (2.9) has also lead to establishing interesting connections between long memory and self-similarity, as discussed in the next section.

3. Self-similar processes and long memory

Recall the definition of self-similarity: a stochastic process $(Y(t), t \geqslant 0)$ is called self-similar with exponent $H > 0$ of self-similarity if for all $c > 0$ the processes $(Y(ct), t \geqslant 0)$ and $(c^H Y(t), t \geqslant 0)$ have the same finite-dimensional distributions (*i.e.* scaling of time is equivalent to an appropriate scaling of space).

In applications a self-similar process is often a (continuous time) model for a cumulative input of a system in steady state, hence of a particular interest are self-similar processes reflecting this: processes with stationary increments. The common abbreviation for a self-similar process with stationary increments is SSSI (or H-SSSI if one wants to emphasize the exponent of self-similarity). We refer the reader to Samorodnitsky and Taqqu (1994) and Embrechts and Maejima (2002) for information on the properties of self-similar processes, including some of the facts presented below.

Suppose that $(Y(t), t \geqslant 0)$ is an H-SSSI process with a finite variance. Since we trivially have $Y(0) = 0$ a.s., we see that

$$E(Y(t) - Y(s))^2 = E\Big(Y(t-s) - Y(0)\Big)^2 = EY^2(t-s) = (t-s)^{2H} EY^2(1)^2$$

for all $t > s \geqslant 0$, and so

$$\begin{aligned} \mathrm{Cov}(Y(s), Y(t)) &= \frac{1}{2}\Big[EY^2(t) + EY^2(s) - E(Y(t) - Y(s))^2\Big] \qquad (3.1) \\ &= \frac{EY^2(1)}{2}\Big[t^{2H} + s^{2H} - (t-s)^{2H}\Big]. \end{aligned}$$

Assuming non-degeneracy $(EY(1)^2 \neq 0)$, the expression in the right hand side of (3.1) turns out to be nonnegative definite if and only if $0 < H \leqslant 1$, in which case it is a legitimate covariance function.

In particular, for every $0 < H \leqslant 1$ there is a unique zero mean Gaussian process whose covariance function is consistent with self-similarity with exponent H and stationary increments and, hence, is given by (3.1). Conversely, a Gaussian process with a covariance function given by (3.1) is, clearly, both self-similar with exponent H and has stationary increments. That is, for every $0 < H \leqslant 1$ there is a unique (up to a global multiplicative constant) H-SSSI zero mean Gaussian process. This process is called Fractional Brownian Motion (FBM), and will be denoted by $(B_H(t), t \geqslant 0)$. It is trivial to check that for $H = 1$, $E\big(tB_H(1) - B_H(t)\big)^2 = 0$ for all $t \geqslant 0$, which means that the process is a straight line through the origin and random normal slope. The interesting and nontrivial models are obtained when $0 < H < 1$, which is what we will assume in the sequel.

The increment process $X_n = B_H(n+1) - B_H(n)$, $n \geqslant 0$ of an FBM is a stationary process called Fractional Gaussian noise (FGN). An immediate conclusion from (3.1) is that

$$\rho_n = \mathrm{Corr}(X_0, X_n) \sim 2H(2H-1)n^{-2(1-H)}$$

as $n \to \infty$. Therefore, the correlation function of an FGN with $1/2 < H < 1$ satisfies (2.4) with $d = 2(1-H) < 1$. Fractional Gaussian noises with $H > 1/2$ are commonly viewed as long range dependent. Since the process is a Gaussian one, this is not particularly controversial, because the covariance on whose behavior the term "long range dependent" hinges here, determines the structure of the process.

Fractional Brownian Motion with $H > 1/2$ was used by Mandelbrot and Van Ness (1968) and Mandelbrot and Wallis (1968) to give a probabilistic model consistent with an unusual behaviour of water levels in the Nile river observed by Hurst (1951), and it was noted already there that the realizations of long range dependent Fractional Gaussian noises may exhibit apparently obvious non-stationarity. For example, as one compares the four plots on Figure 2, it appears that, for larger values of H, the plots tend to indicate changing "level", or the mean values, of the process at different time intervals. The phenomenon is dramatic in the case $H = .9$. This is, of course, in spite of the fact that the process is stationary and the mean is always zero.

Notice that for $0 < H < 1/2$ the correlations of a FGN tend to be negative, and for $H = 1/2$ the FGN is simply an i.i.d. sequence.

For the increment process of a self-similar process (as a FGN is) not only does the exponent H of self-similarity determine the asymptotic rate of growth of the *variance* of partial sums, but it clearly determines the *distributional* rate of growth of the partial sums. Indeed, if $(Y(t), t \geqslant 0)$ is an H-SSSI process, and $X_n = Y(n+1) - Y(n)$, $n \geqslant 0$ is its increment process, then $S_n = X_0 + \ldots + X_{n-1} = Y(n) \stackrel{\mathrm{d}}{=} n^H Y(1) = n^H X_0$ (recall that $Y(0) = 0$ for an H-SSSI process). Under the assumption of Gaussianity, the variance does determine the distribution of a partial sum, but this is not the case if the process is not Gaussian. The existence of a clear threshold $H = 1/2$ that separates short and long memory for Fractional Gaussian noises led researchers to concentrate on exponent of self-similarity itself, instead of the variance. This is, obviously, particularly attractive in the infinite variance case.

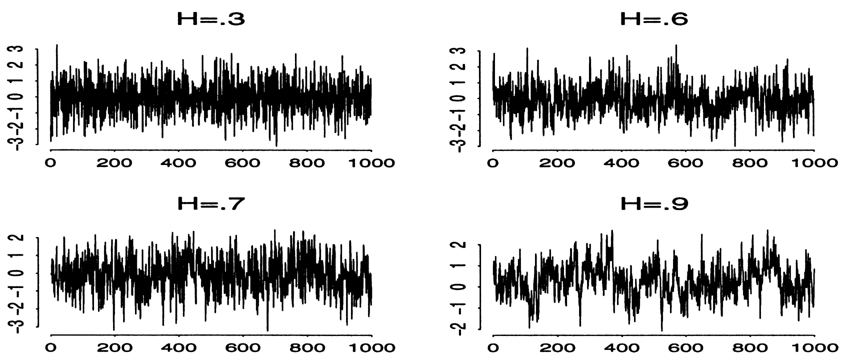

Figure 2. — Different Fractional Gaussian noises.

The case that has generated the most extensive research was that of self-similar stable processes. Recall that a stochastic process $(Y(t),\, t \in T)$ (where T is an arbitrary parameter set) is α-stable, $0 < \alpha < 2$, if for any $A, B > 0$ there is a non-random function $(D(t),\, t \in T)$ such that

$$\Big(AY_1(t) + BY_2(t),\, t \in T\Big) \stackrel{\mathrm{d}}{=} \Big((A^\alpha + B^\alpha)^{1/\alpha}\, Y(t) + D(t),\, t \in T\Big)$$

in terms of equality of finite-dimensional distributions, where $(Y_1(t),\, t \in T)$ and $(Y_2(t),\, t \in T)$ are independent copies of $(Y(t),\, t \in T)$. Note that a Gaussian process is stable, but with $\alpha = 2$, and that an α-stable process with $0 < \alpha < 2$ has infinite second moment (even the first absolute moment is infinite if $\alpha \leqslant 1$); we refer the reader to Samorodnitsky and Taqqu (1994) for more information on α-stable processes. If T is a subset of $\mathbb{R}^n$, a stable process is usually referred to as a *stable random field*; a discussion of symmetric stable random fields and their sample path properties is in Nolan (1988).

If $(Y(t),\, t \geqslant 0)$ is an α-stable Lévy motion (a process with stationary independent increments), then it is an SSSI process with $H = 1/\alpha$ (if $\alpha = 1$, this statement requires the process to be symmetric). If we recall that the value $H = 1/2$ is a critical value for the length of memory of the increments of *Gaussian* SSSI processes, it is not surprising that the value $H = 1/\alpha$ has attracted attention in the α-stable case. Specifically, suppose that $(Y(t),\, t \geqslant 0)$ is an α-stable H-SSSI process. Consider its increment process $X_n = Y(n+1) - Y(n)$, $n \geqslant 0$. Do the properties of the increment process change significantly as the value of H crosses the threshold $1/\alpha$?

To consider this question, we need, first of all, to understand what values the exponent H of self-similarity can take in this case. It turns out that the feasible pairs (α, H) of the index of stability and exponent of self-similarity in an α-stable H-SSSI process lie in the range

$$0 < \alpha < 2,\ 0 < H \leqslant \max\left(1, \frac{1}{\alpha}\right),$$

see e.g. Samorodnitsky and Taqqu (1994). In particular, the value $H = 1/\alpha$ is an interior point of the feasible range only in the case $1 < \alpha < 2$. It is important to keep in mind here that, unlike the situation in the Gaussian case, a feasible pair (α, H) *does not*, in general, determine the (finite-dimensional distributions of) an α-stable H-SSSI process.

In this regard it is interesting to note that, for $0 < \alpha < 1$, the only α-stable H-SSSI process with the highest possible value of $H = 1/\alpha$ is the independent increment Lévy motion discussed above, for which the increment process is an i.i.d. sequence (see Samorodnitsky and Taqqu (1990)). For $\alpha = 1$, assuming symmetry, there are two obvious H-SSSI processes with $H = 1$: the 1-stable Lévy motion, and the straight line process $Y(t) = tY(1)$, $t \geqslant 0$ with $Y(1)$ having a symmetric 1-stable distribution. In the former case the increment process is again an i.i.d. sequence, in the latter case the increment process is a constant (at a random level) sequence. It has been an open question whether the linear combinations of independent Lévy motion and straight line processes exhausted all possible symmetric 1-stable H-SSSI processes with $H = 1$. This question has been recently answered in negative by Cohen and Samorodnitsky (2005) who constructed an entire family of such processes that do not have either a Lévy motion component or a straight line component.

This leaves open the question whether, at least in the case $1 < \alpha < 2$, the value $H = 1/\alpha$ is of a special importance for the length of memory of the increment process.

To address this question we will describe some "standard" families of α-stable H-SSSI processes that have been studied in the literature, and for simplicity we will concentrate on the symmetric case. Henceforth we will use the standard notation SαS for "symmetric α stable". The first standard family of SαS H-SSSI processes is that of Linear Fractional Stable motions. These are processes that have the form

$$Y(t) = \int_{\mathbb{R}} g(t,x)\, M(dx)\,, \quad t \geqslant 0\,, \tag{3.2}$$

where M is a SαS random measure with Lebesgue control on $\mathbb{R}$ and

$$g(t,x) = a\left((t-x)_+^{H-1/\alpha} - (-x)_+^{H-1/\alpha}\right) + b\left((t-x)_-^{H-1/\alpha} - (-x)_-^{H-1/\alpha}\right),$$
$$t \geqslant 0,\ x \in \mathbb{R}\,,$$

where $H \in (0,1)$, $H \neq 1/\alpha$. Here a and b are two real constants, and $0^c = 0$ for all real c. We refer the reader to Samorodnitsky and Taqqu (1994) for information on stable random measures and integrals with respect to these measures.

The second standard family of SαS H-SSSI processes is that of Real Harmonizable Fractional Stable motions. These are processes of the form

$$Y(t) = \mathrm{Re} \int_{\mathbb{R}} \frac{e^{itx}-1}{ix} |x|^{-H+1-1/\alpha} \tilde{M}(dx)\,, \quad t \geqslant 0\,, \tag{3.3}$$

$0 < H < 1$, where $\tilde{M}$ is a complex isotropic SαS random measure with Lebesgue control on $\mathbb{R}$.

Finally, the third standard family of SαS H-SSSI processes is that of sub-Gaussian Fractional Stable motions, having the form

$$Y(t) = A^{1/2} B_H(t)\,, \quad t \geqslant 0\,, \tag{3.4}$$

where $(B_H(t),\, t \geqslant 0)$ is a Fractional Brownian motion independent of a strictly $\alpha/2$-stable positive random variable A. Obviously, here we also have $0 < H < 1$.

It is straightforward to check that all the families of processes defined above are families of SαS H-SSSI processes. All 3 families are restricted to the case $0 < H < 1$ (the Linear Fractional Stable motion requires a special definition in the case $1 < \alpha < 2$ and $H = 1/\alpha$), and their obvious extension to the case $\alpha = 2$ reduces, as it should, all 3 families to the Fractional Brownian motion. The increment process $X_n = Y(n+1) - Y(n)$, $n \geqslant 0$ is called the Linear (Real Harmonizable, sub-Gaussian) Fractional Stable noise, respectively.

Recall that we are considering the case $1 < \alpha < 2$ and trying to understand whether anything special happens to the properties of a Fractional Stable noise when the exponent H of self-similarity crosses the value $H = 1/\alpha$.

There are no correlations to look at, so one possibility is to look for a substitute. One such substitute is the *codifference* defined for a stationary process $(X_n,\, n \geqslant 0)$ by

$$\tau(n) = \log Ee^{i(X_n - X_0)} - \log\left(Ee^{iX_n}Ee^{-iX_0}\right)$$

$$= \log Ee^{i(X_n - X_0)} - \log\left(\left|Ee^{iX_0}\right|^2\right),$$

$n = 0, 1, 2, \ldots$, see e.g. Samorodnitsky and Taqqu (1994). For stationary Gaussian processes $\tau(n)$ is equal twice the covariance at lag n. Surprisingly, codifference carries enough information to ensure that for stationary stable processes (or, even more generally, for stationary infinitely divisible processes whose marginal Lévy measure does not charge the set $\{2\pi k,\, k = \pm 1, \pm 2, \ldots\}$) a process is mixing if and only if $\tau(n) \to 0$ as $n \to \infty$ (see Rosiński and Żak (1996)). In the paper Astrauskas *et al.* (1991) the authors looked at the asymptotic behavior as $n \to \infty$ of the codifference for the Linear Fractional Stable noise, and found that the this behavior undergoes a change when the exponent of self-similarity crosses the level $H = 1 - 1/\alpha(\alpha - 1)$, not $H = 1/\alpha$! Furthermore, for Real Harmonizable and sub-Gaussian Fractional Stable noises the codifference does not converge to zero at all.

On the other hand, Mikosch and Samorodnitsky (2000) have looked at the *ruin probability* instead of codifference. For a stationary ergodic process $(X_n,\, n \geqslant 0)$ with a finite mean μ and a number $c > \mu$, the (infinite horizon) ruin probability is defined by

$$P_{\text{ruin}}(\lambda) = P\left(S_n > c\,n + \lambda \ \text{ for some } \ n = 1, 2, \ldots\right),$$

$\lambda > 0$. If $(X_n,\, n \geqslant 0)$ is a Linear Fractional Stable noise, Mikosch and Samorodnitsky (2000) showed that

$$P_{\text{ruin}}(\lambda) \sim \begin{cases} C\,\lambda^{-(\alpha-1)} & \text{if } \ 0 < H < 1/\alpha \\ C\,\lambda^{-\alpha(1-H)} & \text{if } \ 1/\alpha < H < 1 \end{cases}$$

as $\lambda \to \infty$, where $C = C(\alpha, H, c, a, b)$ is a finite positive constant. Therefore, one observes a change occurring here as the exponent of self-similarity crosses the level $H = 1/\alpha$.

Finally, the only interesting change one can hope for the sub-Gaussian Fractional Stable noise can occur for $H = 1/2$, the critical value for FGN, since the sub-Gaussian Fractional Stable noise is just a Fractional Gaussian noise with a random scale.

One can see that the evidence is, at best, sketchy that for the increment processes of SαS H-SSSI processes with $1 < \alpha < 2$ the value of $H = 1/\alpha$ is of a special importance as far as long memory of the increment process is concerned.

4. Flows and long memory of stationary stable processes

In this section we indicate an alternative approach to long memory stable processes, that does not depend on a value of a single parameter, as discussed above for the increments of self-similar processes. We will consider, for simplicity, the symmetric case.

Let $(X_n,\, n \geqslant 0)$ be a stationary SαS process. According to a theory developed in Rosiński (1995), such a process has a representation of the form

$$X_n = \int_E f_n(x)\; M(dx)\,, \quad n = 0, 1, 2, \ldots\,, \tag{4.1}$$

where M is a SαS random measure on a measurable space $(E, \mathcal{E})$ with a σ-finite control measure m, and

$$f_n(x) = a_n(x) \left(\frac{dm \circ \phi^n}{dm}(x) \right)^{1/\alpha} f \circ \phi^n(x) \;\; x \in E\,, \tag{4.2}$$

for $n = 0, 1, 2, \ldots$, where $\phi : E \to E$ is a one-to-one map with both ϕ and ϕ^{-1} measurable, mapping the control measure m into an equivalent measure (a so-called measurable non-singular map). Further,

$$a_n(x) = \prod_{j=0}^{n-1} u \circ \phi^j(x)\,,\; x \in E\,,$$

for $n = 0, 1, 2, \ldots$, with $u : E \to \{-1, 1\}$ a measurable function and $f \in L^\alpha(m, \mathcal{E})$.

Combining the Hopf decomposition and the null-positive decomposition of the flow, (see Krengel (1958)), Rosiński (1995) and Samorodnitsky (2005) showed that one can decompose the process $(X_n,\, n \geqslant 0)$ into a sum of 3 independent stationary SαS processes

$$X_n = X_n^D + X_n^{CN} + X_n^P,\; n = 0, 1, 2, \ldots, \tag{4.3}$$

where the first component in the right hand side of (4.3) is generated by a dissipative flow (the dissipative component of the process), the second one is generated by a conservative null flow (the conservative null part of the process) and last one in generated by a positive flow (the positive part of the process). Once again, we refer the reader to Krengel (1985) for the ergodic-theoretical notions. The decomposition in (4.3) is unique in distribution.

Here is an illustrative example of a decomposition (4.3). Let

$$E = \Big(\{1,2,3\} \times \mathbb{Z}\Big)^{\mathbb{Z}}$$

endowed with the cylindrical σ-field. Let the control measure m be defined as $m = (m_1 + m_2 + m_3)/3$, where m_1 is the law on $\Big(\{1\} \times \mathbb{Z}\Big)^{\mathbb{Z}}$ of the Markov chain with the entrance law (at time zero) equal to the counting measure on $\{1\} \times \mathbb{Z}$ and transition matrix $p^{(1)}_{(1,i),(1,i+1)} = 1$ for all $i \in \mathbb{Z}$, m_2 is the law on $\Big(\{2\} \times \mathbb{Z}\Big)^{\mathbb{Z}}$ of the Markov chain with the entrance law equal to the counting measure on $\{2\} \times \mathbb{Z}$ and transition matrix of the simple symmetric random walk on $\mathbb{Z}$. Finally, m_3 is the law on $\Big(\{3\} \times \mathbb{Z}\Big)^{\mathbb{Z}}$ of a positive recurrent Markov chain with the entrance law equal to a stationary distribution of that Markov chain.

Let the kernel f_n is given by $f_n\big(x_j,\, j \in \mathbb{Z}\big) = \mathbf{1}\big(x_n = 0\big)$. Let ϕ be the backward shift operator $\phi\big(\ldots, x_{-1}, x_0, x_1, \ldots\big) = \big(\ldots, x_0, x_1, x_2, \ldots\big)$. Since the measure m is, obviously, shift invariant, we obtain (4.2) with $a_n \equiv 1$, the Radon-Nykodim derivative equal identically to 1, and $f = f_0$. Here the 3 components in the decomposition (4.3) correspond to the restrictions of the integral in (4.1) to $\Big(\{k\} \times \mathbb{Z}\Big)^{\mathbb{Z}}$ with $k = 1, 2, 3$ correspondingly.

It has been known for some time that the nature of the flow affects the probabilistic properties of the stable process, see Rosiński and Samorodnitsky (1996), Resnick et al. (1999, 2000) and Mikosch and Samorodnitsky (2000). Recently it has become even clearer that the boundaries between dissipative and conservative flows and between null and positive flows are crucial for the length of memory of a stable process. For example, it turns out that

$$\begin{aligned} &n^{-1/\alpha} \max\left(|X_0, |X_1|, \ldots, |X_{n-1}|\right) \\ &\qquad\qquad \Rightarrow \begin{cases} 0 & \text{if } \mathrm{X}^D \equiv 0 \text{ in (4.3)} \\ \text{a non-degenerate limit} & \text{otherwise} \end{cases} \end{aligned}$$

See Samorodnitsky (2004). On the other hand, it turns out that the process $(X_n,\, n \geqslant 0)$ is ergodic if and only if $X^P \equiv 0$ in (4.3); see Samorodnitsky (2004b). One interpretation is that the the boundary between dissipative and conservative flows results in the boundary between stationary SαS processes with short and long memory (with processes generated by conservative flows having long memory), while the boundary between null and positive flows results in the boundary between stationary SαS processes with finite and infinite memory (with processes generated by positive flows having infinite memory).

This appears to be a much more promising approach to long memory for stationary stable processes than that based on, say, the exponent of self-similarity in the case of the increment process of an SαS H-SSSI process, discussed in the previous section. In that regard it is interesting and important to note that the Linear Fractional Stable noise is generated by a dissipative flow, while both the Real Harmonizable and sub-Gaussian Fractional Stable noises are generated by positive flows. Therefore, it reasonable to say that the latter processes have much longer memory than Linear Fractional Stable noises, *regardless* of the value of the exponent of self-similarity.

It is interesting that no examples of SαS H-SSSI processes whose increment processes are generated by conservative null flows seem to had been known, until recently Cohen and Samorodnitsky (2005) described a class of such processes based on local times of Fractional Brownian motions.

We would also like to mention that there is a different connection between SαS H-SSSI processes and flows, due to Pipiras and Taqqu (2002a,b). This decomposition is based on multiplicative, not additive, flows, and it applies to SαS H-SSSI processes whose increment processes are generated by dissipative (additive) flows. Its implications for the probabilistic properties of the process are still unclear.

5. Conclusions

This survey has been written to give the reader a way to look at the issue of long memory and its relation to scaling and self-similarity. Much has been written on the subject, and even more will, undoubtedly, be written in the future; the area is highly active. If the reader has been convinced that concentrating on a few numbers, like exponent of self-similarity, or the rate of decay of correlations (or their substitutes) is not the right way to think about long memory, then the author has been successful in his task.

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