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Girsanov's transformation for SLE(κ, ρ) processes, intersection exponents and hiding exponents (*)

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ABSTRACT. — We relate the formulas giving Brownian (and other) intersection exponents to the absolute continuity relations between Bessel process of different dimensions, via the two-parameter family of Schramm-Loewner Evolution processes SLE(κ, ρ) introduced in [23]. This allows us also to compute the value of some new exponents ("hiding exponents") related to SLEs, planar Brownian motions and the conjectured scaling limit of two-dimensional critical systems.

RÉSUMÉ. — Nous faisons le lien entre les formules donnant les valeurs des exposants d'intersection entre mouvements browniens plans et les relations d'absolue continuité entre processus de Bessel de différentes dimensions, via la famille à deux paramètres de processus de Loewner-Schramm SLE(κ, ρ) introduite dans [23]. Ceci permet en particulier de déterminer la valeur de nouveaux exposants critiques pour le mouvement brownien plan et les SLE.

1. Introduction

The value of the intersection exponents between planar Brownian motions has been derived in the series of papers [15, 16, 17, 18] using the relationship with the exponents for the Schramm-Loewner Evolution process with parameter 6 (in short SLE_6) that can be computed directly. For instance, if $B^1, \ldots, B^p$ denote p independent planar Brownian motions started from p fixed different points on the unit circle, the probability that the p traces $B^1[0,t], \ldots, B^p[0,t]$ remain disjoint and all stay in the same (fixed) half-plane decays like $t^{-\tilde{\zeta}_p/2}$ as t tends to infinity. The exponent $\tilde{\zeta}_p$ is called

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the half-space intersection exponent between p Brownian motions and it is proved in [15] that $\tilde{\zeta}_p = p(2p+1)/3$, as conjectured in [7].

Before SLE allowed to determine the value of these exponents, it was shown in [25] that in order to understand these Brownian exponents, it is convenient to introduce "generalized" Brownian exponents $\tilde{\xi}_p(a_1, \dots, a_p)$ that correspond (in the case where all a_i's are integers) to the decay of the probability of non-intersection between p unions of planar Brownian motions in a half-plane containing respectively $a_1, \dots, a_p$ paths. For instance $\tilde{\xi}_p(1, \dots, 1) = \tilde{\zeta}_p$. In particular, one can define the function

$$U(a) = \lim_{p\to\infty} \sqrt{\tilde{\xi}_p(a, a, \dots, a)/p^2}$$

and show (this is not mysterious, it is basically a consequence of conformal invariance of planar Brownian motion) that

$$\tilde{\xi}_p(a_1, \dots, a_p) = U^{-1}(U(a_1) + \dots + U(a_p)). \tag{1.1}$$

This, combined with the conjectures by Duplantier-Kwon [7] for $\tilde{\zeta}_p$ allowed to predict the value of U and of the generalized exponents $\tilde{\xi}$. Duplantier [4] then observed that this type of equation can also be viewed as coming from the quantum gravity formalism, which provided yet another way to predict the exact form of the function U.

In [26], the relation between the Brownian exponents and the exponents for self-avoiding walks and critical percolation was pointed out. More precisely, a "universality" argument was presented that showed that all conformally invariant models that possess a certain locality condition must basically have the same exponents i.e. the same function U. This allowed to recover the predictions (see [26] and the references therein) for the critical exponents for self-avoiding walks or critical percolation from the above-mentioned prediction for U, and conversely to show that the value of the Brownian exponents would follow from the computation of the exponents for any other local conformally invariant object. This is the strategy that was successfully used in [15]: Show that SLE_6 is local, and compute its exponents. The derivation of the SLE_6 exponents (in the half-plane) is in fact a computation related to the (real) Bessel flow. This gave the rigorous proof of the fact that indeed $U(x) = \sqrt{x + 1/24} - \sqrt{1/24}$ as predicted in [25].

In the recent paper [23], the same basic idea was developed in a different setting. There, the family of random sets satisfying the so-called conformal restriction property is fully described and classified (the corresponding probability measures are called "restriction measures"). This leads [22] to the precise conjecture that $SLE_{8/3}$ is the scaling limit of the half-plane self-avoiding walk. It also proves [23] that the boundary of planar Brownian motion, the boundary of the scaling limit of critical percolation cluster interfaces (that Smirnov [32] proved to be indeed corresponding to SLE_6) and the (conjectured) scaling limit of the self-avoiding walk, do not only have the same exponents but are in fact the same random object. The family of restriction measures is parametrized by a positive real parameter a that can be interpreted as the number of planar Brownian motions that this restriction measure is equivalent to. More precisely, when a is a positive integer, one can construct the restriction measure with exponent a by considering the union of a independent Brownian excursions (i.e. in the half-plane, Brownian motions started from the origin that are "conditioned" to stay forever in the upper half-plane). This shows that the half-plane intersection exponents $\tilde{\xi}$ correspond to intersection exponents between restriction measure samples. Note (but this will not be directly relevant here even if it provides one additional motivation, since one would wish to also understand the relation with the intersection exponents) that the restriction measures are closely related to highest-weight representations of some infinite-dimensional Lie algebras (see [10]).

As shown in [23], the restriction measures (more precisely, their outer boundary) can be described via variants of $SLE_{8/3}$ called SLE($8/3, \rho$) (each ρ corresponds to a value of a). As we shall briefly recall in the next section, SLE($8/3, \rho$) is defined as $SLE_{8/3}$ except that the driving Brownian motion is replaced by a (multiple of) a Bessel process (actually, it is a little bit more complicated than that). We shall see in the present paper that with this SLE(κ, ρ) approach, the computation of the intersection exponents can be interpreted as the standard absolute continuity relations between Bessel processes of different dimensions (following from Girsanov's theorem).

This provides the value of various new exponents, some of which describe probabilities of events that are associated to planar Brownian motions: For instance, consider $n + m$ independent Brownian motions in the complex plane that are started from i, and stopped at their first hitting of the line $\{\Im(z) = R\}$. What is the probability that they all stay in the upper half-plane and that none of the n first Brownian motions contributes to the "right-hand" boundary of the union of the $n + m$ paths restricted to the strip $\{\Im(z) \in [1, R]\}$ (i.e. the n paths are hidden from $+\infty$ by the m other paths – note that this does not imply non-intersection between the paths)? When $R \to \infty$, the probability that this happens decays like a negative power of R and the corresponding exponent is

$$n + m + \frac{1}{4}\left(\sqrt{24n + (\sqrt{1 + 24m} - 3)^2} - (\sqrt{1 + 24m} - 3)\right)$$

(the $n+m$ part is just corresponding to the fact that the $n+m$ paths remain in the upper half-plane). Let us comment that just as for the generalized intersection exponents, the values of these "hiding" exponents are rational only for exceptional values of n, m. For instance, even the exponent for $m = n = 1$ is the irrational number $(3 + \sqrt{7})/2$. However, for $m = 1$ and $n = 4$, the exponent is 7. These "hiding" exponents do not seem to have appeared before in the theoretical physics literature.

The fact that such exponents can be determined can seem somewhat surprising. It is due to the fact that the SLE$(8/3, \rho)$ approach makes it possible to separate the information given by the boundary of the random sets (i.e. the law of the exterior boundary of a union of Brownian paths) from what happens "in the inside". An example of such facts is the symmetry of the Brownian frontier as described in [23].

Last but not least, this description not only provides the values of the intersection exponents, but it also gives directly the law of the paths that are conditioned not to intersect. Of course, all this is very closely related to the computations of the exponents in [15] as principal eigenvalues of some differential operators, and to the corresponding eigenfunction (and the underlying stationary diffusion, for instance the diffusion conditioned to never hit the boundary of the domain), but it is simply formulated in terms of these SLE(κ, ρ) processes. This is much less involved than the corresponding non-intersection conditioning in the Brownian case (see [13, 20]).

The results are not restricted to the $\kappa = 8/3$ case. Hence, one obtains also "hiding/intersection exponents" in the general case. In particular, a non-intersection exponent between p SLE$_\kappa$'s (with some Brownian loops added in a proper way) turns out to be simply $p(p-1)/\kappa$. These exponents are conjectured to be relevant in the study of two-dimensional critical systems from statistical physics. Recall in particular [29] that SLE$_\kappa$ for all $\kappa \in [4, 8]$ are supposed to correspond to the scaling limit of two-dimensional critical models, and that their outer boundaries are conjectured (see [2]) to be closely related to the SLE$(16/\kappa, \rho)$ processes.

As explained at the end of the paper, it also gives a new and simple interpretation of the "quantum gravity function" from [12] predicted by Knizhnik, Polyakov and Zamolodchikov that has been used in various forms to predict the values of exponents of two-dimensional critical systems by theoretical physicists (see e.g. [5] and the references therein). Loosely speaking, the "quantum gravity exponent" (conjectured to correspond to the same system but on a random lattice) is just the value ρ that appears in the SLE(κ, ρ).

When writing up this paper, I had basically the choice between on the one hand being sloppy at times, but with reasonable heuristic intuition, or giving precise complete statements and proofs that would hide the intuition behind stochastic calculus considerations and technical setups. I deliberately chose the first option, since I believe that the gaps left are reasonable.

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2. Background

2.1. (One-sided) restriction

We now recall some facts and notation from [23]. Define the family $\mathcal{A}$ of closed subsets A of the closed upper half-plane $\overline{\mathbb{H}}$ such that

1. $\mathbb{H} \setminus A$ is simply connected.

2. A is bounded and bounded away from the negative half-line.

To each such A, associate the conformal mapping ϕ_A from $\mathbb{H} \setminus A$ onto $\mathbb{H}$ such that $\phi_A(0) = 0$ and $\phi_A(z) \sim z$ when $z \to \infty$.

We say that a closed set $K \subset \overline{\mathbb{H}}$ is left-filled if

- K and $\mathbb{H} \setminus K$ are both simply connected and unbounded
- $K \cap \mathbb{R} = \mathbb{R}_-$

We say that a random left-filled set satisfies one-sided restriction if for all $A \in \mathcal{A}$, the law of K is identical to that of $\phi_A(K)$ given the event $\{K \cap A = \emptyset\}$. It is not very difficult (see [23]) to prove that this implies that for some positive constant α, one has for all A,

$$\mathbf{P}[K \cap A = \emptyset] = \phi_A'(0)^\alpha. \tag{2.1}$$

Conversely (see [23]), for each $\alpha > 0$, there exists a unique random left-filled set satisfying this identity. Its law is called the one-sided restriction measure with exponent α. It can be explicitly constructed using the SLE($8/3, \rho$) processes that will be described below.

Define now the family $\mathcal{A}_t$ just as $\mathcal{A}$ except that condition 2. is replaced by the condition that A is bounded and bounded away from 0. This immediately leads to the definition of "two-sided" restriction properties (see [23]) that we shall also use in the present paper. These are the random sets (which are no longer left-filled) such that for all $A \in \mathcal{A}_t$, the law of K is identical to that of $\phi_A(K)$ conditionally on $\{K \cap A = \emptyset\}$. Again, (2.1) has to hold for all $A \in \mathcal{A}_t$ and some fixed α. It is proved in [23] that this can be realized if and only if $\alpha \geqslant 5/8$. Furthermore, the only random simple path that satisfies the two-sided restriction property is $\mathrm{SLE}_{8/3}$ for which the corresponding exponent is $\alpha = 5/8$.

2.2. Absolute continuity relation between Bessel processes

Suppose that $(X_t, t \geqslant 0)$ is a Bessel process of dimension $d \geqslant 1$ started from $x > 0$ (see e.g. [28] for more details on the content of this subsection). In other words,

$$X_t = x + B_t + \int_0^t \frac{d-1}{2X_s} ds$$

where $(B_t, t \geqslant 0)$ is a standard one-dimensional Brownian motion started from 0. As customary, we will also use the index ν related to the dimension d by

$$d = 2 + 2\nu.$$

Recall that X hits the origin if and only if $d < 2$.

Suppose for a moment that $d = 2$ and that μ is some non-negative real number. Then Itô's formula shows immediately that

$$\log X_t = \log x + \int_0^t \frac{dB_s}{X_s}$$

is a local martingale. It is then possible to apply Girsanov's theorem to understand (for each fixed $t > 0$) the behavior of X under the new probability measure Q_t defined by

$$dQ_t/dP = (X_t/x)^\mu \exp\left(-\mu^2 \int_0^t \frac{ds}{2X_s^2}\right)$$

(it is standard to check that in this particular case, the exponential local martingale $\exp(\mu \log X_t - \mu^2 \langle \log X \rangle_t / 2)$ is a martingale, so that Q_t is indeed a probability measure): Under this new probability measure,

$$\tilde{B}_s := B_s - \mu \int_0^s \frac{ds}{X_s}$$

for $s \in [0, t]$ is a Brownian motion. In other words, as

$$X_s = x + \tilde{B}_s + (1/2 + \mu) \int_0^t \frac{ds}{X_s}$$

it follows that under this new probability measure, $(X_s, s \in [0, t])$ is a Bessel process of dimension $d' = 2 + 2\mu$ i.e. of index μ.

Note that the probability measures Q_t are compatible in the sense that the Q_T probability of any $\sigma(B_s, s \in [0, t])$ measurable set is independent of $T > t$. Hence, one can in fact define a probability measure Q that coincides with Q_t on $\sigma(B_s, s \in [0, t])$ for all t. Under this probability measure Q, the process $(X_s, s \geqslant 0)$ is a Bessel process of index μ.

Conversely, suppose now that $d > 2$ (i.e. $\nu > 0$) and define

$$dQ_t/dP = (X_t/x)^{-\nu} \exp\left(\nu^2 \int_0^t \frac{ds}{2X_s^2}\right).$$

Then, under the new probability measure Q, the process X is a two-dimensional Bessel process.

Plugging in these two facts together shows that if X is a Bessel process of dimension $2 + 2\nu \geqslant 2$ started from $x > 0$, then under the probability measure Q that is induced by the probability measures Q_t defined by

$$dQ_t/dP = (X_t/x)^{\mu-\nu} \exp\left(-(\mu^2 - \nu^2) \int_0^t \frac{ds}{2X_s^2}\right),$$

the process X is a Bessel process of index μ (instead of ν) started from x. As we shall see, this relation between $\mu - \nu$ and the exponent $\mu^2 - \nu^2$ will basically be the reason for the particular form of the critical exponents (i.e. the fact that the function U^{-1} is quadratic) in our two-dimensional context.

Note that (unless $\mu = \nu$), Q is not absolutely continuous with respect to P (the limiting behavior of X when $t \to \infty$ depends on its dimension). Similarly, one can let x go to zero, and interpret heuristically the result as the relation between Bessel processes of different dimension that are started from zero. This is not formally true since dQ_t/dP is not well-defined anymore (Q_t is singular with respect to P because the almost sure behavior of X at time $0+$ depends on its dimension).

2.3. The SLE(κ, ρ) processes

We now recall the definition of the SLE (κ, ρ) processes. Suppose that $\kappa > 0$ and $\rho > -2$. Let X denote a Bessel process of dimension

$$d = 1 + \frac{2(\rho+2)}{\kappa}$$

that is started from $X_0 = x := a/\sqrt{\kappa} \geqslant 0$.

Define $Y = \sqrt{\kappa} X$ and

$$O_t = \int_0^t \frac{-2ds}{Y_s},$$

and also

$$W_t = Y_t + O_t$$

so that

$$W_t = a + \sqrt{\kappa} B_t + \int_0^t \frac{\rho ds}{W_s - O_s}.$$

Then, one defines the Loewner chain g_t with driving function W_t i.e. for all $t \geqslant 0$ and z in the closed upper half-plane $\overline{\mathbb{H}}$,

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t} \tag{2.2}$$

(as long as $g_t(z)$ does not hit W_t). For each t, g_t is a conformal map from a domain H_t onto $\mathbb{H}$, where H_t is the set of points $z \in \mathbb{H}$ such that $|g_s(z) - W_s| > 0$ for $s \in [0, t]$. We call this process SLE(κ, ρ). When $\rho = 0$, this is the (usual) chordal SLE$_\kappa$ process.

We will for the time being assume that $d \geqslant 2$ (so that X does not hit 0). This means that

$$\rho \geqslant -2 + \frac{\kappa}{2}.$$

When $\rho = 0$, this corresponds to the fact that $\kappa \leqslant 4$.

Suppose that $Y_0 = a > 0$. Then, for all fixed $t > 0$, the law of $W[0, t]$ is absolutely continuous (even if the density may not be bounded, or bounded away from zero) with respect to that of $\sqrt{\kappa} B_t$, and therefore the law of the Loewner chain up to time t is absolutely continuous with respect to that of SLE$_\kappa$. In particular, it is almost surely generated by a continuous curve (see [29, 21]). If $Y_0 = 0$, then this does not hold directly, but one can apply the same reasoning to the chains $g_{t_0+t} \circ g_{t_0}^{-1}$ to deduce that SLE(κ, ρ) is generated by a continuous curve, that we shall denote by γ. In other words, g_t is the conformal map from the unbounded connected component of $\mathbb{H} \setminus \gamma[0,t]$ onto $\mathbb{H}$ that is normalized at infinity by $g_t(z) = z + o(1)$. This curve is simple if and only if $\kappa \leqslant 4$ ([29]).

The process

$$O_t = 2 \int_0^t \frac{ds}{O_s - W_s}$$

should be understood as the left-image $g_t(0-)$ of the origin under g_t (when $Y_0 = 0$, then the origin can correspond to two prime ends in $g_t^{-1}(\mathbb{H})$ i.e. the origin has two images corresponding to the limit from the left and from the right of the curve). The fact that $d \geqslant 2$ ensures that the left-image of 0 is never "swallowed" by the SLE(κ, ρ) curve, i.e. that the curve never hits the negative half-line. On the other hand, the SLE(κ, ρ) hits the positive half-line if $\kappa > 4$.

The intuition behind the drift term when $\rho \neq 0$ is the following: It is a repulsion from the origin (more precisely from $0-$) if $\rho > 0$ or an attractive force toward the origin if $\rho < 0$. The fact that $d \geqslant 2$ ensures that the repulsion/attraction is such that the SLE curve never hits the negative half-line: For instance, when $\kappa = 6$, the repulsion has to be sufficiently strong so that the SLE does not swallow the origin (i.e. one must have $\rho \geqslant 1$). When $\kappa = 8/3$, the process can be attracted toward zero without swallowing it (all the values $\rho \geqslant -2/3$ work).

Here since $O_0 = 0$ and $W_0 = a$, we say that the SLE(κ, ρ) process is started from $(0, a)$. Similarly, for any $o \leqslant w$, one can define an SLE(κ, ρ) started from (o, w) by translating the SLE started from $(0, w - o)$ by o.

Note that the SLE(κ, ρ) curve is obtained deterministically (via the Loewner chain) from the process Y (or X).

2.4. Restriction and SLE(κ, ρ) processes

In [23], it is proved that the boundary of the sample of a one-sided restriction measure of exponent η is an SLE($8/3, \rho$) process where

$$\eta = \bar{\eta}(8/3, \rho) = \frac{(\rho + 2)(3\rho + 10)}{32}$$

(here and in the sequel, we will use the bars to indicate that this is a function and not a parameter). It is also shown that if one adds (or decorates) an SLE$_\kappa$ curve with parameter $\kappa \leqslant 8/3$ with a Poisson cloud of Brownian loops with intensity λ, where

$$\lambda = \lambda_\kappa = \bar{\lambda}(\kappa) = \frac{(8 - 3\kappa)(6 - \kappa)}{12\kappa} \tag{2.3}$$

(of course, this depends on the actual normalizing factor in the definition of the loop soup), and then "left-fills" the obtained set, one obtains a sample of a (one-sided) restriction measure of exponent

$$\bar{\eta}(\kappa, 0) = \frac{6-\kappa}{2\kappa}.$$

The same argument is generalized in [2], where it is shown that for all $\kappa \leqslant 8/3$, one can decorate the SLE(κ, ρ) curve with a Poisson cloud of Brownian loops of intensity λ_κ and obtain a one-sided restriction measure sample with exponent

$$\bar{\eta}(\kappa, \rho) = \frac{(\rho+2)(\rho+6-\kappa)}{4\kappa} \tag{2.4}$$

We refer to [23, 2] for further details.

The Brownian loop decoration procedure can be roughly summarized as follows: There exists an infinite measure M supported on (unrooted) Brownian loops in the half-plane. A realization of the Brownian loop-soup with intensity λ is a Poisson point process with intensity λM. A sample of the loop-soup is therefore an infinite countable collection of Brownian loops in the upper half-plane. One decorates a curve with the loop-soup by adding to the curve all the loops of the loop-soup that it intersects. See [27, 23] for more details. When $\kappa = 2$, this is also closely related to the fact that SLE$_2$ is the scaling limit of loop-erased random walk as proved in [21].

2.5. Conditioned Bessel processes

In this paper, we will interpret the probability measure Q defined in Subsection 2.2 in terms of conditioning (i.e. "Q is P conditioned on some event"). This conditioning is singular (it is with respect to an event of zero probability), so that this interpretation has to be made more precise in order to be rigorous. It is very similar to the interpretation of the three-dimensional Bessel process as one-dimensional Brownian motion "conditioned to remain positive". More generally, when $d < 2$, it is well-known that the d dimensional Bessel process "conditioned to remain positive" is a $4-d$ dimensional Bessel process. In order to clarify what we will mean, it is worthwhile to briefly recall these classical facts (see e.g. [28]):

Suppose in the present subsection that $\hat{X}$ is a d dimensional Bessel process with dimension $d < 2$, that is started from $x > 0$. Let $\hat{P}$ denote its law and T its hitting time of the origin (T is a.s. finite because $d < 2$). It is easy to check that $(\hat{X}^{2-d}_{\min(t,T)}, t \geqslant 0)$ is a martingale, and it follows that if one defines for any $t > 0$, the probability measure $\hat{Q}_t$ by

$$d\hat{Q}_t/d\hat{P} = (\hat{X}_t/x)^{2-d} 1_{t<T}$$

on the σ-field $\hat{\mathcal{F}}_t$ generated by $(\hat{X}_s, s \leqslant t)$, one gets a compatible family of probability measures (just as in Subsection 2.2), which in turn defines a probability measure $\hat{Q}$, such that for any t, $\hat{Q} = \hat{Q}_t$ on $\hat{\mathcal{F}}_t$. Clearly, under $\hat{Q}$, the process $\hat{X}$ does never hit the origin (because for each t, $\hat{X}[0,t]$ does not hit the origin $\hat{Q}_t$ almost surely).

The probability measure $\hat{Q}$ will be interpreted as the law of the Bessel process $\hat{X}$ conditioned never to hit the origin. The weighting in the definition of $\hat{Q}_t$ can be decomposed as follows: The $1_{t<T}$ term corresponds to the conditioning by the event that $\hat{X}[0,t]$ does not hit the origin, and the $(\hat{X}_t/x)^{2-d}$ term corresponds to the "renormalized probability" that the future of $\hat{X}$ does not hit the origin.

A way to make this interpretation rigorous goes as follows: Consider for any $R > x$, the hitting time T_R of R by $\hat{X}$. Then, the law of $\hat{X}[0, T_R]$ conditionally on the event $T_R < T$ is identical to the law of $\hat{X}[0, T_R]$ under the measure Q (and this holds for each R, so that the corresponding laws are compatible). Here, there is no weighting in terms of the future of $\hat{X}$ after T_R because $\hat{X}_{T_R} = R$ so that all paths contribute the same way.

Recall also (and this is a classical consequence of Girsanov's theorem) that the law of $\hat{X}$ under the probability measure $\hat{Q}$ is that of a $4-d$ dimensional Bessel process. This fact will be useful later on.

3. Absolute continuity between SLE(κ, ρ)'s

We are now going to combine the previous considerations. Consider an SLE(κ, ρ) with $\nu \geqslant 0$ that is started from $(0, a)$ as before, where $a > 0$. Recall that

$$\nu = \frac{\rho + 2}{\kappa} - \frac{1}{2}$$

i.e.,

$$\rho = \kappa(\nu + \frac{1}{2}) - 2.$$

Define for $\mu \geqslant \nu$, the probability measure Q induced by the measures Q_t as before i.e. for all $t > 0$

$$\begin{aligned} dQ_t/dP &= (X_t/x)^{\mu-\nu} \exp\left(\frac{\nu^2 - \mu^2}{2} \int_0^t \frac{ds}{X_s^2}\right) \\ &= (Y_t/a)^{\mu-\nu} \exp\left(\frac{\kappa(\nu^2 - \mu^2)}{2} \int_0^t \frac{ds}{Y_s^2}\right) \end{aligned}$$

where

$$X_s = \frac{Y_t}{\sqrt{\kappa}} = \frac{W_t - O_t}{\sqrt{\kappa}}.$$

Then, under the probability measure Q, the process X is a Bessel process of dimension $2 + 2\mu$ (instead of $2 + 2\nu$) started from $x = a/\sqrt{\kappa}$. Hence, under this new probability measure, the Loewner chain g_t corresponds to that of an SLE$(\kappa, \overline{\rho})$, where

$$\bar{\rho} = \kappa(\mu + \frac{1}{2}) - 2.$$

Recall that

$$\partial_t g_t'(z) = \frac{-2g_t'(z)}{(g_t(z) - W_t)^2}$$

(this formally follows from the differentiation of (2.2) with respect to z). Therefore,

$$\partial_t \log g_t'(0) = \frac{-2}{Y_t^2}$$

so that

$$\exp\left(\frac{\nu^2 - \mu^2}{2} \int_0^t \frac{ds}{X_s^2}\right) = g_t'(0)^{(\mu^2 - \nu^2)\kappa/4}.$$

Hence,

$$dQ_t/dP = g_t'(0)^\alpha (X_t/x)^{\mu-\nu} \tag{3.1}$$

where

$$\alpha = \frac{(\mu^2 - \nu^2)\kappa}{4}.$$

In order to interpret (3.1), it is convenient to introduce an auxiliary independent sample K of a one-sided restriction measure of exponent α. Then,

$$\mathbf{P}[K \cap \gamma[0,t] = \emptyset] = E[g_t'(0)^\alpha] E_Q[(x/X_t)^{\mu-\nu}] = E[(x/\tilde{X}_t)^{(\mu-\nu)}] \tag{3.2}$$

where $\tilde{X}$ is a Bessel process of dimension $2+2\mu$ started from x. Let us stress that this is an exact identity and not just an asymptotic expansion.

We can let $a \to 0$ for fixed t. The previous formula shows readily that

$$\mathbf{P}[K \cap \gamma[0,t] = \emptyset] \sim c a^{\mu-\nu}, \tag{3.3}$$

where

$$c = E[(\sqrt{\kappa}\tilde{X}_t)^{\nu-\mu}] = (\kappa t)^{(\nu-\mu)/2} E[\tilde{X}_1^{\nu-\mu}]$$

with $\tilde{X}_0 = 0$ (the density of $\tilde{X}_1$ near 0 behaves like $y^{1+2\mu}$ so that this expectation is finite). This gives the value $\mu - \nu$ for the *intersection exponent* between a one-sided restriction measure with exponent α and the SLE(κ, ρ) process.

Furthermore, as in Subsection 2.5, it is possible to interpret Q as the law of this SLE conditioned to never intersect K. For instance for any R, the law of $X[0, T_R]$ conditionally on $X[0, T_R] \cap K = \emptyset$ is Q (when T_R is the hitting time of R by the underlying Bessel process). This conditioned SLE(κ, ρ) is therefore an SLE($\kappa, \overline{\rho}$).

To avoid notational confusion and for future reference, let us sum up the relation between the exponents, ρ's, α's etc. Even if for ease, the statements are loosely stated, they are rigorous when formulated in the way that we have just described.

- An SLE$_\kappa$ conditioned to avoid a sample of a one-sided restriction measure of exponent α is an SLE($\kappa, \bar{\rho}$) where

$$\bar{\rho} = \bar{\rho}(\kappa, 0, \alpha) := \kappa \sqrt{\frac{4\alpha}{\kappa} + \left(\frac{2}{\kappa} - \frac{1}{2}\right)^2} + \frac{\kappa}{2} - 2. \tag{3.4}$$

  Conversely, an SLE (κ, ρ) can be viewed as an SLE$_\kappa$ conditioned not to intersect a one-sided restriction sample of exponent

$$\bar{\alpha} = \bar{\alpha}(\kappa, \rho) := \frac{\rho(\rho + 4 - \kappa)}{4\kappa}. \tag{3.5}$$

- An SLE(κ, ρ) conditioned to avoid a one-sided restriction sample of exponent α is an SLE($\kappa, \bar{\rho}$) where

$$\bar{\rho} = \bar{\rho}(\kappa, \rho, \alpha) = \kappa \sqrt{\frac{4\alpha}{\kappa} + \left(\frac{\rho + 2}{\kappa} - \frac{1}{2}\right)^2} + \frac{\kappa}{2} - 2. \tag{3.6}$$

- The exponent associated to the non-intersection event between an SLE(κ, ρ) and a one-sided restriction sample of exponent α is

$$\bar{\sigma} = \bar{\sigma}(\kappa, \rho, \alpha) = \sqrt{\frac{4\alpha}{\kappa} + \left(\frac{\rho + 2}{\kappa} - \frac{1}{2}\right)^2} - \left(\frac{\rho + 2}{\kappa} - \frac{1}{2}\right). \tag{3.7}$$

  More precisely, if an SLE(κ, ρ) is started from $a > 0$ and runs up to time 1, the probability that it does not intersect the one-sided restriction sample of exponent α decays like a constant times a^σ when $a \to 0$.

Note that

$$\bar{\rho}(\kappa, \rho, \beta) = \bar{\rho}(\kappa, 0, \beta + \overline{\alpha}(\kappa, \rho)),$$

which is not surprising: Conditioning the SLE to avoid a restriction sample of exponent α and then to avoid a restriction sample of exponent β is the same as conditioning to avoid a restriction sample of exponent $\alpha + \beta$.

Let us briefly insist on the following fact: The conditioning is a conditioning of both the SLE and the restriction sample (i.e. the pair is conditioned so that they do not intersect). In the previous statements, we consider the marginal law of this conditioned SLE, and we do not describe the law of the couple. However, the restriction property shows easily how to recover the law of the restriction sample conditioned not to intersect that SLE (i.e. it gives the conditional law of the restriction sample given the (conditioned) SLE). More precisely, consider an SLE(κ, ρ) as before. Let Γ_- be the connected component of $\mathbb{H} \setminus \gamma$ which has the negative half-line on its boundary. Define also a conformal map Ψ_- from $\mathbb{H}$ onto Γ_- that fixes both the origin and infinity, and let K denote an independent sample of the one-sided restriction measure of exponent $\bar{\alpha}(\kappa, \rho)$. Then, the joint law of a restriction measure sample conditioned not to intersect an SLE$_\kappa$ (in the sense described above) is just that of $(\Psi_-(K), \gamma)$.

4. Exponents

This implies a variety of results concerning the value of critical exponents. To illustrate this, we now briefly describe some of them, leaving details and further exponents for the interested reader.

4.1. Hiding exponents between one-sided restriction measures

Let us first focus on the case $\kappa = 8/3$. In this case, SLE$(8/3, \rho)$ is itself the right-boundary of a one-sided restriction sample of exponent $\eta = \bar{\eta}(8/3, \rho)$. Suppose that $d \geqslant 2$ i.e. $\eta \geqslant 1/3$ and $\rho \geqslant -2/3$. Then, the intersection exponent between this right-boundary and another one-sided restriction sample of exponent β is

$$\begin{aligned}\sigma = \sigma(\eta \text{ hides } \beta) &= \bar{\sigma}(8/3, \rho, \beta) \\ &= \frac{1}{4}\left(\sqrt{10 + 24\eta + 24\beta - 6\sqrt{1+24\eta}} - \sqrt{10 + 24\eta - 6\sqrt{1+24\eta}}\right) \\ &= \frac{1}{4}\left(\sqrt{24\beta + (\sqrt{1+24\eta} - 3)^2} - (\sqrt{1+24\eta} - 3)\right).\end{aligned}$$

This can be interpreted as a hiding exponent between one-sided restriction measures of exponents η and β: Consider two independent samples K_η and K_β of one-sided restriction measures with respective exponents $\eta \geqslant 1/3$ and

$\beta \geqslant 0$. Consider the probability that the right-boundary of $K_\eta \cup K_\beta$ in the strip $\{\Im(z) \in [1, R]\}$ consists only of points in K_η. This probability decays like $R^{-\sigma}$ as $R \to \infty$.

In the special case where $\eta = 5/8$, the right-boundary of the restriction measure sample is the $\text{SLE}_{8/3}$ curve itself. Hence, non-intersection between the right-boundary of $K_{5/8}$ and K_β is just non-intersection between K_β and the SLE curve, so that the exponent σ in this case is the same as the non-intersection exponent $\hat{\xi}(5/8, \beta) = \tilde{\xi}(5/8, \beta) - 5/8 - \beta$ between restriction measures. This gives another way (if one combines the obtained value of $\tilde{\xi}(5/8, \beta)$ with the cascade formula (1.1)) to recover the Brownian half-plane exponents that were derived in [15, 17] using computations involving the SLE_6 processes.

In the very special case where $\eta = \beta = 5/8$, one gets a description of the right-boundary of the union of two Brownian excursions in terms of one $\text{SLE}_{8/3}$ conditioned not to intersect another independent one. We will come back to this is the two-sided case.

When $\eta > 5/8$, the hiding exponent σ is smaller than $\hat{\xi}(\eta, \beta)$, which is not surprising since the corresponding events are larger.

Note again, that for the values of η such that $1 + 24\eta$ is a perfect square (for instance $\eta = 1$ and $\eta = 2$ corresponding to one or to the union of two Brownian excursion), the obtained exponents are simpler.

Let us insist on the fact that this is valid for all $\beta \geqslant 0$, and $\eta \geqslant 1/3$ (corresponding to the $d \geqslant 2$ assumption) but that it does a priori not hold for $\eta < 1/3$; we shall see in the next subsection what to do in this case. Recall that $\eta = 1/3$ corresponds to the scaling limit of conditioned percolation cluster boundaries (see [23]). The exponent σ is in this special case equal to $\sqrt{3\beta/2}$.

There exist various alternative ways to formulate the "hiding events" since restriction measures can be described in terms of Brownian excursions or conditioned SLE_6's (see [23]). For example, one way to phrase this in terms of planar Brownian motions is described at the end of the introduction. The proof is a consequence of the previous considerations, and of the relation between exponents for Brownian excursions and for Brownian motions as developed for instance in [20].

Note that the existence of these hiding exponents itself is a non-trivial fact (sub-multiplicativity does not simply hold as it does for non-intersection events).

4.2. When $\eta < 1/3$

As we have just mentioned, the previous expression for the hiding exponent σ is not valid when $\eta < 1/3$. One way to circumvent the difficulty is to first condition the boundary of the one-sided restriction sample of exponent η not to hit the negative half-line (it cannot hide another restriction measure if it hits the negative half-line). Recall (see Subsection 2.5) that a Bessel process X of dimension $d < 2$ started from $x \in [0, 1]$ hits 1 before 0 with probability x^{2-d} (because X^{2-d} is a local martingale), and that the process "conditioned" not to intersect 0 is a Bessel process of dimension $4 - d$.

Suppose now that $\rho \in (-2, -2/3)$, so that $d < 2$. The probability that an SLE$(8/3, \rho)$ started from $(0, a)$ does not intersect the negative half-line before its capacity (the Loewner time-parametrization) reaches one, decays like a constant times $a^{2-d} = a^{-1/2-3\rho/4}$ when $a \to 0$. Furthermore, it follows from the relation between the d and $4-d$ dimensional Bessel processes that the conditioned process is an SLE$(8/3, \rho^*)$ where

$$\rho^* = \kappa - 4 - \rho = -\frac{4}{3} - \rho$$

(see [2] for a similar facts). The corresponding exponents $\eta = \bar{\eta}(8/3, \rho)$ and $\eta^* = \bar{\eta}(8/3, \rho^*)$ satisfy

$$\sqrt{1 + 24\eta} + \sqrt{1 + 24\eta^*} = 6.$$

Straightforward computations then show that for all $\beta > 0$ and $\eta \in (0, 1/3]$,

$$\begin{aligned} &\sigma(\eta \text{ hides } \beta) \\ &= \bar{\sigma}(\bar{\eta}(8/3, \rho^*) \text{ hides } \beta) + 2 - d \\ &= \frac{1}{4}\left(\sqrt{24\beta + (3 - \sqrt{1 + 24\eta})^2} - (3 - \sqrt{1 + 24\eta}) + (6 - 2\sqrt{1 + 24\eta})\right). \end{aligned}$$

Hence, one can sum up things by saying that the formula

$$\sigma(\eta \text{ hides } \beta) = \frac{1}{4}\left(\sqrt{24\beta + (3 - \sqrt{1 + 24\eta})^2} + (3 - \sqrt{1 + 24\eta})\right) \quad (4.1)$$

in fact holds for all $\eta > 0$ and $\beta > 0$. Let us note that when $\eta \to 0_+$, one gets a non-trivial limit:

$$\sigma(0_+ \text{ hides } \beta) = \frac{1 + \sqrt{1 + 6\beta}}{2},$$

which is somewhat surprising (one might have guessed at first sight that the exponent should blow up when $\eta \to 0_+$). Indeed, for each fixed large

R, the probability that $K_{1/N}$ hides K_1 (with obvious notation) in the strip $\{\Im(z) \in [1, R]\}$ is anyway smaller that $1/(N+1)$. This is due to the fact that a restriction measure of exponent $(N+1)/N$ can be viewed as the union of $N+1$ independent copies of $K_{1/N}$, so that the probability that $K_{1/N}$ hides all N others is no larger than $1/(N+1)$. However, when $\eta \to 0_+$ (i.e. $N \to \infty$), even if the probabilities (for fixed R) go to zero, this does not affect the exponents (only the "multiplicative constants" vanish).

4.3. Iterations

The description of conditioned SLE(κ, ρ) as another SLE($\kappa, \bar{\rho}$) allows to iterate the procedure (this is for instance apparent in the formulation with the hitting times T_R), and to obtain exponents describing the joint behavior of more than two restriction measures. For instance, in the simplest case where $\kappa = 8/3$, one gets readily the exponents describing the non-intersection between p $\mathrm{SLE}_{8/3}$'s (these are the exponents corresponding [22] to the non-intersection of self-avoiding walks in a half-plane):

For each positive integer p, consider p independent $\mathrm{SLE}_{8/3}$'s that are conditioned not to intersect (appropriately defined). Define η_p the exponent of the obtained restriction measure, and define ρ_p such that the right-most SLE is an SLE($8/3, \rho_p$). Clearly, $\eta_p = \bar{\eta}(8/3, \rho_p)$. Furthermore, for each $p \geqslant 0$,

$$\rho_{p+1} = \bar{\rho}(8/3, 0, \eta_p) = \bar{\rho}(8/3, 0, \bar{\eta}(8/3, \rho_p))$$

(where $\rho_1 = 0$). This shows readily that

$$\rho_p = 2(p-1) \tag{4.2}$$

and

$$\eta_p = \frac{p(3p+2)}{8}. \tag{4.3}$$

Hence, the exponent describing the probability that p independent chordal $\mathrm{SLE}_{8/3}$ (up to time 1) started at distance a of each other are mutually avoiding is

$$\eta_p - p\frac{5}{8} = \frac{3p(p-1)}{8}.$$

This result is not new since (in the notation of [15, 17]) $\eta_p = \tilde{\xi}_p(5/8, \ldots, 5/8)$; these exponents also correspond to those conjectured in [8] for self-avoiding walks (see [22] for the conjectured relation between self-avoiding walks and $\mathrm{SLE}_{8/3}$).

4.4. Other κ's

One can easily generalize the iterative procedure for other κ's. Suppose for instance that we consider the conditioned measure for p SLE_κ's for $\kappa < 8/3$ that are conditioned to mutually avoid each other and by the event that no Brownian loop in the Brownian loop-soup with intensity λ_κ intersects two different paths. The right-most path is then an $\mathrm{SLE}(\kappa, \rho_p)$ for some ρ_p that a priori depends on κ, but it turns out that

$$\rho_p = 2(p-1).$$

If one adds another independent Brownian loop-soup with intensity λ_κ to this right-most path and looks at the obtained right-most boundary, one obtains a restriction measure with exponent

$$\eta_p(\kappa) = \bar{\eta}(\kappa, \rho_p) = p\frac{(2p+4-\kappa)}{2\kappa}. \tag{4.4}$$

For $\kappa = 2$, the exponents correspond to those for loop-erased random walks derived by Kenyon [11] and Fomin [9] (previously conjectured in [3]). This is not surprising since loop-erased random walks converge to SLE_2 in the scaling limit (see [21]).

One equivalent way to describe the corresponding event goes as follows: Run p independent chordal SLE_κ's $S_1, \dots, S_p$ started from nearby points (for instance from the points a, $2a$, $\dots$, pa) up to time one. Consider p independent Brownian loop-soups of intensity λ_κ, and define for each $j \leqslant p$, the union $\mathcal{S}_j$ of the loops in the j-th soup that intersect S_j. Consider now the event that for $j=2$ up to $j=p$,

$$S_j \cap (S_{j-1} \cup \mathcal{S}_{j-1}) = \emptyset.$$

Then, the probability of this event decays like a^σ when $a \to 0$, where

$$\sigma = \eta_p(\kappa) - p\bar{\eta}(\kappa, 0) = \frac{p(p-1)}{\kappa}. \tag{4.5}$$

In the special case $\kappa = 2$ that we just mentioned, the relation between SLE_2 and loop-erased random walks [21] and Wilson's algorithm [36] gives to this event a natural interpretation in terms of uniform spanning trees.

For $\kappa \geqslant 8/3$, the previous description does not make much sense (the density of the loop-soup is negative), and it raises the interesting problem to find a simple geometric way to interpret the exponent in terms of a physical model. Conjecturally, the exponents correspond to the (asymptotic) probability of occurrence of the corresponding configuration in a sample of the corresponding critical FK-percolation model.

4.5. The "quantum gravity" function

As the formulas show, $\bar{\rho}(\kappa, 0, \alpha)$ is in fact the same as the quantum gravity function U (this actually also holds for $\kappa \neq 8/3$), if one compares with the "KPZ relation" [12] used e.g. in Duplantier [5] (see [6] for a survey and references). Hence, the SLE(κ, ρ) approach does give another interpretation of the "quantum gravity equations," and also permits (using the relation with restriction measures) to identify precisely what exponents (i.e. what events) are given by this quantum gravity formalism. When $\kappa \neq 8/3$, this was not so obvious.

On a rigorous level, since the exponents computed via SLE (for instance in the present paper) are rigorously derived, while the KPZ relation is not, one may view the SLE derivation of the exponents as a derivation of the KPZ relation (modulo the assumption that the critical exponents for statistical mechanics systems on a random planar graph exist and are universal).

4.6. Negative α's

In fact (but we prefer to emphasize it in this separate paragraph), the absolute continuity relation and the derivation of the hiding exponents also apply for (some) negative α. In order for the absolute continuity between Bessel processes to hold, the condition is that both have a dimension not smaller than 2. In other words, if one starts with an SLE(κ, ρ) such that

$$\rho \geqslant -2 + \frac{\kappa}{2}$$

then, the arguments developed in Section 3 go through except that there is no interpretation of the weighting as a non-intersection probability (the weighting is here an unbounded function of the path). The constraint that the obtained conditioned Bessel process has dimension at least 2 means that

$$\alpha \geqslant -\frac{(4-\kappa)^2}{16\kappa}$$

(note that this does not depend on ρ). Loosely speaking, when α is too negative, then the SLE is not able to compensate the weighting (so that Q is still a probability measure). This basically shows that - as one might have expected from the formulas - that the hiding exponents make sense on the interval of values of α for which it can be extended analytically (as a function of α).

In the special case where $\kappa = 8/3$, the lower bound on α is $-1/24$. In the special case $\rho = 0$, the hiding exponent is the intersection exponent

$\tilde{\xi}(5/8, \alpha) - 5/8 - \alpha$. We have just argued that a to this power describes indeed the asymptotic behavior of the quantity $E[g_1'(0)^\alpha]$ for an $\mathrm{SLE}_{8/3}$ started from a as a vanishes, for all values of $\alpha \geqslant -1/24$.

If one then applies the cascade ideas, as developed in [25] say, it is then simple to see that this for instance enables to deduce that exponents $\tilde{\xi}(1, \alpha) - 1 - \alpha$ for instance describe the asymptotic behavior of $E[g_1'(0)^\alpha]$ when g_1 corresponds this time to the conformal map associated to a Brownian excursion started from a, up to time 1, when $a \to 0$. Recall that

$$\tilde{\xi}(u, \alpha) = \frac{(\sqrt{24u+1} + \sqrt{24\alpha+1} - 1)^2 - 1}{24}.$$

In particular,

$$\tilde{\xi}(1, -1/24) = 5/8 \text{ and } \tilde{\xi}(5/8, -1/24) = 1/3.$$

5. The two-sided picture

5.1. The SLE(κ, ρ) martingales

Before turning our attention to the two-sided picture, let us point out the following by-product of the description of the SLE(κ, ρ)'s as an SLE_κ conditioned not to intersect a one-sided restriction sample of exponent $\alpha(\rho)$. It is a simple heuristic explanation to the (useful) martingales associated to SLE(κ, ρ) derived and used in [23, 2]. Let us first focus on the $\kappa = 8/3$ case studied in [23].

Let $A \in \mathcal{A}$. Consider the event that the SLE($8/3, \rho$) does avoid A. Let us now focus on the conditional probability of this event given the path up to time t. This is a function of W_t, O_t and of the image of A under g_t. Define as in [23] the conformal map h_t from $\mathbb{H} \setminus g_t(A)$ onto $\mathbb{H}$ that is normalized by $h_t(z) = z + o(1)$ when $z \to \infty$ (this is just a real shift of $\phi_{g_t(A)}$ as defined in the preliminary section). If one views the SLE($8/3, \rho$) as an $\mathrm{SLE}_{8/3}$ conditioned to avoid a restriction sample K, the conditional probability can be decomposed as follows. First, the $\mathrm{SLE}_{8/3}$ started from W_t has to avoid $g_t(A)$: This event has probability $h_t'(W_t)^{5/8}$. Second, the restriction sample has to avoid the set $g_t(A)$ as well. This occurs with probability $h_t'(O_t)^\alpha$. Conditionally on these two events, the image under h_t of the $\mathrm{SLE}_{8/3}$ is an $\mathrm{SLE}_{8/3}$ in $\mathbb{H}$ started from $h_t(W_t)$ and the image of the restriction measure sample is a restriction measure sample in $\mathbb{H}$ started from $h_t(O_t)$. The "probability" of non-intersection between these two sets is going to be affected by the scaling factor given by the non-intersection exponent $\nu - 1/4 = \rho/\kappa$ i.e.

$$\left(\frac{h_t(W_t) - h_t(O_t)}{W_t - O_t}\right)^{3\rho/8}.$$

Hence, the quantity

$$M_t = h_t'(O_t)^\alpha h_t'(W_t)^{5/8} \left(\frac{h_t(W_t) - h_t(O_t)}{W_t - O_t}\right)^{3\rho/8}$$

is a martingale. This is proved analytically in [23].

The same argument can be used for the local martingales associated to SLE(κ, ρ)'s for $\kappa \neq 8/3$ as derived in [2] (with an additional "loop-soup term").

5.2. The two-sided case

In fact, M_t is a martingale also in the two-sided case. More precisely, suppose that A is the symmetric image with respect to the imaginary line of a set in $\mathcal{A}$ (i.e. it is attached to the negative half-line). We will suppose in this subsection that $\kappa = 8/3$ and $\rho > 0$. Then, M_t is still a bounded martingale (this is proved in [23]), that is well-defined up to the (possibly infinite) time T at which the SLE curve hits A. Just as when $A \in \mathcal{A}$ (see [23]):

- If T is finite, then there exists a sequence $t_n \to T$ such that $\lim_{n\to\infty} M_{t_n} = 0$.
- If T is infinite, then there exists an unbounded sequence t_n such that

$$\lim_{n\to\infty} h_{t_n}'(W_{t_n}) = \lim_{n\to\infty} \frac{h_{t_n}(W_{t_n}) - h_{t_n}(O_{t_n})}{W_{t_n} - O_{t_n}} = 1.$$

  This is basically due to the fact that $g_t(A)$ becomes smaller and smaller, so that h_t becomes closer to the identity.

However, the term $h_t'(O_t)$ does not tend to one (when $t \to T = \infty$), because even if $g_t(A)$ becomes smaller, O_t gets closer and closer to $g_t(A)$. But since the SLE path is transient, the term $h_t'(O_t)$ has a (non-trivial) limit when $t \to \infty$ (if $T = \infty$) that can be interpreted as follows:

The SLE($8/3, \rho$) is a simple curve γ that separates the upper half-plane into two connected components Γ_- and Γ_+ (defined in such a way that the negative half-line is on the boundary of Γ_-). We now focus on Γ_-. Let Φ_-

denote a anti-conformal map (i.e. $\Phi_-(\bar{z})$ is analytic) from $\mathbb{H}$ onto Γ_- such that $\Phi_-(0) = 0$ and $\Phi_-(\infty) = \infty$ (i.e. $\Phi_-(x+iy) = \Psi_-(-x+iy)$ where Ψ_- is as before). In particular, the image of the positive half-line is the negative half-line, and the image of the negative half-line is the curve γ. Consider a sample K of a one-sided restriction measure of exponent α that is independent of the SLE γ. Define

$$\mathcal{K} = \overline{\Phi_-(K)}.$$

Note that the set $\mathcal{K}$ consists of γ and of a subset of Γ_-. In particular, its "right-boundary" is γ. Since, K is scale-invariant, the actual choice of Φ_- does not change the law of $\mathcal{K}$. Then, almost surely on the event $T = \infty$,

$$\lim_{t\to\infty} h_t'(O_t)^\alpha = \mathbf{P}[\Phi_-(K) \cap A = \emptyset | \gamma].$$

In particular, this implies that almost surely,

$$\lim_{t\to T} M_t = 1_{T=\infty} \mathbf{P}[\mathcal{K} \cap A = \emptyset | \gamma].$$

Since the martingale is bounded (by one), the optional stopping theorem shows that

$$\mathbf{P}[\mathcal{K} \cap A = \emptyset] = \mathbf{E}[M_T] = \mathbf{E}[M_0] = h_0'(0)^\eta = \phi_A'(0)^\eta.$$

But, since γ satisfies one-sided restriction (to the right) with exponent η, it follows that in fact

$$\mathbf{P}[\mathcal{K} \cap A = \emptyset] = \phi_A'(0)^\eta$$

for all $A \in \mathcal{A}_t$. In other words, $\mathcal{K}$ is a sample of the two-sided restriction measure with exponent η.

In the special case where $\eta = 2$, we see that the restriction measure with exponent 2 corresponds to two $\mathrm{SLE}_{8/3}$'s conditioned not to intersect. This is closely related to the predictions concerning the scaling limits of self-avoiding polygons [22].

5.3. Two-sided exponents

Two-sided hiding

This description of the two-sided restriction measures leads naturally to the following exponent that describe the probability that if one considers two independent two-sided restriction measure samples K_η and K_β of exponents

η and β (where $\eta > 5/8$ and $\beta \geqslant 5/8$), then $K_\beta \cap \{\Im(z) \in [1, R]\} \subset K_\eta$ decays like $R^{-\tau}$ as $R \to \infty$, where

$$\begin{aligned}\tau &= \tau(\eta \text{ hides } \beta) \\ &= \tilde{\xi}(5/8, \sigma(\alpha \text{ hides } \beta) + \beta + \alpha) - \eta - \beta.\end{aligned}$$

Hence,

$$\tau(\eta \text{ hides } \beta) = \frac{\sqrt{24\beta + (\sqrt{1+24\eta} - 6)^2} - (\sqrt{1+24\eta} - 6)}{2}. \tag{5.1}$$

Note in particular that

$$\tau(1 \text{ hides } 1) = 3 \text{ and } \tau(2 \text{ hides } 1) = 2. \tag{5.2}$$

In particular, in both these cases, the exponent of the conditioned restriction measure is 5.

No cut-points

A by-product of these calculations is the exponent that describes the decay of the probability that a two-sided restriction measure of exponent $\eta > 5/8$ has no cut-point. More precisely, when $\eta \in (5/8, 35/24)$, the probability that a sample K_η of the two-sided restriction sample of exponent η has no cut-point inside the strip $\{\Im(z) \in [1, R]\}$ decays like $R^{-\delta(\eta)}$ when $R \to \infty$, where

$$\delta(\eta) = 6 - \sqrt{1 + 24\eta}. \tag{5.3}$$

Furthermore, the conditional law is that of the two-sided restriction measure with exponent

$$\eta' = \eta + \delta(\eta) = \eta + 6 - \sqrt{1 + 24\eta}; \tag{5.4}$$

i.e.

$$\sqrt{1 + 24\eta} + \sqrt{1 + 24\eta'} = 12. \tag{5.5}$$

The reason is that the absence of cut-points means that the left-boundary does not intersect the right-boundary. In the interpretation described above, this occurs if K does not hit the negative half-line (or more precisely a segment on the negative half-line). The exponent that describes the probability that a d-dimensional Bessel process does not hit the origin is $2 - d$ and the conditioned process is a $4 - d$ dimensional Bessel process. It follows that the restriction exponent of the conditioned set is

$$\tilde{\xi}(5/8, \bar{\eta}(8/3, -4/3 - \rho))$$

where ρ is chosen so that $\tilde{\xi}(5/8, \bar{\eta}(8/3, \rho)) = \eta$.

When $\eta \geqslant 35/24$, the restriction measure sample has a.s. no cut-point, so that the problem is not relevant. When $\eta < 5/8$, the two-sided restriction measure does not exist. When $\eta = 5/8$, then K_η is almost surely a simple path, so that the probability that it has no cut point in an annulus is 0. However, when $\eta \to 5/8+$, one sees that δ tends to 2, and that the conditional law "tends" to that of a restriction measure of exponent 21/8, that can therefore be viewed as the filling of an $\mathrm{SLE}_{8/3}$ conditioned to have no cut-point! Of course, since $\mathrm{SLE}_{8/3}$ is a.s. a simple curve, this depends a lot on the limiting procedure used to define this conditioned object (here: first replace $\mathrm{SLE}_{8/3}$ by a restriction measure of exponent $5/8+\epsilon$, then condition it to have no cut point (in larger and larger annuli), and finally let ϵ tend to zero).

It is worthwhile stressing the special case where $\eta = 1$. The exponent δ is equal to 1 and it is related to Bálint Virág's Brownian beads [34]. It gives a description of the restriction measure of parameter $\eta' = 2$ as the filling of one single path. More precisely: "The filling of a Brownian excursion conditioned to have no cut point has the same law as the filling of the union of two Brownian excursions." It raises the question whether this conditioned Brownian excursion has something to do with the path that is obtained by considering the appropriate SLE_κ to which one chronologically attaches Brownian loops as in [23] in order to construct a restriction measure sample of exponent 2.

Note also that the two-sided measure obtained if one conditions K_η to hide K_β, is the same as the one obtained if one conditions $K_{\eta'}$ to hide K_β. This is not surprising: One first conditions K_η to have no cut point, and then weights it by the "space" it leaves in its inside.

Mixed two-sided hiding

One can also define exponents associated to "mixed" two-sided hiding: Consider the exponent $\hat{\tau}(\eta, \beta)$ that is associated to the fact that the left-boundary of $K_\eta \cup K_\beta$ consists only of points in K_η while the right-boundary consists of points in K_β. This time

$$\hat{\tau} = \tilde{\xi}(5/8, \sigma(\beta \text{ hides } \alpha) + \alpha + \beta) - \eta - \beta,$$

where as before $\tilde{\xi}(5/8, \alpha) = \eta$. This leads to

$$\hat{\tau}(\eta, \beta) = \frac{9 - B - E + 2\sqrt{(B-3)^2 + (E-3)^2 - 1}}{4} \tag{5.6}$$

where $B = \sqrt{1 + 24\beta}$ and $E = \sqrt{1 + 24\eta}$. For instance $\hat{\tau}(1,1) = (2\sqrt{7}-1)/4$.

Radial hiding

All two-sided hiding exponents yield readily the corresponding exponent in the radial setting, using the mapping described for example in [26] and the disconnection exponents computed in [16, 17] (see also [24]).

For instance, consider $n+p$ independent Brownian motions started from the origin and stopped when they hit the unit circle. Consider the event that the union of these $n+p$ paths do not disconnect the circle of radius r from 1, and if the boundary of the connected component of $\mathbf{U} \setminus (B^1 \cup \ldots B^{n+p})$ that contains 1 consists of points of $B^1 \cup \ldots \cup B^n$. Then, the probability of this event decays like r^ρ when $r \to 0$, where

$$\rho = \rho(n \text{ hides } p) = \frac{\left(\sqrt{24p + (\sqrt{1+24n} - 6)^2} + 5\right)^2 - 4}{48} \tag{5.7}$$

Note that when $n = 2$ or $n = 1$, the hiding exponent is just

$$\rho(2 \text{ hides } \beta) = \rho(1 \text{ hides } \beta) = \xi(2, \beta)$$

(in the notation of [16]), which is not surprising because of the inside/outside symmetry of the Brownian frontier pointed out in [23]. The inside/outside symmetry of the Brownian frontier also shows that a single Brownian motion started from the origin, "conditioned not to disconnect the origin from infinity and to have no cut point" also separates the plane into the "inside" I and the "outside" O in such a way that (I, O) and (O, I) have the same law.

When the half-plane exponent is 5, then the radial exponent is 2. For instance, $\rho(1 \text{ hides } 1) = 2$, so that the corresponding existence problem is "critical": Are there points B_T on the outer boundary of a planar Brownian path $(B_t, t \in [0, 1])$ such that (locally) the outer boundary consists only of the future after B_T (or only of the past before B_T)?

Bibliography

[1] DUBÉDAT (J.), SLE and triangles, *Elect. Comm. Probab.* **8**, p. 28-42 (2003).

[2] DUBÉDAT (J.), SLE(κ, ρ) martingales and duality, preprint (2003).

[3] DUPLANTIER (B.), Loop-erased random walks in two dimensions: exact critical exponents and winding numbers, *Phys. A***191**, p. 516-522 (1992).

[4] DUPLANTIER (B.), Random walks and quantum gravity in two dimensions, *Phys. Rev. Lett.* **81**, p. 5489-5492 (1998).

[5] DUPLANTIER (B.), Conformally invariant fractals and potential theory, *Phys. Rev. Lett.* **84**, p. 1363-1367 (2000).

[6] DUPLANTIER (B.), Conformal fractal geometry and boundary quantum gravity, preprint (2003).

[7] DUPLANTIER (B.), KWON (K.-H.), Conformal invariance and intersection of random walks, *Phys. Rev. Lett.* **61**, p. 2514-2517 (1988).

[8] DUPLANTIER (B.), SALEUR (H.), Exact surface and wedge exponents for polymers in two dimensions, *Phys. Rev. Lett.* **57**, p. 3179-3182 (1986).

[9] FOMIN (S.), Loop-erased random walks and total positivity, *Trans. Amer. Math. Soc.* **353**, p. 3563-3583 (2001).

[10] FRIEDRICH (R.), WERNER (W.), Conformal fields, restriction properties, degenerate representations and SLE, *C.R. Acad. Sci. Paris* **335**, p. 947-952 (2002).

[11] KENYON (R.), Long-range properties of spanning trees, *J. Math. Phys.* **41**, p. 1338-1363 (2000).

[12] KNIZHNIK (V.G.), POLYAKOV (A.M.), ZAMOLODCHIKOV (A.B.), Fractal structure of 2-D quantum gravity, *Mod. Phys. Lett.* **A3**, p. 819 (1988).

[13] LAWLER (G.F.), Non-intersecting Brownian motions, *Math. Phys. El. J.* **1**, paper no.4 (1995).

[14] LAWLER (G.F.), An introduction to the stochastic Loewner evolution, to appear (2001).

[15] LAWLER (G.F.), SCHRAMM (O.), WERNER (W.), Values of Brownian intersection exponents I: Half-plane exponents, *Acta Mathematica* **187**, p. 237-273 (2001).

[16] LAWLER (G.F.), SCHRAMM (O.), WERNER (W.), Values of Brownian intersection exponents II: Plane exponents, *Acta Mathematica* **187**, p. 275-308 (2001).

[17] LAWLER (G.F.), SCHRAMM (O.), WERNER (W.), Values of Brownian intersection exponents III: Two-sided exponents, *Ann. Inst. Henri Poincaré* **38**, p. 109-123 (2002).

[18] LAWLER (G.F.), SCHRAMM (O.), WERNER (W.), Analyticity of planar Brownian intersection exponents, *Acta Mathematica* **189**, p. 179-201 (2002).

[19] LAWLER (G.F.), SCHRAMM (O.), WERNER (W.), One-arm exponent for critical 2D percolation, *Electronic J. Probab.* **7**, paper no.2 (2002).

[20] LAWLER (G.F.), SCHRAMM (O.), WERNER (W.), Sharp estimates for Brownian non-intersection probabilities, in *In and out of equilibrium,* V. Sidoravicius Ed., Prog. Probab. **51**, Birkhäuser, p. 113-131 (2002).

[21] LAWLER (G.F.), SCHRAMM (O.), WERNER (W.), Conformal invariance of planar loop-erased random walks and uniform spanning trees, *Ann. Probab.*, to appear (2001).

[22] LAWLER (G.F.), SCHRAMM (O.), WERNER (W.), On the scaling limit of planar self-avoiding walks, AMS Proc. Symp. Pure Math. *Fractal Geometry and Applications,* to appear (2002).

[23] LAWLER (G.F.), SCHRAMM (O.), WERNER (W.), Conformal restriction. The chordal case, *J. Amer. Math. Soc.* **16**, p. 917-955 (2003).

[24] LAWLER (G.F.), SCHRAMM (O.), WERNER (W.), Conformal restriction. The radial case, in preparation (2003).

[25] LAWLER (G.F.), WERNER (W.), Intersection exponents for planar Brownian motion, *Ann. Prob.* **27**, p. 1601-1642 (1999).

[26] LAWLER (G.F.), WERNER (W.), Universality for conformally invariant intersection exponents, *J. Eur. Math. Soc.* **2**, p. 291-328 (2000).

[27] LAWLER (G.F.), WERNER (W.), The Brownian loop-soup, *Probab. Th. Rel. Fields*, to appear (2003).

[28] REVUZ (D.), YOR (M.), *Continuous martingales and Brownian motion*, Springer, (1991).

[29] ROHDE (S.), SCHRAMM (O.), Basic properties of SLE, *Ann. Math.*, to appear (2001).

[30] SCHRAMM (O.), Scaling limits of loop-erased random walks and uniform spanning trees, *Israel J. Math.* **118**, p. 221-288 (2000).

[31] SCHRAMM (O.), A percolation formula, *Electr. Comm. Prob.* **6**, p. 115-120 (2001).

[32] SMIRNOV (S.), Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits, *C. R. Acad. Sci. Paris Ser. I Math.* **333**, p. 239-244 (2001).

[33] SMIRNOV (S.), WERNER (W.), Critical exponents for two-dimensional percolation, *Math. Res. Lett.* **8**, p. 729-744 (2001).

[34] VIRÁG (B.), Brownian beads, *Probab. Th. Rel. Fields*, to appear (2003).

[35] WERNER (W.), *Random planar curves and Schramm-Loewner Evolutions*, Lecture Notes of the 2002 St-Flour summer school, Springer, to appear (2002).

[36] WILSON D.B., Generating spanning trees more quickly than the cover time, *Proc. 28th ACM Symp.*, p. 296-303 (1996).