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Semilinear wave equation on manifolds (*)

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Dedicated to M. Milla Miranda in the occasion of his 60th. anniversary.

1. Introduction

Let \( \Omega \) be a bounded open set of \( \mathbb{R}^n \) \((n \geq 1)\) with smooth boundary \( \Gamma \). Let \( \nu \) be the outward normal unit vector to \( \Gamma \) and \( T > 0 \) a real number. We consider the cylinder \( Q = \Omega \times ]0, T[ \) with lateral boundary \( \Sigma = \Gamma \times ]0, T[ \).

We investigate existence and asymptotic behaviour of weak solution for the problem

\[
\begin{align*}
w'' + \frac{\partial w}{\partial \nu} + F(w) + \beta(x) w' = 0 & \quad \text{on} \quad \Sigma, \\
w(0) = w_0, \quad w'(0) = w_1 & \quad \text{on} \quad \Gamma,
\end{align*}
\]

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where the prime means the derivative with respect to $t$, normal derivative of $w$ and $F : \mathbb{R} \to \mathbb{R}$ is a function that satisfies

$$F \text{ continuous and } sF(s) > 0, \forall s \in \mathbb{R}. \quad (1.2)$$

It is important to call the attention to the reader that the idea employed in this work comes from Lions [8], pg. 134. The main point consists in adding to (1.1) an elliptic equation in $\Omega$ to reduce the problem to a canonical model of Mathematical Physics, but in this case on a manifold which is the lateral boundary $\Sigma$ of the cylinder $Q$. A Similar type of problem, also motivated by Lions [8], can be seen in Cavalvanti and Domingos Cavalcanti [2].

The plan of this article is the following: In the section 2, we give notations, terminology and we treat the linear case associated to (1.1). In the section 3, we prove existence for weak solution when $F$ satisfies the condition (1.2), approximating $F$ by Lipschtz functions. In this Lipschitz case, we employ Picard’s successive approximations and then we apply the Strauss’ method [9]. Finally in the section 4, we obtain the asymptotic behaviour by the method of pertubation of energy as in Zuazua [10].

2. Notations, Assumptions and Results

Denote by $|\cdot|$, $(\cdot, \cdot)$ and $\|\cdot\|$, $((\cdot, \cdot))$ the inner product and norm, respectively, of $L^2(\Omega)$ and $H^{\frac{1}{2}}(\Gamma)$.

For

$$G(s) = \int_0^s F(\sigma)d\sigma$$

we will denote a primitive of $F$.

We consider the following assumption on $\beta$ in (1.1):

$$\beta \in L^\infty(\Gamma) \text{ such that } \beta(x) \geq \beta_0 > 0, \text{ a.e. on } \Gamma. \quad (2.1)$$

As was said in the introduction, for $\lambda > 0$, let us consider the problem

$$\begin{align*}
-\Delta w + \lambda w &= 0 \quad \text{in } Q, \\
w'' + \frac{\partial w}{\partial \nu} + F(w) + \beta(x) w' &= 0 \quad \text{on } \Sigma, \\
w(0) &= w_0, \quad w'(0) = w_1 \quad \text{on } \Gamma.
\end{align*} \quad (2.2)$$

From elliptic theory, we know that for $\varphi \in H^{\frac{1}{2}}(\Gamma)$, the solution $\Phi$ of the boundary value problem

$$\begin{align*}
-\Delta \Phi + \lambda \Phi &= 0 \quad \text{in } \Omega, \\
\Phi &= \varphi \quad \text{on } \Gamma,
\end{align*} \quad (2.3)$$
belongs to $H^1(\Omega, \Delta) = \{ u \in H^1(\Omega) ; \Delta u \in L^2(\Omega) \}$. By the trace theorem, it follows that $\frac{\partial \Phi}{\partial \nu} \in H^{-\frac{1}{2}}(\Gamma)$.

Formally, we have by (2.3) that
\[
0 = \int_\Omega \nabla \Phi \nabla \Psi dx + \lambda \int_\Omega \Phi \Psi dx - \int_{\Gamma} \frac{\partial \Phi}{\partial \nu} \Psi d\Gamma.
\]

We take $\Psi \in H^1(\Omega, \Delta)$ and we define
\[
a(\Phi, \Psi) = \int_\Omega \nabla \Phi \nabla \Psi dx + \lambda \int_\Omega \Phi \Psi dx
\]
(2.4)

Thus, by (2.4)
\[
a(\Phi, \Psi) = \langle \gamma_1 \Phi, \gamma_0 \Psi \rangle,
\]
where $\gamma_0$ and $\gamma_1$ are the traces of order zero and one, respectively, and $\langle \cdot, \cdot \rangle$ represents the duality pairing between $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$.

We consider the scheme
\[
\varphi \in H^{\frac{1}{2}}(\Gamma) \quad \xrightarrow{A} \quad \Phi \in H^1(\Omega, \Delta)
\]
\[
\frac{\partial \Phi}{\partial \nu} \in H^{-\frac{1}{2}}(\Gamma)
\]

Thus
\[
A = \gamma_1 \circ \gamma_0^{-1} : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma), \quad A \varphi = \frac{\partial \Phi}{\partial \nu}.
\]

Therefore $A$ is self-adjoint and $A \in \mathcal{L} \left(H^{\frac{1}{2}}(\Gamma), H^{-\frac{1}{2}}(\Gamma)\right)$.

Moreover, we have
\[
\langle A \varphi, \varphi \rangle = a(\Phi, \Phi)
\]
(2.5)
and so by (2.4) we get
\[
\langle A \varphi, \varphi \rangle = \int_\Omega |\nabla \Phi|^2 dx + \lambda \int_\Omega |\Phi|^2 dx \geq \min \{1, \lambda\} \|\Phi\|^2_{H^1(\Omega)} \geq \alpha \|\gamma_0 \Phi\|^2 = \alpha \|\varphi\|^2,
\]
proving that $A$ is positive.
We formulate now the problem on $E$. For this, we define
\[ w(t)|_\Gamma = u(t) \quad \text{and} \quad \frac{\partial w(t)}{\partial \nu}|_\Gamma = Au(t). \]

In this way, the problem (1.2) is reduced to find a function $u : \Sigma \to \mathbb{R}$ such that
\[
\begin{align*}
    u'' + Au + F(u) + \beta(x)u' &= 0 \quad \text{on} \quad \Sigma, \\
    u(0) &= u_0, \quad u'(0) = u_1 \quad \text{on} \quad \Gamma,
\end{align*}
\]
which will be investigated in the section 3.

Firstly we will state a result that guarantees the existence and uniqueness of solution for the linear problem associated the (1.1).

**THEOREM 2.1.** Given $(u_0, u_1, f) \in H^{\frac{1}{2}}(\Gamma) \times L^2(\Gamma) \times L^2(\Sigma)$, there exists a unique function $u : \Sigma \to \mathbb{R}$ such that
\[
    u \in C^0 \left( 0, T; H^{\frac{1}{2}}(\Gamma) \right) \cap C^1 \left( 0, T; L^2(\Gamma) \right), \tag{2.6}
\]
\[
    u'' + Au + \beta u' = f \quad \text{in} \quad L^2 \left( 0, T; H^{-\frac{1}{2}}(\Gamma) \right), \tag{2.7}
\]
\[
    u(0) = u_0, \quad u'(0) = u_1 \quad \text{on} \quad \Gamma. \tag{2.8}
\]
Moreover we have the energy inequality
\[
    \frac{1}{2} |u'(t)|^2 + \frac{\alpha}{2} \|u(t)\|^2 \leq \frac{1}{2} |u_1|^2 + \frac{\alpha}{2} \|u_0\|^2 + \int_0^T (f(s), u'(s)) \, ds, \ a.e \ in \ [0, T]. \tag{2.9}
\]

**Proof.** In the proof of this linear case, we employ the Faedo-Galerkin’s method. \Box

3. Existence of Solution

The goal of this section is to obtain existence of solutions for the problem (1.1).

**THEOREM 3.1.** Consider $F$ satisfying (1.2) and suppose
\[
    (u_0, u_1) \in H^{\frac{1}{2}}(\Gamma) \times L^2(\Gamma) \quad \text{and} \quad G(u_0) \in L^1(\Gamma).
\]
Then there exists a function $u : \Sigma \to \mathbb{R}$ such that
\[
    u \in L^\infty \left( 0, T; H^{\frac{1}{2}}(\Gamma) \right), \tag{3.1}
\]
\[
    u' \in L^\infty \left( 0, T; L^2(\Gamma) \right). \tag{3.2}
\]
To prove the Theorem 3.1, the following Lemma will be used:

**LEMMA 3.1.**

Assume that \((u_0, u_1) \in H^\frac{1}{2} (\Gamma) \times L^2 (\Gamma)\) and suppose that the function \(F\) satisfies

\[
F \in L^\infty \left(0, T; H^{-\frac{1}{2}} (\Gamma) + L^1 (\Gamma) \right), \quad (3.3)
\]

\[
u(0) = u_0, \quad u'(0) = u_1 \quad \text{on} \quad \Gamma. \quad (3.4)
\]

To prove the Theorem 3.1, the following Lemma will be used:

**LEMMA 3.1.** — Assume that \((u_0, u_1) \in H^\frac{1}{2} (\Gamma) \times L^2 (\Gamma)\) and suppose that the function \(F\) satisfies

\[
F : \mathbb{R} \to \mathbb{R} \text{ be Lipschitz function such that } sF(s) \geq 0, \forall s \in \mathbb{R}. \quad (3.5)
\]

Then there exists only one function \(u : \Sigma \to \mathbb{R}\) satisfying the conditions

\[
u \in L^\infty \left(0, T; H^\frac{1}{2} (\Gamma) \right), \quad (3.6)
\]

\[
u' \in L^\infty \left(0, T; L^2 (\Gamma) \right), \quad (3.7)
\]

\[
u'' + Au + F(u) + \beta u' = 0 \quad \text{in} \quad L^2 \left(0, T; H^{-\frac{1}{2}} (\Gamma) \right), \quad (3.8)
\]

\[
u(0) = u_0, \quad u'(0) = u_1 \quad \text{on} \quad \Gamma. \quad (3.9)
\]

Furthermore

\[
\frac{1}{2} |u'(t)|^2 + \frac{\alpha}{2} \|u(t)\|^2 + \int_\Gamma G(u(x, t)) \, d\Gamma \leq \frac{1}{2} |u_1|^2 + \\
+ \frac{\alpha}{2} \|u_0\|^2 + \int_\Gamma G(u_0(x)) \, d\Gamma, \ a.e \ in \ [0, T]. \quad (3.10)
\]

**Proof of Lemma 3.1.** — The proof will be done employing the Picard successive approximations method. Let us consider the sequence of successives approximations

\[
u_0, \nu_1, \nu_2, \ldots, \nu_n, \ldots \quad (3.11)
\]

defined as the solutions of the linear problems

\[
\left| \begin{array}{l}
u''_n + Au_n + F(u_{n-1}) + \beta u'_n = 0 \quad \text{on} \quad \Sigma, \\
u_n(0) = u_0, \quad u'_n(0) = u_1 \quad \text{on} \quad \Gamma.
\end{array} \right. \quad (3.12)
\]

Using that \(F\) is Lipschitz and from Theorem 2.1, one can prove, using induction, that (3.12) has a solution for each \(n \in \mathbb{N}\) with the regularity claimed in the Theorem 2.1. We will prove now that the sequence (3.11) converges to a function \(u : \Sigma \to \mathbb{R}\) in the conditions of the Lemma 3.1.
For this end, we define \( v_n = u_n - u_{n-1} \) which is the unique solution of the problem

\[
\begin{align*}
\frac{d^2 v_n}{dt^2} + Av_n + F(u_{n-1}) - F(u_{n-2}) + \beta v_n' &= 0 \quad \text{on} \quad \Sigma, \\
v_n(0) = 0, \quad v_n'(0) &= 0 \quad \text{on} \quad \Gamma.
\end{align*}
\] (3.13)

By the energy inequality (2.9), we have

\[
\frac{1}{2} |v_n'(t)|^2 + \frac{\alpha}{2} \|v_n(t)\|^2 \leq - \int_0^t (F(u_{n-1}) - F(u_{n-2}), v_n'(s)) \, ds. \quad (3.14)
\]

Set

\[
e_n(t) = \text{ess sup}_{s \in [0,t]} \left\{ \frac{1}{2} |v_n'(s)|^2 + \frac{\alpha}{2} \|v_n(s)\|^2 \right\}. \quad (3.15)
\]

Thus, since \( F \) is Lipschitz, we have

\[
- \int_0^t (F(u_{n-1}) - F(u_{n-2}), v_n'(s)) \, ds \leq C \int_0^t |v_{n-1}(s)|^2 \, ds + \frac{1}{2} e_n(t). \quad (3.16)
\]

We have also

\[
|v_{n-1}(s)|^2 \leq C e_{n-1}(s). \quad (3.17)
\]

Combining (3.14) – (3.17), we get

\[
e_n(t) \leq C \int_0^t e_{n-1}(s) \, ds,
\]

and, by iteration, we obtain, for \( n = 1, 2, \ldots \), that

\[
e_n(t) \leq e_0 C_T \frac{(Ct)^n}{n!},
\]

hence, we conclude that the series \( \sum_{n=1}^\infty e_n(t) \) is uniformly convergent on \( [0,T] \). By the definition of \( e_n(t) \), see (3.15), it follows that the series \( \sum_{n=1}^\infty (u'_n - u'_{n-1}) \) and \( \sum_{n=1}^\infty (u_n - u_{n-1}) \) are convergents in the norms of \( L^\infty(0,T;L^2(\Gamma)) \) and \( L^\infty(0,T;H^{1/2}(\Gamma)) \), respectively. Therefore, there exists \( u : \Sigma \to \mathbb{R} \) such that

\[
\begin{align*}
u_n \to u \quad &\text{strong in} \quad L^\infty(0,T;H^{1/2}(\Gamma)), \\
u'_n \to u' \quad &\text{strong in} \quad L^\infty(0,T;L^2(\Gamma)).
\end{align*}
\] (3.18) (3.19)
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Since $F$ is Lipschitz, we have by (3.18) that

$$F(u_n) \to F(u) \text{ strong in } L^\infty(0,T;L^2(\Gamma)).$$  \hspace{1cm} (3.20)

Then, by the convergences (3.18) – (3.20), we can pass to the limit in (3.12) and we obtain, by standard procedure, a unique function $u$ satisfying (3.6) – (3.10). \hfill \Box

We will prove now the main result.

Proof of Theorem 3.1. — By Strauss [9], there exists a sequence of functions $(F_\nu)_{\nu \in \mathbb{N}}$, such that each $F_\nu : \mathbb{R} \to \mathbb{R}$ is Lipschitz and $(F_\nu)_{\nu \in \mathbb{N}}$ approximates $F$ uniformly on bounded sets of $\mathbb{R}$. Since the initial data $u_0$ is not necessarily bounded, we have to approximate $u_0$ by bounded functions of $H^{\frac{1}{2}}(\Gamma)$. We consider the functions $\xi_j : \mathbb{R} \to \mathbb{R}$ defined by

$$\xi_j(s) = \begin{cases} -j, & \text{if } s < -j, \\ s, & \text{if } |s| \leq j, \\ j, & \text{if } s > j. \end{cases}$$

Considering $\xi_j(u_0) = u_{0j}$, we have by Kinderlehrer and Stampacchia [5] that the sequence $(u_{0j})_{j \in \mathbb{N}} \subset H^{\frac{1}{2}}(\Gamma)$ is bounded a.e. in $\Gamma$ and

$$u_{0j} \to u_0 \text{ strong in } H^{\frac{1}{2}}(\Gamma).$$  \hspace{1cm} (3.21)

Thus, for $(u_{0j}, u_1) \in H^{\frac{1}{2}}(\Gamma) \times L^2(\Gamma)$, the Lemma 3.1 says that there exists only one solution $u_{j\nu} : \Sigma \to \mathbb{R}$ satisfying (3.6) – (3.9) and the energy inequality

$$\begin{aligned}
\frac{1}{2} |u'_{j\nu}(t)|^2 + \frac{\alpha}{2} \|u_{j\nu}(t)\|^2 + \int_\Gamma G_\nu(u_{j\nu}(x,t)) \, d\Gamma &\leq \frac{1}{2} |u_1|^2 + \\
&\quad + \frac{\alpha}{2} \|u_{0j}\|^2 + \int_\Gamma G_\nu(u_{0j}(x)) \, d\Gamma.
\end{aligned}$$  \hspace{1cm} (3.22)

We need an estimate for the term $\int_\Gamma G_\nu(u_{0j}(x)) \, d\Gamma$. Since $u_{0j}$ is bounded a.e. in $\Gamma$, $\forall j \in \mathbb{N}$, it follows that

$$F_\nu(u_{0j}) \to F(u_{0j}) \text{ uniform in } \Gamma.$$  

So

$$\int_\Gamma G_\nu(u_{0j}(x)) \, d\Gamma \to \int_\Gamma G(u_{0j}(x)) \, d\Gamma \text{ uniform in } \mathbb{R}.$$  \hspace{1cm} (3.23)
From (3.21), there exists a subsequence of \((u_{0j})_{j \in \mathbb{N}}\), which will still be denoted by \((u_{0j})_{j \in \mathbb{N}}\), such that
\[
u_{0j} \to u_0 \text{ a.e. in } \Gamma.
\]
Hence, by continuity of \(G\), we have that \(G(u_{0j}) \to G(u_0)\) a.e. in \(\Gamma\). We also have that \(G(u_{0j}) \leq G(u_0) \in L^1(\Gamma)\). Thus, by the Lebesgue’s dominated convergence theorem, we get
\[
G(u_{0j}) \to G(u_0) \text{ strong in } L^1(\Gamma). \tag{3.24}
\]
Then, by (3.23) and (3.24), we obtain that
\[
\int_{\Gamma} G_{\nu}(u_{0j}(x)) \, d\Gamma \leq C, \tag{3.25}
\]
where \(C\) is independent of \(j\) and \(\nu\). In this way, using (3.21) and (3.25), we have from (3.22) that
\[
|u_{j\nu}'|^2 + \|u_{j\nu}\|^2 + \int_{\Gamma} G(u_{j\nu}(x,t)) \, d\Gamma \leq C, \tag{3.26}
\]
where \(C\) is independent of \(j\), \(\nu\) and \(t\).

From (3.26), we obtain that
\[
(u_{j\nu}) \text{ is bounded in } L^\infty(0, T; H^{1/2}(\Gamma)), \tag{3.27}
\]
\[
(u_{j\nu}') \text{ is bounded in } L^\infty(0, T; L^2(\Gamma)). \tag{3.28}
\]
We have that (3.27) and (3.28) are true for all pairs \((j, \nu) \in \mathbb{N}^2\), in particular, for \((i, i) \in \mathbb{N}^2\). Thus, there exists a subsequence of \((u_{ii})\), which we denote by \((u_i)\), and a function \(u : \Sigma \to \mathbb{R}\), such that
\[
u_i \to u \text{ weak star in } L^\infty(0, T; H^{1/2}(\Gamma)), \tag{3.29}
\]
\[
u_i' \to u' \text{ weak in } L^\infty(0, T; L^2(\Gamma)). \tag{3.30}
\]
We also have by (3.8) that
\[
u_{ii}'' + Au_i + F_i(u_i) + \beta u_i = 0 \quad \text{in } L^2(0, T; H^{-1/2}(\Gamma)). \tag{3.31}
\]
From (3.29), (3.30) and observing that the injection of \(H^1(\Sigma)\) in \(L^2(\Sigma)\) is compact, there exists a subsequence of \((u_i)\), which we still denote by \((u_i)\), such that
\[
u_i \to u \text{ a.e. in } \Sigma.
\]
Since $F$ is continuous
\[ F(u_i) \rightarrow F(u) \text{ a.e. in } \Sigma. \]

Furthermore, since $u_i(x,t)$ is bounded in $\mathbb{R}$,
\[ F_i(u_i) - F(u_i) \rightarrow 0 \text{ a.e. in } \Sigma. \]

Therefore, we conclude
\[ F_i(u_i) \rightarrow F(u) \text{ a.e. in } \Sigma. \quad (3.32) \]

Taking duality between (3.31) and $u_i$ we obtain
\[
\int_0^T (F_i(u_i), u_i(t)) dt = \int_0^T |u_i'(t)|^2 dt - \alpha \int_0^T \|u_i(t)\|^2 dt - \\
- (u_i'(T), u_i(T)) + (u_1, u_{0j}) - \int_0^T (\beta u_i'(t), u_i(t)) dt.
\]

Using (2.1), (3.6) and (3.7), we have by (3.33) that
\[
\int_0^T (F_i(u_i), u_i(t)) dt \leq C, \quad (3.34)
\]
where $C$ is independent of $i$.

Thus, from (3.32) and (3.34), it follows by Strauss’ theorem, see Strauss [9], that
\[ F_i(u_i) \rightarrow F(u) \text{ strongly in } L^1(\Sigma). \quad (3.35) \]

By (3.29), (3.30) and (3.35) it is permissible to pass to the limit in (3.31) obtaining a function $u : \Sigma \rightarrow \mathbb{R}$ satisfying (3.1) – (3.4). $\square$

4. Asymptotic Behaviour

In this section we study the exponential decay for the energy $E(t)$ associated to the weak solution $u$ given by the Theorem 3.1. This energy is given by
\[
E(t) = \frac{1}{2} |u'(t)|^2 + \alpha \|u(t)\|^2 + \int_\Gamma G(u(x,t))d\Gamma, \quad t \geq 0. \quad (4.1)
\]

We consider the followings additional hypothesis:
\[ 0 \leq G(s) \leq sF(s), \quad \forall s \in \mathbb{R} \quad (4.2) \]
THEOREM 4.1. — Let $F$ satisfying (1.2) and (4.2). Then the energy (4.1) satisfies
\[ E(t) \leq 4E(0)e^{-\frac{\varepsilon}{2}t}, \tag{4.3} \]
where $\varepsilon$ is a positive constant.

Proof. — For an arbitrary $\varepsilon > 0$, we define the perturbed energy
\[ E_{\varepsilon}(t) = E_{\nu}(t) + \varepsilon \eta(t) \tag{4.4} \]
where $E_{\nu}(t)$ is the energy similar to (4.1) associated to the solution obtained in the Lemma 3.1 and
\[ \eta(t) = (u_{\nu}(t), \nu_{\nu}(t)). \]

Note that
\[ |\eta(t)| \leq C_2 E_{\nu}(t), \]
where $C_2 = \max \left\{ C_1, \frac{1}{\alpha} \right\}$, and $C_1$ is the immersion constant of $H^{1/2}(\Gamma)$ into $L^2(\Gamma)$.

Then,
\[ |E_{\varepsilon}(t) - E_{\nu}(t)| \leq \varepsilon C_2 E_{\nu}(t), \]
or
\[ (1 - \varepsilon C_2) E_{\nu}(t) \leq E_{\varepsilon}(t) \leq (1 + \varepsilon C_2) E_{\nu}(t). \]

Taking $0 < \varepsilon \leq \frac{1}{2C_2}$, we get
\[ \frac{E_{\nu}(t)}{2} \leq E_{\varepsilon}(t) \leq 2E_{\nu}(t), \forall t \geq 0. \tag{4.5} \]

Multiplying the equation in (3.8) for $u_{\nu}'$, using (2.1) and the fact of $A$ to be positive, we obtain
\[ E_{\nu}'(t) \leq -\beta_0 |u_{\nu}'(t)|^2 \leq 0. \tag{4.6} \]

Differentiating the function $\eta(t)$ and using (3.8), (4.2) and the fact of $A$ to be positive comes that
\[ \eta'(t) \leq \left(1 + \frac{\beta_1}{2\mu}\right) \left|u_{\nu}'(t)\right|^2 + \left(\frac{\beta_1 \mu C_1}{2} - \alpha\right) \left\|u_{\nu}(t)\right\|^2 - \int_{\Gamma} G_{\nu}(u_{\nu})d\Gamma, \tag{4.7} \]
where $\beta_1 = \|\beta\|_{L^\infty(\Gamma)}$ and $\mu > 0$ to be chosen.
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It follows by (4.4), (4.6) and (4.7) that

\[
E_{\nu\epsilon}'(t) \leq \left[ \epsilon \left( 1 + \frac{\beta_1}{2\mu} \right) - \beta_0 \right] |u_\nu'(t)|^2 - \epsilon \left( \alpha - \frac{\beta_1 \mu C_1}{2} \right) \|u_\nu(t)\|^2 - \epsilon \int_{\Gamma} G_\nu(u_\nu)d\Gamma.
\]  
(4.8)

Taking \( \mu = \frac{\alpha}{\beta_1 C_1} \) and \( 0 < \epsilon \leq \frac{2\alpha \beta_0}{3\alpha + \beta_1^2 C_1} \), we get

\[
E_{\nu\epsilon}'(t) \leq -E_\nu(t).
\]  
(4.9)

Choosing \( \epsilon \leq \min \left\{ \frac{1}{2C_2}, \frac{2\alpha \beta_0}{3\alpha + \beta_1^2 C_1} \right\} \) then (4.5) and (4.9) occur simultaneously, therefore

\[
E_{\nu\epsilon}'(t) + \frac{\epsilon}{2} E_{\nu\epsilon}(t) \leq 0,
\]

that is,

\[
E_\nu(t) \leq 4E_\nu(0)e^{-\frac{\epsilon}{2}t}.
\]  
(4.10)

From (3.29), (3.30) and since \( G_\nu \) is continuous, we have

\[
G_\nu(u_\nu(\cdot,t)) - G_\nu(u(\cdot,t)) \to 0 \text{ a.e. in } \Gamma, \forall t \geq 0.
\]  
(4.11)

But we know that \( F_\nu \to F \) uniformly on bounded sets of \( \mathbb{R} \). Then

\[
G_\nu(u(\cdot,t)) \to G(u(\cdot,t)) \text{ a.e. in } \Gamma, \forall t \geq 0.
\]  
(4.12)

Thus, by (4.11) and (4.12)

\[
G_\nu(u_\nu(\cdot,t)) \to G(u(\cdot,t)) \text{ a.e. in } \Gamma, \forall t \geq 0.
\]  
(4.13)

Moreover, we have, by (4.10), that

\[
\int_{\Gamma} G_\nu(u_\nu(x,t))d\Gamma \leq 4E_\nu(0).
\]

Therefore, using (3.21), (3.23) and (3.24), we get

\[
\liminf_{\nu \to \infty} \int_{\Gamma} G_\nu(u_\nu(x,t))d\Gamma \leq 4E(0).
\]  
(4.14)
By (4.13), (4.14) and Fatou’s lemma, we have

\[
\int_{\Gamma} G(u(x,t)) \, d\Gamma \leq \liminf_{\nu \to \infty} \int_{\Omega} G_{\nu}(u_{\nu}(x,t)) \, d\Gamma.
\]

Hence, passing \( \liminf \) in (4.10), we get (4.3). \( \square \)

**Remark.** — In the existence we can take \( \lambda = 0 \). For this end, we define in \( H^1(\Omega) \) the norm

\[
[v]^2 = \int_{\Omega} |\nabla v|^2 \, dx + \int_{\Gamma} |\gamma_0 v|^2 \, d\Gamma, \quad \forall v \in H^1(\Omega),
\]

obtaining now the positivity of operator \( A + \zeta I \), for \( \zeta > 0 \) arbitrary, like in Lions [8]. For the asymptotic behaviour, we need the additional hypothesis \( \beta_0 > \zeta \).

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**Bibliography**


