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## Critical boundary constants and Pohozaev identity <sup>(\*)</sup>

OULD AHMED-IZID-BIH ISSELKOU <sup>(1)</sup>

*à mes deux filles Kénizé et Maöna*

**RÉSUMÉ.** — La première partie de ce travail concerne le problème,

$$(P\epsilon) \begin{cases} \Delta u + u^{\frac{n+2}{n-2}} = 0 \text{ dans } B_1, \\ u > 0 \text{ dans } B_1, \\ u = \epsilon \text{ sur } \partial B_1, \end{cases}$$

où  $B_1 = \{x \in \mathbb{R}^n, \|x\| < 1\}$ ,  $n \geq 3$  et  $\epsilon > 0$ .

On démontre qu'il existe une constante critique  $\epsilon^* = \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}}$ , telle que le problème  $(P\epsilon)$  admet exactement deux solutions  $u_{\epsilon 1}$  et  $u_{\epsilon 2}$  ( $u_{\epsilon 1} < u_{\epsilon 2}$ ) si  $0 < \epsilon < \epsilon^*$ , une solution unique si  $\epsilon = \epsilon^*$  et n'admet pas de solution si  $\epsilon > \epsilon^*$ . Toutes ces solutions seront données explicitement. Il est démontré que quand  $\epsilon \rightarrow 0$ ,

$$\frac{u_{\epsilon 1}(x) - \epsilon}{\epsilon} \rightarrow 0 \text{ sur } \overline{B_1} \text{ et } \frac{u_{\epsilon 2}(x) - \epsilon}{\epsilon} \rightarrow \|x\|^{2-n} - 1, \text{ sur } \overline{B_1} \setminus \{O\}.$$

Au cours de la seconde partie, on s'intéresse au problème

$$(Q\epsilon) \begin{cases} \Delta u + f(x, u) = 0 \text{ dans } \Omega, \\ u > 0 \text{ dans } \Omega, \\ u = \epsilon \text{ sur } \partial\Omega, \end{cases}$$

où  $\Omega$  est un domaine borné, régulier et étoilé par rapport à l'origine,  $f$  est continue et dépend asymptotiquement de  $u$  "comme  $u^\alpha$ ",  $1 < \alpha$  et  $\alpha \neq \frac{n+2}{n-2}$ . Différents résultats d'existence de constante au bord critique  $\epsilon^*$  pour le problème  $(Q\epsilon)$  sont donnés.

**ABSTRACT.** — The first part of this work deals with the problem

$$(P\epsilon) \begin{cases} \Delta u + u^{\frac{n+2}{n-2}} = 0 \text{ in } B_1, \\ u > 0 \text{ in } B_1, \\ u = \epsilon \text{ on } \partial B_1. \end{cases}$$

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where  $B_1 = \{x \in \mathbb{R}^n, \|x\| < 1\}$ ,  $n \geq 3$  and  $\epsilon > 0$ .

We show that there exists a critical constant  $\epsilon^* = \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}}$ , such that the problem  $(P\epsilon)$  admits two solutions  $u_{\epsilon 1}$  and  $u_{\epsilon 2}$  ( $u_{\epsilon 1} < u_{\epsilon 2}$ ) if  $0 < \epsilon < \epsilon^*$ , only one solution if  $\epsilon = \epsilon^*$  and no solution if  $\epsilon > \epsilon^*$ . We give all these solutions explicitly. We show that, when  $\epsilon \rightarrow 0$ ,

$$\frac{u_{\epsilon 1}(x) - \epsilon}{\epsilon} \rightarrow 0 \text{ on } \overline{B_1} \text{ and } \frac{u_{\epsilon 2}(x) - \epsilon}{\epsilon} \rightarrow \|x\|^{2-n} - 1, \text{ if } x \in \overline{B_1} \setminus \{O\}.$$

The second part is devoted to the following problem

$$(Q\epsilon) \begin{cases} \Delta u + f(x, u) = 0 \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = \epsilon \text{ on } \partial\Omega, \end{cases}$$

where  $\Omega$  is a regular bounded domain which is starshaped about the origin,  $f$  is continuous and behaves like  $u^\alpha$  in the second variable,  $1 < \alpha$  and  $\alpha \neq \frac{n+2}{n-2}$ . We give different existence results for a boundary critical datum  $\epsilon^*$  for  $(Q\epsilon)$ .

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## 1. The Sobolev Exponent Growth

Let us consider the problem

$$(P\epsilon) \begin{cases} \Delta u + u^{\frac{n+2}{n-2}} = 0 \text{ in } B_1, \\ u \geq 0 \text{ in } B_1, \\ u = \epsilon \text{ on } \partial B_1. \end{cases}$$

In [9], it is shown that there exists a critical boundary datum  $\epsilon^*$  such that  $\forall 0 < \epsilon < \epsilon^*$ ,  $(P\epsilon)$  admits -at least- one  $C^2$ -solution. There is no solution when  $\epsilon > \epsilon^*$ . It is known that  $(P0)$  does not admit a nontrivial solution (see [13]). According to [5], every regular solution of  $(P\epsilon)$  is spherically symmetric. Let  $u_\epsilon$  ( $\epsilon > 0$ ), be a solution of  $(P\epsilon)$ , then

$$v_\epsilon = \frac{u_\epsilon - \epsilon}{\epsilon}$$

is a solution of

$$(A\lambda) \begin{cases} \Delta w + \lambda(1+w)^{\frac{n+2}{n-2}} = 0 \text{ in } B_1, \\ w > 0 \text{ in } B_1, \\ w = 0 \text{ on } \partial B_1, \end{cases}$$

where

$$\lambda = \epsilon^{\frac{4}{n-2}}.$$

The maximum principle implies

$$v_\epsilon > 0, \text{ in } B_1.$$

It is known (see [11],[4] and [3]) that there exists a constant  $\lambda^*$  such that (A $\lambda$ ) admits just two  $C^3$ -spherically symmetric solutions when  $0 < \lambda < \lambda^*$ , only one solution for  $\lambda = \lambda^*$  and no solution if  $\lambda > \lambda^*$ . We give here the value of  $\lambda^*$  and the explicit solutions for every  $\lambda \leq \lambda^*$ . We show that when  $\lambda \rightarrow 0$ , the “small” solution tends to  $v = 0$  the trivial solution of (A0) and the “big” one tends to  $H(x) = \|x\|^{2-n} - 1$ ,  $x \neq O$ .

PROPOSITION 1

$$\lambda^* = (\epsilon^*)^{\frac{4}{n-2}} = \frac{n(n-2)}{4}.$$

*Proof.* — For

$$0 < \epsilon < \epsilon^* = \left( \frac{n(n-2)}{4} \right)^{\frac{n-2}{4}},$$

let  $u_\epsilon$  be a regular ( $P\epsilon$ ) solution. As in [13], let us put

$$g(x) = \sum_{i=1}^n x_i D_i v_\epsilon \nabla v_\epsilon \quad (\text{where } v_\epsilon = \frac{u_\epsilon - \epsilon}{\epsilon}),$$

and use the Divergence Theorem, to get

$$\begin{aligned} \epsilon^{\frac{4}{n-2}} \left\{ \left(1 - \frac{1}{2}n\right) \int_{B_1} [1 + v_\epsilon(x)]^{\frac{n+2}{n-2}} v_\epsilon(x) dx + \frac{n-2}{2} \int_{B_1} \left[ (1 + v_\epsilon(x))^{\frac{2n}{n-2}} - 1 \right] dx \right\} \\ + \frac{1}{2} \int_{\partial B_1} [x \cdot \nu] \|\nabla v_\epsilon(x)\|^2 ds = \int_{\partial B_1} [x \cdot \nabla v_\epsilon(x)] [\nabla v_\epsilon(x) \cdot \nu] ds. \end{aligned}$$

From the identity

$$\int_{B_1} [1 + v_\epsilon(x)]^{\frac{2n}{n-2}} dx = \int_{B_1} [1 + v_\epsilon(x)]^{\frac{n+2}{n-2}} v_\epsilon(x) dx + \int_{B_1} [1 + v_\epsilon(x)]^{\frac{n+2}{n-2}} dx,$$

we infer that

$$\begin{aligned} (*) \quad \epsilon^{\frac{4}{n-2}} \left\{ \frac{2-n}{2} \int_{B_1} dx + \frac{n-2}{2} \int_{B_1} [1 + v_\epsilon(x)]^{\frac{n+2}{n-2}} dx \right\} \\ + \frac{1}{2} \int_{\partial B_1} [x \cdot \nu] \|\nabla v_\epsilon(x)\|^2 ds = \int_{\partial B_1} [x \cdot \nabla v_\epsilon(x)] [\nabla v_\epsilon(x) \cdot \nu] ds. \end{aligned}$$

Using again the Divergence Theorem, we get

$$\epsilon^{\frac{4}{n-2}} \int_{B_1} (1 + v_\epsilon)^{\frac{n+2}{n-2}} dx = - \int_{B_1} \Delta v_\epsilon(x) dx = - \int_{\partial B_1} \frac{\partial v_\epsilon(x)}{\partial \nu} ds,$$

where  $\nu$  denotes the unit outward normal to  $\partial B_1$ .

The Maximum Principle implies that

$$\frac{\partial v_\epsilon}{\partial \nu} < 0 \text{ on } \partial B_1.$$

As  $v_\epsilon$  is spherically symmetric and vanishes on  $\partial B_1$ , we get

$$x \cdot \nabla v_\epsilon(x) = x \cdot \nu \frac{\partial v_\epsilon}{\partial \nu}, \text{ on } \partial \Omega, \text{ and}$$

$$\frac{\partial v_\epsilon(x)}{\partial \nu} = -\|\nabla v_\epsilon(x)\| = l, \text{ on } \partial B_1,$$

where  $l$  is a constant on  $\partial B_1$ . Using the fact that  $x \cdot \nu = 1$  on  $\partial B_1$ , we obtain from (\*)

$$\epsilon^{\frac{4}{n-2}} \frac{2-n}{2} \int_{B_1} dx - \frac{n-2}{2} l \int_{\partial B_1} ds = \frac{1}{2} l^2 \int_{\partial B_1} ds.$$

This equation is equivalent to

$$|S_1| l^2 + (n-2) |S_1| l + \epsilon^{\frac{4}{n-2}} (n-2) |B_1| = 0.$$

$$|B_1| = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}, \text{ and } |S_1| = n|B_1|,$$

where  $|B_1|$  is the Lebesgue measure of the unit ball of  $\mathfrak{R}^n$  and  $|S_1| = n|B_1|$  is the surface measure of the unit sphere.

We obtain the following equation in  $l = v'_\epsilon(1)$

$$(1) \quad nl^2 + n(n-2)l + \epsilon^{\frac{4}{n-2}}(n-2) = 0.$$

When

$$0 < \epsilon < \left( \frac{n(n-2)}{4} \right)^{\frac{n-2}{4}},$$

this equation admits two negative solutions

$$l_1(\epsilon) = \frac{n(2-n) + \sqrt{n^2(n-2)^2 - 4n(n-2)\epsilon^{\frac{4}{n-2}}}}{2n}$$

$$l_2(\epsilon) = \frac{n(2-n) - \sqrt{n^2(n-2)^2 - 4n(n-2)\epsilon^{\frac{4}{n-2}}}}{2n}.$$

When

$$\epsilon = \left( \frac{n(n-2)}{4} \right)^{\frac{n-2}{4}},$$

equation (1) admits a unique negative solution

$$l_1 = \frac{2-n}{2},$$

and no real solution if

$$\epsilon > \left( \frac{n(n-2)}{4} \right)^{\frac{n-2}{4}}.$$

So it is clear that

$$\lambda^* = (\epsilon^*)^{\frac{4}{n-2}} \leq \frac{n(n-2)}{4}.$$

The proof will be complete, if one shows that

$$\forall 0 < \epsilon < \left( \frac{n(n-2)}{4} \right)^{\frac{n-2}{4}},$$

there exists just two regular solutions of  $(P\epsilon)$ . Let us recall that the problem

$$\Delta u + u^{\frac{n+2}{n-2}} = 0 \text{ in } \mathfrak{R}^n,$$

admits the radial solutions (see [12])

$$u_\lambda(\|x\|) = \lambda^{\frac{n-2}{4}} (n(n-2))^{\frac{n-2}{4}} (\lambda^2 + \|x\|^2)^{\frac{2-n}{2}}, \lambda > 0.$$

Let us put

$$\phi_\lambda = u_{\lambda|_{B_1}}.$$

$$\max_{\lambda > 0} \phi_\lambda(1) = \phi_1(1) = \left( \frac{n(n-2)}{4} \right)^{\frac{n-2}{4}}.$$

It is immediate to verify that

$$\phi_\lambda(1) = \phi_{\frac{1}{\lambda}}(1), \forall \lambda > 0.$$

$$\phi_\lambda \neq \phi_{\frac{1}{\lambda}}, \forall \lambda \neq 1.$$

In particular,

$$\phi'_\lambda(1) \neq \phi'_{\frac{1}{\lambda}}(1), \forall \lambda \neq 1.$$

As the function

$$\lambda \rightarrow \phi_\lambda(1) = \lambda^{\frac{n-2}{2}} (n(n-2))^{\frac{n-2}{4}} (1 + \lambda^2)^{\frac{2-n}{2}},$$

is continuous on  $[1, \infty[$ , with

$$\lim_{\lambda \rightarrow \infty} \lambda^{\frac{n-2}{2}} (n(n-2))^{\frac{n-2}{4}} (1 + \lambda^2)^{\frac{2-n}{2}} = 0,$$

we obtain two distinct solutions of  $(P\epsilon)$ , when

$$0 < \epsilon < \left( \frac{n(n-2)}{4} \right)^{\frac{n-2}{4}},$$

and one solution when

$$\epsilon = \left( \frac{n(n-2)}{4} \right)^{\frac{n-2}{4}},$$

which is  $\phi_1$ .

So the proof of Proposition 1 is complete.

Let us study the behavior of solutions when  $\epsilon \rightarrow 0$ . Let  $u_{\epsilon 1}$  and  $u_{\epsilon 2}$  be the two solutions of  $(P\epsilon)$ , with

$$u'_{\epsilon 1}(1) = l_1(\epsilon) = \frac{n(2-n) + \sqrt{n^2(n-2)^2 - 4n(n-2)\epsilon^{\frac{4}{n-2}}}}{2n},$$

and

$$u'_{\epsilon 2}(1) = l_2(\epsilon) = \frac{n(2-n) - \sqrt{n^2(n-2)^2 - 4n(n-2)\epsilon^{\frac{4}{n-2}}}}{2n}.$$

$$\forall 0 < \epsilon < \epsilon^*, \exists! \lambda(\epsilon) > 1,$$

such that

$$u_{\epsilon 1} = \phi_{\lambda(\epsilon)} \quad \text{and} \quad u_{\epsilon 2} = \phi_{\frac{1}{\lambda(\epsilon)}}.$$

$$\lambda(\epsilon) = \frac{[n(n-2)]^{\frac{1}{2}} + \sqrt{n(n-2) - 4\epsilon^{\frac{4}{n-2}}}}{2\epsilon^{\frac{2}{n-2}}}.$$

Let us put

$$\psi_{\epsilon 1} = \frac{u_{\epsilon 1} - \epsilon}{\epsilon} \quad \text{and} \quad \psi_{\epsilon 2} = \frac{u_{\epsilon 2} - \epsilon}{\epsilon}.$$

PROPOSITION 2

- (i)  $\psi_{\epsilon 1} \rightarrow \psi_1 = 0$ , in  $C^1(\overline{B_1})$ , as  $\epsilon \rightarrow 0$ .
- (ii)  $\psi_{\epsilon 2}(x) \rightarrow \psi_2 = \|x\|^{2-n} - 1$ , in  $C^1_{loc}(\overline{B_1} \setminus \{O\})$ , as  $\epsilon \rightarrow 0$ .

*Proof.* — Let us remark the following

$$l_1(\epsilon) \rightarrow 0, \text{ as } \epsilon \rightarrow 0, \text{ and } l_2(\epsilon) \rightarrow 2 - n, \text{ as } \epsilon \rightarrow 0.$$

We give here a direct proof, using the explicit knowledge of  $\psi_{\epsilon i}$ ,  $i \in \{1, 2\}$ . As we have seen, for every  $0 < \epsilon < \epsilon^*$ , there exists a unique

$$\lambda(\epsilon) = \frac{[n(n-2)]^{\frac{1}{2}} + \sqrt{n(n-2) - 4\epsilon^{\frac{4}{n-2}}}}{2\epsilon^{\frac{2}{n-2}}} > 1,$$

$$(\lambda(\epsilon) \rightarrow \infty \text{ as } \epsilon \rightarrow 0)$$

such that we have

$$\psi_{\epsilon 1}(r) = \frac{[\lambda(\epsilon)]^{\frac{n-2}{2}} [n(n-2)]^{\frac{n-2}{4}} \left\{ ([\lambda(\epsilon)]^2 + r^2)^{\frac{2-n}{2}} - ([\lambda(\epsilon)]^2 + 1)^{\frac{2-n}{2}} \right\}}{[\lambda(\epsilon)]^{\frac{n-2}{2}} [n(n-2)]^{\frac{n-2}{4}} \left\{ [\lambda(\epsilon)]^2 + 1 \right\}^{\frac{2-n}{2}}},$$

$$\psi_{\epsilon 2}(r) = \frac{[\lambda(\epsilon)]^{\frac{2-n}{2}} [n(n-2)]^{\frac{n-2}{4}} \left\{ ([\lambda(\epsilon)]^{-2} + r^2)^{\frac{2-n}{2}} - ([\lambda(\epsilon)]^{-2} + 1)^{\frac{2-n}{2}} \right\}}{[\lambda(\epsilon)]^{\frac{2-n}{2}} (n(n-2))^{\frac{n-2}{4}} \left\{ [\lambda(\epsilon)]^{-2} + 1 \right\}^{\frac{2-n}{2}}}.$$

We finally get

$$\psi_{\epsilon 1}(r) = \left( \frac{[\lambda(\epsilon)]^2 + 1}{[\lambda(\epsilon)]^2 + r^2} \right)^{\frac{n-2}{2}} - 1; \quad \psi_{\epsilon 2}(r) = \left\{ \frac{1 + [\lambda(\epsilon)]^2}{1 + [\lambda(\epsilon)]^2 r^2} \right\}^{\frac{n-2}{2}} - 1.$$

It is immediate to verify that

$$\psi_{\epsilon 1}(\|x\|) = \left( \frac{[\lambda(\epsilon)]^2 + 1}{[\lambda(\epsilon)]^2 + \|x\|^2} \right)^{\frac{n-2}{2}} - 1 \rightarrow 0, \text{ in } C^1(\overline{B_1}), \text{ as } \epsilon \rightarrow \infty$$

and

$$\psi_{\epsilon 2}(\|x\|) = \left\{ \frac{1 + [\lambda(\epsilon)]^2}{1 + [\lambda(\epsilon)]^2 \|x\|^2} \right\}^{\frac{n-2}{2}} - 1 \rightarrow \|x\|^{2-n} - 1, \text{ on } \overline{B_1} \setminus \{O\}.$$

*Remark 1.* — According to Theorem 1.1 in [7], it is, in general, false that every positive solution  $u$  in  $B_1$  of  $\Delta u + u^\alpha = 0$ , is a restriction of a positive solution  $v$  of this problem in  $\mathfrak{R}^n$ .

## 2. Nonlinearities with Noncritical Growth

### 2.1. The Subcritical Behavior

We deal here with the following problem

$$(Q\epsilon) \begin{cases} \Delta u + f(x, u) = 0 \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = \epsilon \text{ on } \partial\Omega, \end{cases}$$

Let us suppose the following

(i)  $\Omega$  is a bounded regular domain of  $\mathbb{R}^n$ , which is starshaped about the origin.

(ii)  $f \in C^0(\overline{\Omega} \times \mathbb{R}_+, \mathbb{R}_+)$ ,

(iii) there exist positive constants  $c_1, \gamma, \alpha$  and a positive function  $a \in C^0(\overline{\Omega})$  such that

$$1 < \gamma \leq \alpha < \frac{n+2}{n-2}; \quad c_1 t^\gamma \leq f(x, t), \quad \forall x \in \overline{\Omega}, t > 0,$$

$$\lim_{t \rightarrow \infty} \frac{f(x, t)}{t^\alpha} = a(x) \text{ and } f(x, t) = o(t) \text{ near } t = 0, \text{ uniformly in } x \in \overline{\Omega}.$$

PROPOSITION 3. — Under the previous hypotheses on  $\Omega$  and  $f$ , there exists a positive constant  $\epsilon^*(\Omega, f)$ , such that for every  $0 \leq \epsilon \leq \epsilon^*(\Omega, f)$ , the problem  $(Q\epsilon)$  admits, at least, one solution  $u_\epsilon \in C^{1,\delta}(\overline{\Omega})$ ,  $0 \leq \delta < 1$ . There is no bounded solution of  $(Q\epsilon)$  if  $\epsilon > \epsilon^*(\Omega, f)$ .

*Proof.* — The proof is nearly the same as in (Theorem 1 in [9]). The only difference is that the subsolutions and supersolutions are considered as elements of  $H_0^1(\Omega) \cap L^\infty(\Omega)$ , and the inequalities are in the sense of duality  $H^{-1}(\Omega)$ ,  $H_0^1(\Omega)$ .

Let us recall the main steps for this proof.

1. We use the hypothesis (iii) and Théorème 3.1 in [2], to show that the problem  $(Q\epsilon)$  admits -at least- one solution  $u \in H_0^1(\Omega)$ , when  $\epsilon$  is “small” enough. Using the  $L^p$ -estimates (see [1]), we infer that  $u \in W^{2,p}(\Omega)$ ,  $\forall p > 1$ . One can use embedding results (see [8]) to deduce that  $u \in C^{1,\alpha}(\overline{\Omega})$ .

2. We show that if  $(Q\bar{\epsilon})$  admits a solution, so does  $(Q\epsilon)$  for every  $\epsilon \leq \bar{\epsilon}$ .

3. Using the a priori estimate in [6], we show that  $(Q\epsilon)$  does not admit a solution, if  $\epsilon$  is great enough.

From these steps, we infer that the set  $I$  of  $\epsilon$ , for which  $(Q\epsilon)$  admits a solution, is a bounded interval.

4. Let  $\epsilon^*(\Omega, f)$  be the upper bound of  $I$ . The blow-up argument used in [6], can be applied to show that there exists no increasing sequence  $(\epsilon_j)$  in  $I$ , such that

$$\lim_{j \rightarrow \infty} \epsilon_j = \epsilon^*(\Omega, f), \text{ with } \lim_{j \rightarrow \infty} \max_{x \in \Omega} u_{\epsilon_j}(x) = \infty.$$

This last a priori  $L^\infty$ -estimate of the solutions  $u_\epsilon$  near  $\epsilon^*(\Omega, f)$ , leads to a solution of  $(Q\epsilon^*(\Omega, f))$ .

*Remark 2.* — When  $\Omega = B_r = \{x \in \mathbb{R}^n; \|x\| < r\}$  and  $f(x, u) = u^\alpha$ , then

1. every solution of  $(Q\epsilon)$  is spherically symmetric (see [5]),
2.  $\epsilon^*(B_r, \alpha) = r^{\frac{2}{1-\alpha}} \epsilon^*(B_1, \alpha)$ , (see [10]).

## 2.2. The Supercritical Growth Case

Let us consider the following problem

$$(T\epsilon) \begin{cases} \Delta u + a(x)u^\beta = 0 \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = \epsilon \text{ on } \partial\Omega. \end{cases}$$

We suppose that

(i)  $\Omega$  is a bounded regular domain, which is starshaped about the origin.

(ii)  $a \in C^0(\overline{\Omega}, \mathbb{R}_+^*)$  and  $\beta > \frac{n+2}{n-2}$ .

Under appropriate hypotheses, the following problem

$$(P) \begin{cases} \Delta u + a(x)u^\beta = 0 \text{ in } \mathbb{R}^n, \\ u > 0 \text{ in } \mathbb{R}^n, \\ u \in C^2(\mathbb{R}^n), \end{cases}$$

admits solutions (see [14]).

**PROPOSITION 4.** — *Let us suppose that the problem  $(P)$  admits a solution, then under hypotheses (i) and (ii), there exists a positive constant  $\epsilon^*(\Omega, a)$  such that  $(T\epsilon)$  admits, at least, one solution  $u_\epsilon \in C^{1,\delta}(\overline{\Omega})$ ,  $0 \leq \delta < 1$ , when  $0 < \epsilon < \epsilon^*(\Omega, a)$ . There is no  $L^\infty$ -solution of  $(T\epsilon)$  for  $\epsilon > \epsilon^*(\Omega, a)$ .*

*Proof.* — The proof is similar to the demonstration of Theorem 2 in [9].

**Remark 3.** — *The hypothesis concerning the existence of a solution of (P) is justified by the critical growth case (see section 1).*

**The  $\epsilon^*(\Omega, a)$ -limit case.**

Before dealing with this case, let us state the following lemma.

**LEMMA 1.** — *Under the hypotheses (i) and (ii), assume that  $(u_j)$  is a sequence of  $C^2(\overline{\Omega})$  – functions and  $(\epsilon_j)$  is a real sequence, such that*

$$(P_j) \begin{cases} \Delta u_j + a(x)u_j^\beta = 0 \text{ in } \Omega, \\ u_j > 0 \text{ in } \Omega, \\ u = \epsilon_j > 0 \text{ on } \partial\Omega. \end{cases}$$

*Then, if the real sequence  $(\epsilon_j)$  is bounded in  $\mathfrak{R}$ , so is  $(u_j)$  in  $H_0^1(\Omega)$ .*

*Proof.* — Using Pohozaev Identity, we get

$$\begin{aligned} (1 - \frac{n}{2}) \int_{\Omega} \|\nabla u_j(x)\|^2 dx + \frac{1}{2} \int_{\partial\Omega} (x.\nu) \|\nabla u_j(x)\|^2 ds + \frac{n}{\beta + 1} \int_{\Omega} a(x)u_j^{\beta+1}(x) dx \\ - \int_{\partial\Omega} (x.\nu)a(x)\epsilon_j^{\beta+1} dx = \int_{\partial\Omega} (x.\nabla u_j(x))(\nabla u_j(x).\nu) ds. \end{aligned}$$

Using the Green’s first identity, we get

$$\int_{\Omega} a(x)u_j^{\beta+1} dx = \int_{\Omega} \|\nabla u_j(x)\|^2 dx - \int_{\partial\Omega} \epsilon_j \frac{\partial u_j(x)}{\partial \nu} ds.$$

So we infer that

$$\begin{aligned} (*) \quad \left(1 - \frac{n}{2} + \frac{n}{\beta + 1}\right) \int_{\Omega} \|\nabla u_j(x)\|^2 dx = \int_{\partial\Omega} (x.\nabla u_j(x))(\nabla u_j(x).\nu) ds \\ - \frac{1}{2} \int_{\partial\Omega} (x.\nu) \|\nabla u_j(x)\|^2 ds + \int_{\partial\Omega} x.\nu a(x)\epsilon_j^{\beta+1} ds + \frac{n}{\beta + 1} \int_{\partial\Omega} \epsilon_j \frac{\partial u_j(x)}{\partial \nu} ds. \end{aligned}$$

Using the maximum principle, and the fact that  $u_j = \epsilon_j$ , on  $\partial\Omega$ , we obtain

$$\begin{aligned} (1 - \frac{n}{2} + \frac{n}{\beta + 1}) \int_{\Omega} \|\nabla u_j(x)\|^2 dx = \frac{1}{2} \int_{\partial\Omega} \|\nabla u_j(x)\|^2 x.\nu ds \\ + \int_{\partial\Omega} x.\nu a(x)\epsilon_j^{\beta+1} ds - \frac{n}{\beta + 1} \int_{\partial\Omega} \epsilon_j \|\nabla u_j(x)\| ds. \end{aligned}$$

As  $\Omega$  is regular and starshaped, we get

$$\begin{aligned} \left(1 - \frac{n}{2} + \frac{n}{\beta + 1}\right) \int_{\Omega} \|\nabla u_j(x)\|^2 dx &\geq c_0 \int_{\partial\Omega} \|\nabla u_j(x)\|^2 ds \\ &\quad - \frac{n}{\beta + 1} \int_{\partial\Omega} \epsilon_j \|\nabla u_j(x)\| ds - c_1, \end{aligned}$$

where,

$$c_0 = \frac{1}{2} \min_{x \in \partial\Omega} x \cdot \nu > 0 \text{ and } \int_{\partial\Omega} x \cdot \nu a(x) \epsilon_j^{\beta+1} ds \leq c_1.$$

As,

$$\beta > \frac{n+2}{n-2} \iff 1 - \frac{n}{2} + \frac{n}{\beta+1} < 0,$$

we get

$$\int_{\Omega} \|\nabla u_j(x)\|^2 dx \leq c_2 \int_{\partial\Omega} \|\nabla u_j(x)\|^2 ds + c_3 \int_{\partial\Omega} \|\nabla u_j(x)\| ds + c_4,$$

where  $c_2 < 0 < c_3$  and  $c_i, i = 2, \dots, 4$  are constants not depending on  $j$ . Using Hölder's Inequality, we obtain

$$\begin{aligned} \int_{\Omega} \|\nabla u_j(x)\|^2 dx &\leq c_2 \int_{\partial\Omega} \|\nabla u_j(x)\|^2 ds + c_5 \left( \int_{\partial\Omega} \|\nabla u_j(x)\|^2 ds \right)^{\frac{1}{2}} + c_4 \\ &\leq \sup_{t \in \mathfrak{R}} c_2 t^2 + c_5 t + c_4 < \infty. \end{aligned}$$

Let us put  $v_j = u_j - \epsilon_j$ , then  $v_j \in H_0^1(\Omega)$  and

$$\|\nabla v_j\|_{L^2(\Omega)} = \|\nabla u_j\|_{L^2(\Omega)}.$$

Using Poincaré Inequality, we get

$$\exists c_0 > 0 ; \|u_j - \epsilon_j\|_{H_0^1(\Omega)} \leq c_0, \forall j.$$

As

$$u_j^2(x) = [u_j(x) - \epsilon_j + \epsilon_j]^2 \leq 2 \left\{ [u_j(x) - \epsilon_j]^2 + \epsilon_j^2 \right\},$$

and  $\epsilon_j$  is bounded in  $\mathfrak{R}$ , this completes the proof of Lemma 1.

*Remark 4.* — The a priori estimate in Lemma 1 remains true for nonlinearities such that, there exist constants  $c$  and  $\gamma$ , with

$$c + uf(x, u) \leq \gamma F(x, u), \text{ where } F(x, u) = \int_0^u f(x, t) dt, \gamma > 2^* = \frac{2n}{n-2}.$$

PROPOSITION 5. — Under the hypotheses of Proposition 4, if  $a \in C^{0,\delta}(\overline{\Omega})$ ,  $0 < \delta \leq 1$ , then  $(T\epsilon^*(\Omega, a))$  admits a solution.

*Proof.* — Let  $(\epsilon_j)$  be an increasing real sequence such that

$$0 < \epsilon_j < \epsilon_{j+1} < \lim_{i \rightarrow \infty} \epsilon_i = \epsilon^*(\Omega, a).$$

For every  $j$ , let  $u_j$  be the solution of  $(T\epsilon_j)$  (see Proposition 4). As  $u_j \in C^2(\overline{\Omega})$ , one can use Lemma 1 to obtain

$$\exists c > 0, \|u_j\|_{H^1(\Omega)} \leq c, \quad \forall j.$$

Then, up to a subsequence,  $u_j \rightharpoonup u$  in  $H^1(\Omega)$ -weak,  $u_j \rightarrow u$  in  $L^2(\Omega)$  — strong and  $u_j \rightarrow u$ , a.e. in  $\Omega$ . One can multiply  $(P_j)$  by  $u_j$  to verify that

$$a(x)u_{\epsilon_j}^\beta \in L^{\frac{\beta+1}{\beta}}(\Omega).$$

By using the  $L^p$ -estimates and a bootstrap argument, one can show that  $u$  is a solution of  $(T\epsilon^*(\Omega, a))$ .

PROPOSITION 6. — Let  $u$  be a spherically symmetric  $L_{loc}^\infty(\mathbb{R}^n)$ — solution of

$$\begin{cases} \Delta u + u^\beta = 0 & \text{in } \mathbb{R}^n \\ u > 0 & \text{in } \mathbb{R}^n, \end{cases}$$

then  $u \in C^2(\mathbb{R}^n)$  and  $u \in L^p(\mathbb{R}^n)$ ,  $\forall p > \frac{n(\beta-1)}{2}$ .

*Proof.* — Let us choose  $r > 0$ , such that  $u(r) < \infty$ . As  $u \in L^\infty(B_r)$ , one can use the  $L^p$ -estimates (see [1]) to infer that  $u \in W^{2,p}(B_r)$ ,  $\forall p > 1$ . We infer that (see [8])

$$u \in C^{1,\delta}(\overline{B_r}), \quad \forall 0 < \delta < 1.$$

From the previous line, we see that  $u^\beta \in C^{0,\delta}(\overline{B_r})$ ,  $\forall 0 < \delta < 1$ . So we can use the Schauder Estimates to deduce that  $u \in C^{2,\delta}(\overline{B_r})$ . We can use Proposition 4, to infer that

$$\exists \epsilon^*(B_r, \beta) \text{ such that } u(r) \leq \epsilon^*(B_r, \beta).$$

It is easy to verify( see [10]) that

$$\epsilon^*(B_r, \beta) \leq \epsilon^*(B_1, \beta)r^{\frac{2}{1-\beta}},$$

so we deduce that, if  $p > \frac{n(\beta-1)}{2}$ , then  $u \in L^p(\mathbb{R}^n)$ .

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## Bibliography

- [1] AGMON (S.), DOUGLIS (A.) and NIRENBERG (L.). — *Estimates near the Boundary for Solutions of Elliptic Partial Differential Equations satisfying General Boundary Value Conditions I*. Comm. Pure Appl. Math., 12, pp. 623-727 (1959).
- [2] BOCCARDO (L.), MURAT (F.) and PUEL (J.P.). — *Quelques Opérateurs Quasi-linéaires*. C. R. Acad. Sc. Paris, t. 307, Série I, pp. 749-752 (1988).
- [3] BREZIS (H.) and NIRENBERG (L.). — *Positive Solutions of Nonlinear Elliptic Equations Involving Critical Sobolev Exponent*. Comm. Pure Appl. Math. 36, pp. 437-477 (1983).
- [4] CRANDALL (M. G.) and RABINOWITZ (P.H.). — *Some Continuation and Variational Methods for Positive Solutions of Nonlinear Elliptic Eigenvalue Problems*. Arch. Rational Mech. Anal. 58, pp.207-218 (1975).
- [5] GIDAS (B.), NI (W.-M.) and NIRENBERG (L.). — *Symmetry and Related Properties via the Maximum Principle*. Comm. Math. Phys. 68, pp.209-243 (1979).
- [6] GIDAS (B.) and SPRUCK (J.). — *A Priori Bounds for Positive Solutions of Nonlinear Elliptic Equations*. Comm. Partial Differential Equations 6, pp.883-901 (1981).
- [7] GIDAS (B.) and SPRUCK (J.). — *Global and Local Behavior of Positive Solutions of Nonlinear Elliptic Equations*. Comm. Pure Appl. Math., Vol.34, pp.525-598 (1981).
- [8] GILBARG (D.) and TRUDINGER (N.S.). — *Elliptic Partial Differential Equations of Second Order*. Springer Verlag (1977).
- [9] ISSELKOU (O.A.-I.-B.). — *A Critical Value for the Boundary Datum of a Dirichlet's Problem*. Funkcialaj Ekvacioj 41, pp.207-214 (1998).
- [10] ISSELKOU (O.A.-I.-B.). — *Donnée au Bord Critique pour un Problème de Dirichlet*. Revue URED No 8 and 9 (Dakar, 1999).
- [11] JOSEPH (D.D.) and LUNDGREN (T.S.). — *Quasilinear Dirichlet Problems Driven by Positive Sources*. Arch. Rational Mech. Anal. 49, pp. 241-269 (1973).
- [12] LOEWNER (C.) and NIRENBERG (L.). — *Partial Differential Equations Invariant under Conformal or Projective Transformation*. Contribution to Analysis (L. Ahlfors ed.), Academic Press, New York, pp. 245-272 (1974).
- [13] POHOZAEV (S.I.). — *Eigenfunctions of the Equations  $\Delta u + \lambda f(u) = 0$* . Soviet Math. Dokl. 6, pp.1408-1411 (1965).
- [14] POHOZAEV (S.I.). — *On Entire Solutions of Semilinear Elliptic Equations*. Research Note Math. 266, Pitman, pp.56-69, London (1992).