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Rational points on some pencils of conics with 6 singular fibres (*)

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Abstract. — Let \( k \) be an algebraic number field, let \( c \) be a non-square in \( k \) and let \( f_4, f_2 \) be homogeneous polynomials in \( X, Y \) of degrees 4 and 2 respectively. Necessary and sufficient conditions are obtained for the solubility in \( k \) of
\[
U^2 - cv^2 = f_4(X, Y)f_2(X, Y).
\]

Let \( \mathcal{Y} \to \mathbb{P}^1 \) be a pencil of conics defined over an algebraic number field \( k \). It is conjectured that the only obstruction to the Hasse principle on \( \mathcal{Y} \), and also to weak approximation, is the Brauer-Manin obstruction; and it was shown in [3] that this follows from Schinzel’s Hypothesis. Descriptions of the Brauer-Manin obstruction and of Schinzel’s Hypothesis can be found in [3]. It is of interest that arguments which show that the Brauer-Manin obstruction is the only obstruction to the Hasse principle for particular classes of \( \mathcal{Y} \) normally fall into two parts:

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(i) the proof that some comparatively down-to-earth obstruction is the only obstruction to the Hasse principle;

(ii) the identification of that obstruction with the Brauer-Manin obstruction.

The theorem in this paper is entirely concerned with (i); the equivalence of the obstruction in the theorem with the Brauer-Manin one has already been proved in a much more general context in [1], §2.6b and Chapter 3.

If one does not assume Schinzel’s Hypothesis, little is known. The only promising-looking line of attack is through the geometry of the universal torseurs on $\mathcal{Y}$; and these are much easier to study when $\mathcal{Y}$ has the special form

$$U^2 - cV^2 = P(W)$$

(1)

where $c$ is a non-square in $k$ and $P(W)$ is a separable polynomial in $k[W]$. By writing $W = X/Y$ we can take the solubility of (1) into the equivalent (though ungeometric) problem of the solubility of

$$U^2 - cV^2 = f(X, Y)$$

(2)

in $k$, where $f$ is homogeneous of even degree; here $\deg f$ is $1 + \deg P$ or $\deg P$. The simplest non-trivial case is that of Châtelet surfaces, when $P(W)$ has degree 3 or 4; in this case the conjecture was proved in [2]. The object of this paper is to prove the conjecture when $\deg f = 6$ and $f = f_4f_2$ over $k$, where $\deg f_4 = 4$ and $\deg f_2 = 2$.

Until the statement of the main theorem, we make no assumption about (2) other than that $f(X, Y)$ has even degree $n$ and no repeated factor. After multiplying $X, Y$ by suitable integers in $k$, we can assume that

$$f(X, Y) = a \prod_{i=1}^{n} (X + \lambda_i Y)$$

where $a$ is an integer in $k$ and the $\lambda_i$ are integers in $\overline{k}$; the $\lambda_i$ form complete sets of conjugates over $k$. For convenience we write $\gamma = \sqrt{c}$. We can assume that $\gamma$ does not lie in any $k(\lambda_i)$; for otherwise $f(X, Y)$ would have a non-trivial factor of the form $F^2 - cG^2$ with $F, G$ in $k[X, Y]$ and we could instead consider the simpler equation

$$U^2 - cV^2 = g(X, Y) = f(X, Y)/(F^2 - cG^2).$$

We can clearly also assume that the $\lambda_i$ are all distinct; for otherwise we can remove a squared factor from $f(X, Y)$ and reduce to a simpler problem
which has already been solved in [2]. To avoid trivialities, we shall also rule out solutions for which each side of (2) vanishes.

Let $a_1, \ldots, a_h$ be a set of representatives for the ideal classes in $k$; then it is enough to look for solutions $u, v, x, y$ of (2) for which $x, y$ are integers whose highest common factor is some $a_m$. (To move from rational to integral solutions may appear unnatural; but in fact it greatly simplifies the argument which follows, because it means that our intermediate equations do not have to be homogeneous.)

**Lemma 1.** — There is a finite computable list of n-tuples $(\alpha_1^{(r)}, \ldots, \alpha_n^{(r)})$ not depending on $u, v, x, y$, where $\alpha_i^{(r)}$ is in $k(\lambda_i)$ and conjugacy between $\lambda_i$ and $\lambda_j$ extends to conjugacy between $\alpha_i^{(r)}$ and $\alpha_j^{(r)}$, with the following property. If (2) has a solution with $x, y$ integers whose highest common factor is some $a_m$, then for some $r$ the system

$$u_i^2 - cv_i^2 = \alpha_i^{(r)}(x + \lambda_iy) \quad (1 \leq i \leq n)$$

has solutions with $u_i, v_i$ in $k(\lambda_i)$ for each $i$.

**Proof.** — We postulate once for all that the manipulations which follow are to be carried out in such a way as to preserve conjugacy. A prime factor $p$ of $x + \lambda_iy$ in $k(\lambda_i)$ which also divides $f(x, y)/(x + \lambda_iy)$ must divide

$$a \prod_{j \neq i} (-\lambda_iy + \lambda_jy) = y^5a \prod_{j \neq i} (\lambda_j - \lambda_i),$$

and for similar reasons it must divide

$$a \prod_{j \neq i} (\lambda_ix - \lambda_jx) = -x^5a \prod_{j \neq i} (\lambda_j - \lambda_i).$$

Hence it divides $aa_m \prod_{j \neq i} (\lambda_j - \lambda_i)$ and must therefore belong to a finite computable list; and any prime ideal not in this list which divides some $x + \lambda_iy$ to an odd power must split or ramify in $k(\lambda_i, \gamma)/k(\lambda_i)$. As ideals, $(x + \lambda_iy) = b_i c_i$ where $b_i$ only contains the prime ideals which either lie in the finite computable list above or ramify in $k(\lambda_i, \gamma)/k(\lambda_i)$, and every prime ideal which occurs to an odd power in $c_i$ must split in $k(\lambda_i, \gamma)/k(\lambda_i)$. By transferring squares from $b_i$ to $c_i$ we can assume that each $b_i$ is square-free. Each $b_i$ belongs to a finite list independent of $x, y$, and conorm $c_i = \mathcal{C}_i, \sigma \mathcal{C}_i$ where $\mathcal{C}_i$ is an ideal in $k(\lambda_i, \gamma)$ and $\sigma$ is the non-trivial automorphism of $k(\lambda_i, \gamma)$ over $k(\lambda_i)$. Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_H$ be a set of representatives for the ideal classes in $k(\lambda_i, \gamma)$; then for some $\mathfrak{A}^{(i)}$ from this list $\mathfrak{A}^{(i)} \mathcal{C}_i$ is principal, say

$$\mathfrak{A}^{(i)} \mathcal{C}_i = (\xi_i + \gamma \eta_i)$$

with $\xi_i, \eta_i$ in $k(\lambda_i)$. Thus

$$\left(\xi_i^2 - c\eta_i^2\right) = b_i^{-1} \mathfrak{A}^{(i)} \sigma \mathfrak{A}^{(i)}(x + \lambda_iy)$$

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as ideals. This implies \( \xi_i^2 - \eta_i^2 = \alpha_i(x + \lambda_i y) \) where the ideal \((\alpha_i)\) belongs to a finite computable list; and as we can clearly vary \( \alpha_i \) by any squared factor, this ensures the same property for \( \alpha_i \).

Strictly speaking, the elements of our list consist of equivalence classes of \( n \)-tuples (where the formulation of the equivalence relation is left to the reader); but we shall need to fix which representatives we choose. However, in what follows we shall also need to know that we can take the \( u_i, v_i \) to be integers without thereby imposing an uncontrolled extra factor in the \( \alpha_i^{(r)} \). For this purpose we need the following result:

**Lemma 2.** — Let \( K \) be an algebraic number field and \( C \) a non-square in \( D_K \). Then there exists \( A = A(K, C) \) in \( D_K \) such that if \( D \) is in \( D_K \) with

\[
U^2 - CV^2 = D
\]

soluble in \( K \), and if \( A^2 | D \), then (4) is soluble with \( U, V \) in \( D_K \).

**Proof.** — Write \( L = K(\sqrt{C}) \), let \( \sigma \) be the non-trivial automorphism of \( L/K \) and let \( \mathfrak{A}_1, \ldots, \mathfrak{A}_H \) be a set of integral representatives for the ideal classes of \( L \). Let \( d \) be any non-zero integer of \( K \) such that \( u^2 - Cv^2 = d \) for some \( u, v \) in \( K \), and write

\[
(u + C^{1/2}v) = m/n
\]

where \( m, n \) are coprime ideals in \( L \). Thus \( (u - C^{1/2}v) = \sigma m/\sigma n \), so that \( \sigma n|m \). Choose \( r \) so that \( \mathfrak{A}_r n \) is principal — say equal to \( (B) \). If \( u_1, v_1 \) are defined by

\[
u_1 + C^{1/2}v_1 = B(u + C^{1/2}v)/\sigma B
\]

then the denominator of \( u_1 + C^{1/2}v_1 \) divides \( \sigma \mathfrak{A}_r \) and \( u_1^2 - Cv_1^2 = d \). If \( A \) in \( K \) is divisible by \( 2C^{1/2}\mathfrak{A}_r \cdot \sigma \mathfrak{A}_r \) for every \( r \), then \( A^2d = (Au_1)^2 - C(Av_1)^2 \) where \( Au_1 \) and \( Av_1 \) are integers. \( \square \)

Since we can multiply each \( \alpha_i^{(r)} \) by the square of any nonzero integer in \( k(\lambda_i) \), subject to the preservation of conjugacy, we can assume that \( \alpha_i^{(r)} \) is divisible by \( (A(k(\lambda_i), c))^2 \) in the notation of Lemma 2; thus if (3) is soluble at all for given integers \( x, y \) then it is soluble in integers. Moreover

\[
\prod_{i=1}^n \alpha_i^{(r)} = \left( a \prod_{i=1}^n (u_i^2 - cv_i^2) \right) / (u^2 - cv^2),
\]

so that \( \prod \alpha_i^{(r)} = a(u(r)^2 - cv(r)^2) \) for some \( u(r), v(r) \) in \( k \). Conversely, any solution of (3) gives rise to a solution of (2); and for this we do not require any condition on \( (x, y) \).
If the system (3) has solutions at all, it has solutions for which conjugacy between \( \lambda_i \) and \( \lambda_j \) extends to conjugacy between \( u_i, v_i \) and \( u_j, v_j \); such solutions have the form
\[
 u_i = \lambda_i^{n-1} \xi_0 + \ldots + \xi_{n-1}, \quad v_i = \lambda_i^{n-1} \eta_0 + \ldots + \eta_{n-1} \quad (5)
\]
for some \( \xi_\nu, \eta_\nu \) in \( k \). Thus we can replace (3) by the system
\[
 (\lambda_i^{n-1} X_0 + \ldots + X_{n-1})^2 - c(\lambda_i^{n-1} Y_0 + \ldots + Y_{n-1})^2 = \alpha_i^{(r)} (X + \lambda_i Y) \quad (6)
\]
which is to be solved in \( k \). If we eliminate \( X, Y \) these become \( n - 2 \) homogeneous quadratic equations in \( 2n \) variables, which give a variety \( \mathcal{V}^{(r)} \) defined over \( k \). In the special case \( n = 4 \) it was shown in [2], §7 that the \( \mathcal{V}^{(r)} \) are factors of the universal torses for (1); and the same argument works for all even \( n > 2 \). However, we shall not need to know this.

In the following theorem all the statements about (2) can be trivially translated into statements about (1).

**Theorem 1.** Suppose that \( n = 6 \) and that \( f(X, Y) \) in (2) has the form
\[
f(X, Y) = f_4(X, Y) f_2(X, Y) \quad (7)
\]
where \( f_4, f_2 \) are defined over \( k \) and have degrees 4, 2 respectively. Assume also that \( f(X, Y) \) has no repeated factor. If there is an \( a \) which is soluble in every completion of \( k \) then that is soluble in \( k \); and if this holds for some \( a \) then (2) contains a Zariski dense set of points defined over \( k \).

**Proof.** We first rewrite the equations for \( \mathcal{V}^{(r)} \) in a form which makes better use of the decomposition (7). We can suppose that the linear factors of \( f_4 \) are the \( X + \lambda_i Y \) with \( i = 1, 2, 3, 4 \). The system (6) is equivalent to (3); but instead of (5) we now make the substitution
\[
 u_i = \lambda_i^3 \xi_0 + \ldots + \xi_3, \quad v_i = \lambda_i^3 \eta_0 + \ldots + \eta_3 \quad (i = 1, 2, 3, 4),
\]
\[
 u_i = \lambda_i \xi_4 + \xi_5, \quad v_i = \lambda_i \eta_4 + \eta_5 \quad (i = 5, 6)
\]
in (3). Correspondingly we replace (6) by
\[
 U_i^2 - cV_i^2 = \alpha_i^{(r)} (X + \lambda_i Y) \quad (i = 1, 2, 3, 4), \quad (8)
\]
\[
 (\lambda_i X_4 + X_5)^2 - c(\lambda_i Y_4 + Y_5)^2 = \alpha_i^{(r)} (X + \lambda_i Y) \quad (i = 5, 6), \quad (9)
\]
where we have written
\[
 U_i = \lambda_i^3 X_0 + \ldots + X_3, \quad V_i = \lambda_i^3 Y_0 + \ldots + Y_3 \quad (i = 1, 2, 3, 4).
\]
By eliminating \( X, Y \) between the four equations (8), we obtain two homogeneous quadratic equations in the eight variables \( U_i, V_i \); we treat these as
defining a projective variety $X_1 \subset \mathbb{P}^7$. The $U_i, V_i$ are not defined over $k$, but it is clear how $\text{Gal}(\overline{k}/k)$ acts on them.

We can now outline the proof of the theorem. It falls naturally into three steps.

(i) $X_1$ contains a large enough supply of lines defined over $k$.

(ii) We can choose a Zariski dense set of lines each of whose inverse images in $Y^{(r)}$ is everywhere locally soluble.

(iii) $Y^{(r)}$ contains a Zariski dense set of points defined over $k$.

The map $Y^{(r)} \to Y$ then gives the theorem.

By hypothesis, $X_1$ has points in every completion of $k$; hence as in [2], Theorem A, there is a point $P_0$ in $X_1(k)$, and we can take $P_0$ to be in general position on $X_1$. Indeed, we have weak approximation on $X_1$ because $X_1$ contains two conjugate $\mathbb{P}^3$ given by

$$X_i = \gamma Y_i = 0 \quad (i = 1, 2, 3, 4)$$

for either choice of sign, and these have no common point. To a general $k$-point $P$ of $X_1$ we can in an infinity of ways find a $k$-plane which contains $P_0$ and $P$ and which meets both these $\mathbb{P}^3$; for we need only choose a $k$-point $P'$ on $PP_0$ and note that since $P'$ does not lie on either $\mathbb{P}^3$ there is a unique transversal from $P'$ to the two $\mathbb{P}^3$. Conversely, a general $k$-plane through $P_0$ which meets both these $\mathbb{P}^3$ will meet $X_1$ in just one more point, which must therefore be defined over $k$. In this way we obtain a map $\mathbb{P}^6(k) \to X_1(k)$ which is surjective, and this implies weak approximation.

Now let $\Lambda_0$, which is a $\mathbb{P}^5$, be the tangent space to $X_1$ at $P_0$, and write $X_2 = X_1 \cap \Lambda_0$, so that $X_2$ is a cone whose vertex is $P_0$ and whose base $X_3$ is a Del Pezzo surface of degree 4. (The fact that there are 16 lines on a non-singular Del Pezzo surface, and the incidence relations between them, can be read off from [4], Theorem 26.2.) We can give a rather explicit description of $X_3$, and in particular we can identify the 16 lines on it, which turn out to be distinct. Drawing on Cayley’s exhaustive classification of singular cubic surfaces, a sufficiently erudite reader can derive a painless proof that $X_3$ is actually nonsingular. (What we actually use is the much weaker statement that $X_3$ is absolutely irreducible and not a cone, which is not hard to verify.) For $X_2$ contains the line which is the intersection of

$$U_i - \epsilon_i \gamma V_i = 0 \quad (i = 1, 2, 3, 4)$$  \hspace{1cm} (10)
with $\Lambda_0$, where each $\epsilon_i$ is $\pm 1$. (This intersection is proper because $P_0$ is in general position.) We denote this line by $L^* (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ and its projection onto $X_3$ by $L (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$. The latter clearly meets the four lines which are obtained by changing just one sign, because this already happens for the corresponding lines in $X_2$; so by symmetry the fifth line which it meets must be obtained by changing all four signs. This can be checked directly; for if we temporarily drop the notation of (3) and write

$$P_0 = (u_1, v_1, u_2, v_2, u_3, v_3, u_4)$$ in $X_0 \subset \mathbb{P}^7$,

then the join of the two points $(\epsilon_1 c v_1 \pm \gamma u_1, \epsilon_1 u_1 \pm \gamma v_1, \ldots)$ passes through $P_0$, and each point lies on the corresponding $L^* (\pm \epsilon_1, \pm \epsilon_2, \pm \epsilon_3, \pm \epsilon_4)$. Since

$$u_i (\epsilon_i c v_i \pm \gamma u_i) - c v_i (\epsilon_i u_i \pm \gamma v_i) = \pm \gamma (u_i^2 - c v_i^2)$$

and the equations for $X_1$ are given by the vanishing of linear combinations of the $u_i^2 - c v_i^2$, these two points also lie on $\Lambda_0$. The point

$$P_1 = (\epsilon_1 c v_1, \epsilon_1 u_1, \epsilon_2 c v_2, \epsilon_2 u_2, \epsilon_3 c v_3, \epsilon_3 u_3, \epsilon_4 c v_4, \epsilon_4 u_4),$$

lies on the join of these two points; $P_1$ is distinct from $P_0$ unless $P_0$ lies on the $\mathbb{P}^3$ given by (10) or the $\mathbb{P}^3$ derived from it by changing the sign of $\gamma$. Because $P_0$ is in general position, we can assume that neither of these happens. Now a straightforward calculation, using the fact that we can describe $X_1$ by equations which express $U_1^2 - c V_1^2$ and $U_2^2 - c V_2^2$ as linear combinations of $U_3^2 - c V_3^2$ and $U_4^2 - c V_4^2$, shows that $P_1$ is nonsingular on $X_2$ unless $P_0$ lies on one of 12 lines, a typical one of which is given by

$$U_1 = V_1 = U_2 = V_2 = 0, U_3 = \epsilon_3 \gamma V_3, U_4 = -\epsilon_4 \gamma V_4.$$ Under the same condition, the point induced on $X_3$ is nonsingular.

The lines $L(+++++)$ and $L(----)$ are defined over $k(\gamma)$ and conjugate over $k$; thus their intersection is defined over $k$ and $X_3$ does contain a point defined over $k$. Moreover the $u_i^2 - c v_i^2$ cannot all vanish because $\gamma$ is not in any $k(\lambda_i)$; so $P_1$ is nonsingular on $X_2$ and $k$-points are Zariski dense on $X_3$. (See [4], Theorems 30.1 and 29.4.) Henceforth $P_2 \neq P_0$ will always denote a point on $X_2$ defined over $k$ and $P_3$ will denote the corresponding point on $X_3$.

Once we have chosen $P_2$, the general point of the line $P_0 P_2$ is given by setting the $X_i, Y_i$ for $i = 0, 1, 2, 3$ equal to linear forms in $Z_1, Z_2$; and we can suppose that $P_0$ corresponds to $(1,0)$ and $P_2$ to $(0,1)$. The equations for $X_1$ are then satisfied identically, and (8) expresses $X, Y$ as quadratic forms in $Z_1, Z_2$. There remain the equations (9), which now take the form

$$(\lambda_4 X_4 + X_5)^2 - c (\lambda_4 Y_4 + Y_5)^2 = \phi_i (Z_1, Z_2) \quad (i = 5, 6) \quad (11)$$

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for certain quadratic forms $\phi_5, \phi_6$. In view of the remarks in the previous paragraph we can certainly assume that $\phi_5, \phi_6$ are linearly independent and each has rank 2. We need to check that we can choose the line $P_0P_2$ so that the system (11) is everywhere locally soluble. This is of course the crucial step in the proof of the Theorem; but in order not to disrupt the flow of the argument, we postpone the proof of it and of an auxiliary result to Lemma 3 below. Given this, we would like to conclude the argument by appealing to Theorem A of [2]; but unfortunately we are in the exceptional case (E5) of that theorem. Some discussion of this exceptional case can already be found in the literature (for example in [2]); but it is not clear that any published result meets our needs. We therefore proceed as follows.

Suppose first that $\lambda_5, \lambda_6$ are in $k$ and write

$$U_i = \lambda_i X_4 + X_5, \quad V_i = \lambda_i Y_4 + Y_5 \quad (i = 5, 6).$$

The equation (11) for $i = 5$ is $U_5^2 - cV_5^2 = \phi_5(Z_1, Z_2)$, which is everywhere locally soluble, and therefore soluble by the Hasse-Minkowski theorem. Its general solution is given by homogeneous quadratic forms in three variables $W_1, W_2, W_3$. The equation (11) with $i = 6$ now reduces to

$$U_6^2 - cV_6^2 = g(W_1, W_2, W_3)$$

where $g$ is quartic. This is everywhere locally soluble; so all we have to do is to set $W_3$ equal to $e_1 W_1 + e_2 W_2$ where $e_1, e_2$ are integers in $k$ such that

$$U_6^2 - cV_6^2 = g(W_1, W_2, e_1 W_1 + e_2 W_2)$$

is everywhere locally soluble and has no Brauer-Manin obstruction. This is not difficult. Let $S$ consist of the places in $k$ which are either infinite or divide $6c$ or either of the polynomials $g(W_1, 0, W_3)$ or $g(0, W_2, W_3)$; by means of a linear transformation on the $W_i$ if necessary, we can assume that neither of these expressions vanishes identically and hence $S$ is finite. Solubility of (13) at the places in $S$ can be ensured by local conditions on $e_1, e_2$. Choose $e_1$ to satisfy all these local conditions and also $g(1, 0, e_1) \neq 0$. For the local solubility of (13) all we now have to consider are the primes in $S$ and the primes $p$ which divide $g(1, 0, e_1)$. For the former, we need only impose local conditions on $e_2$; for the latter it is enough to ensure that $p \nmid g(0, 1, e_2)$, which we can do because $\text{Norm } p > 3$. Finally, $g(W_1, W_2, W_3)$ is the product of two absolutely irreducible quadratic forms defined over $k$ which correspond to the linear factors of $\phi_6$; so it is irreducible over $k$ by Lemma 3. By Hilbert irreducibility we can ensure that $g(W_1, W_2, e_1 W_1 + e_2 W_2)$ is irreducible over $k$; so the Châtelet equation (13) is soluble, by Theorem B of [2].

If instead $\lambda_5, \lambda_6$ are not in $k$, it follows from Lemma 3 and the linear independence of $\phi_5$ and $\phi_6$ that $\phi_5 \phi_6$ is irreducible over $k$. Hence (11) is
soluble in \( k \) by Theorem 12.1 of [2]. The reader can easily check that the solutions thus constructed are in general position, and therefore Zariski dense on (2).

All that remains to do is to prove the following:

**LEMMA 3.** If \( \varphi(\gamma) \) is everywhere locally soluble there are lines \( P_0P_2 \) such that (11) is everywhere locally soluble and \( \phi_i(Z_1, Z_2) \) is irreducible over \( k(\lambda_i) \) for \( i = 5, 6 \).

**Proof.** We note first that in general \( \phi_i \) is irreducible over \( k(\lambda_i) \). For if we take \( P_2 \) to be \( P_1 \) and \( P_0, P_1 \) to have \( Z \)-coordinates \((1, 0), (0, 1)\) respectively, each \( U_i^2 - cV_i^2 \) with \( i = 1, 2, 3, 4 \) is a multiple of \( Z_i^2 - cZ_i^2 \); hence the same is true of \( X \) and \( Y \), and therefore of \( \phi_5 \) and \( \phi_6 \). The general assertion now follows from Hilbert's Irreducibility Theorem.

The main complication in the proof of this Lemma is that we cannot assume weak approximation on \( X_3 \); indeed weak approximation is probably not even true, since the Brauer group of \( X_3 \) is non-trivial. (See [5].) Let \( S_1 \) be a finite set of places in \( k \) containing the infinite places, all small primes and all primes dividing \( 2c \), any \( a_m \), the discriminant of \( f \) or any of the \( \alpha_i^{(r)} \). Then we can choose \( P_0 \) to be in the image of \( \mathcal{Y}(r) \) under the map \( \mathcal{Y}(r) \rightarrow X_1 \) for each \( v \) in \( S_1 \), by weak approximation on \( X_1 \). Denote by \( u_i, v_i, x, y \) the values of \( U_i, V_i, X, Y \) at \( P_0 \); these values depend on the particular coordinate representation of \( P_0 \) which we choose, so that we can still multiply the \( u_i, v_i \) by an arbitrary \( \mu \neq 0 \) in \( k \) and multiply \( x, y \) by \( \mu^2 \). We can therefore ensure that \( x, y \) are integers and that the ideal \( (x, y) \) is not divisible by the square of any prime ideal outside \( S_1 \). We then re-choose the \( u_i, v_i \) for \( i = 1, 2, 3, 4 \) to satisfy (8) and be integral, which we can do by the remark immediately after the proof of Lemma 2. This of course alters \( P_0 \), but since it leaves \( x, y \) unchanged the equations (9) remain locally soluble at every place in \( S_1 \). Because the old \( P_0 \) was in general position on \( X_1 \), we can assume that the right hand sides of the two equations (9) do not vanish at \( P_0 \).

We do not know the quadratic forms \( \phi_5 \) and \( \phi_6 \) until we have chosen \( P_2 \). But the values of \( \phi_5(1, 0) \) and \( \phi_6(1, 0) \) as elements of \( k^*/k^{*2} \) only depend on \( P_0 \), for they are simply the values of the right hand sides of the two equations (9) at \( P_0 \). We can therefore properly involve these values in the argument in advance of the choice of \( P_2 \). We now have local solubility of (11) for \( i = 5, 6 \) for \( Z_2 = 0 \) except perhaps at primes which are not in \( S_1 \) but which divide \( \phi_5(1, 0)\phi_6(1, 0) \); let \( S_2 \) be the finite set of such primes. We can delete from \( S_2 \) any primes for which \( c \) is a quadratic residue, for (11) is
certainly soluble at such primes. To prove the Lemma, we need only show that we can choose \( P_2 \) so that no prime \( p \) in \( S_2 \) divides \( \phi_5(0,1) \phi_6(0,1) \).

Now let \( p \) be in \( S_2 \) and \( \mathfrak{P} \) be any prime ideal in \( k(\lambda_1, \ldots, \lambda_4, \gamma) \) which divides \( p \), and use a tilde to denote reduction mod \( \mathfrak{P} \); we have \( \mathfrak{P} \parallel p \) because all the primes which ramify lie in \( S_1 \). The two \( P^3 \) given by \( U_i \pm \tilde{\gamma} V_i = 0 \) \( (i = 1, 2, 3, 4) \) are also given by \( X_i \pm \gamma Y_i = 0 \) \( (i = 1, 2, 3, 4) \); so if \( \tilde{P}_0 \) lies on either of them then \( \tilde{\gamma} \) would be equal to the reduction mod \( \mathfrak{P} \) of the value of \( \mp X_i / Y_i \) at \( P_0 \). Since the latter is an element of \( k \), this would mean that \( c \) would be a quadratic residue mod \( p \) — a case which we have already ruled out. Again, if for example \( \tilde{u}_1 = \tilde{v}_1 = \tilde{u}_2 = \tilde{v}_2 = 0 \) then \( x, y \) would be divisible by \( \mathfrak{P}^2 \) and hence by \( p^2 \); and this too we have ruled out. The calculations following (10) now show that \( P_1 \) is nonsingular on \( \tilde{X}_2 \), where \( P_1 \) is as in those calculations.

At most one pair of \( \tilde{\bar{u}}_i, \tilde{\bar{v}}_i \) vanish; if there is such a pair, we can suppose it is given by \( i = 4 \). The equations for \( \tilde{X}_2 \) are

\[
U_1^2 - \bar{c} V_1^2 = \text{homogeneous quadratic form in } U_3, V_3, U_4, V_4, \tag{14}
\]

\[
U_1 \tilde{u}_1 - \bar{c} V_1 \tilde{v}_1 = \text{linear form in } U_3, V_3, U_4, V_4,
\]

and two similar ones involving \( U_2 \) and \( V_2 \). The equation (14) is equivalent to the vanishing of a quadratic form of rank 6, so it cannot have a hyperplane section which is not absolutely irreducible; and it now follows easily that \( \tilde{X}_2 \) is absolutely irreducible. The projection from \( \tilde{X}_2 \) to the \( P^3 \) with coordinates \( U_3, V_3, U_4, V_4 \) is generically onto. Hence there are at most \( O(q^2) \) points in \( \tilde{X}_2(\mathbb{F}_q) \) for which the right hand side of (9) vanishes for \( i = 5 \) or \( i = 6 \). The implied constant here, like \( \lambda \) below, is absolute because it depends only on the degrees of the various maps and varieties involved. Now let \( P \) be the point on \( X_3 \) corresponding to \( P_1 \) on \( X_2 \); thus \( P \) is the intersection of two lines on \( X_3 \). We have already shown that \( P \) is nonsingular for all the \( p \) which still concern us. The construction in the proof of [4], Theorem 30.1 specifies a non-constant map \( \psi : P^1 \to X_3 \); and the reduction mod \( p \) of the image of \( \psi \) is obtained by carrying out the corresponding construction using \( \psi \) and \( \tilde{X}_3 \), so this image has good reduction. Hence there is a point \( Q \) in the image of \( \psi \), defined over \( k \) and such that \( \tilde{Q} \) is nonsingular on \( \tilde{X}_3 \) and does not lie on any of the lines of \( \tilde{X}_3 \). Repeating this process using this time the construction in the proof of [4], Theorem 29.4, we obtain a map \( P^2 \to X_3 \) which has good reduction mod \( p \) for all relevant \( p \). This lifts back to a map \( P^3 \to X_2 \) which is generically onto and has good reduction mod \( p \) for all relevant \( p \). Hence there exists an absolute constant \( \lambda > 0 \) such that \( \tilde{X}_2 \) has at least \( \lambda q^3 \) points which can be lifted back to points of \( \tilde{X}_6(\mathbb{F}_q) \). Provided that \( q \) is large enough, which we ensure by putting all small primes into \( S_1 \),
we can choose such a point \( \tilde{P}_2 \) for which the right hand sides of (9) for \( i = 5 \) and \( i = 6 \), reduced mod\( p \), do not vanish. We lift this \( \tilde{P}_2 \) back to \( \tilde{Q} \) on \( \mathcal{X}_0 \). But we have weak approximation on \( \mathcal{X}_0 \). Hence we can choose a rational point \( Q \) on \( \mathcal{X}_0 \) whose reduction mod\( p \) is \( \tilde{Q} \) for each of the finitely many primes in \( S_2 \). If we choose \( P_2 = \phi(Q) \) this will satisfy all our conditions.

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Bibliography


