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Covariant star-products


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1. Introduction

$\star$-products were defined in [1] by Flato, Fronsdal, Lichnerowicz as a tool for quantizing a classical system, described with a symplectic manifold $(M, \omega)$. Roughly speaking, a $\star$-product is a (formal) deformation of the associative algebra $C^\infty(M)$ provided with usual (pointwise) product starting with the Poisson bracket. The quantum structure is then the deformed structure on the unchanged space of observables.

Each quantization procedure, when applied on a coadjoint orbit $M$ of a Lie group $G$, gives some way to build up unitary irreducible representations of $G$. To use $\star$-products for such a purpose, we need in fact a particular property, the covariance of the $\star$-product:

$$[\tilde{X}, \tilde{Y}]_\star = \{\tilde{X}, \tilde{Y}\} = [\tilde{X}, \tilde{Y}],$$

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if $\tilde{X}$ is, for each $X$ in the Lie algebra $\mathfrak{g}$ of $G$, the function on $M$ defined by

$$x \mapsto \tilde{X}(x) = \langle x, X \rangle.$$  

(See [2] for a discussion on invariance and covariance properties for $*$-products on a coadjoint orbit.)

In this paper, we recall first the theorem of existence of $*$-products on a symplectic manifold. This theorem is due to P. Lecomte and M. de Wilde [3]. Some recent new proofs were given by Maeda, Omori and Yoshioka [4] and Lecomte and de Wilde [5]. We expose here that last proof in a slightly different way, which is direct and totally elementary: we build a $*$-product by gluing together local $*$-products defined on domains of a chart of $M$. That proof follows the idea of Vey, Lichnerowicz, Neroslavsky and Vlassov ([6], [7]) and, of course, Maeda, Onori and Yoshioka. In these approaches, the obstruction to construct $*$-product lies in the third cohomology group $H^3(M)$ of the manifold $M$. Lecomte and de Wilde defined formal deformation of the Lie algebra $(\mathfrak{g}, \{ \cdot, \cdot \})$, for such a deformation, the obstruction is in the group $H^3(\mathfrak{g}, M)$ for the adjoint action, which contains strictly $H^3(M)$. Let us finally mention the construction of Maslov and Karasev [8] who found an obstruction in $H^2(M)$ to construct simultaneously a deformation and a representation of the deformed structure on $C^\infty(M)$. Lecomte and de Wilde proved that all these obstructions can be surrounded, with the use of local conformal vector fields on $M$. We follow here that classical proof, using only local computations and Čech calculus.

Then we use this proof in the case of a coadjoint orbit in the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$. More precisely, we consider a point $z_0$ in $\mathfrak{g}^*$ and suppose there exists in $z_0$ a real polarization. Under this assumption, we prove the existence of a covariant $*$-product on the coadjoint orbit of $z_0$, endowed with its canonical symplectic structure.

2. Existence of $*$-products on a symplectic manifold

Let $(M, \omega)$ be a symplectic manifold. We denote by $\{ \cdot, \cdot \}$ the Poisson bracket defined on $C^\infty(M)$ by the usual relations:

$$\{u, v\} = X_u v \quad \text{if} \quad i_{X_u} \omega = -du.$$
A \(*\)-product is by definition a formal deformation in the sense of Gerstenhaber [9] of the associative algebra $C^\infty(M)$, i.e. a bilinear map:

$$C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)[[\nu]], \quad (u, v) \mapsto v \ast v = \sum_{r \geq 0} \nu^r C_r(u, v),$$

where $C^\infty(M)[[\nu]]$ is the space of formal power series in the variable $\nu$ with coefficients in $C^\infty(M)$, such that each $C_r$ is a bidifferential operator and:

1. $C_0(u, v) = uv$, $C_1(u, v) = \{u, v\}$,
2. $C_r(u, v) = (-1)^r C_r(v, u)$,
3. $C_r(1, u) = 0$, $\forall r > 0$,
4. $\sum_{r+s=t} C_r(C_s(u, v), w) = \sum_{r+s=t} C_r(u, C_s(u, w)), \forall t \geq 0$.

With these properties, $\ast$ defines an associative structure on $C^\infty(M)[[\nu]]$, of whom unity is 1 and:

$$[u, v]_\ast = \sum_{r \geq 0} \nu^{2r} C_{2r+1}(u, v) = \frac{1}{2\nu} (u \ast v - v \ast u)$$

is a Lie bracket (it satisfies Jacobi identity) and a formal deformation of the Poisson bracket.

On a symplectic vector space $\mathbb{R}^{2n}$ and thus on any domain $U$ of a canonical chart in $M$; there exists $\ast$-products, for instance the Moyal $\ast$-product [1].

**Theorem [3].** — On each simplectic manifold $(M, \omega)$, there exists a $\ast$-product.

**Proof.** — Let us first choose a locally finite covering $(U_\alpha)_{\alpha \in A}$ of the manifold such that each $U_\alpha$ is the domain of a canonical chart on $M$ and all the intersections:

$$U_{\alpha_1 \ldots \alpha_n} = U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}$$

are contractible. We fix a total ordering $\leq$ on $A$ and a partition of the unity $\psi_\alpha$ subordinated to $(U_\alpha)_{\alpha \in A}$. If $\ast_\alpha$ is a $\ast$-product on $U_\alpha$ and $\text{Der}(\ast_\alpha)$ the space of derivation of $\ast_\alpha$, there exists a canonical linear mapping:

$$\Phi_\alpha : F(U_\alpha) = C^\infty(U_\alpha)/\{\text{constants}\} \longrightarrow \text{Der}(\ast_\alpha), \quad \Phi_\alpha([f])(v) = [f, v]_\alpha.$$
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Let us denote by \( \text{Con}_f(U_\alpha) \) the space of (conformal) vector fields \( \xi_\alpha \) on \( U_\alpha \) such that:

\[
L_{\xi_\alpha} \omega = \omega \quad \text{on} \ U_\alpha.
\]

\( \text{Con}_f(U_\alpha) \) is an affine space on \( F(U_\alpha) \).

Now we suppose, by induction on \( k \), to have, on each \( U_\alpha \), a \( * \)-product \( *_\alpha \):

\[
u *_\alpha v = \sum_{r \geq 0} \nu^r C_{r,\alpha}(u, v),
\]

with \( C_{r,\alpha} = C_{r,\beta} \) on \( U_{\alpha\beta} \), for all \( r < 2k \) and an affine mapping \( D_\alpha \) form \( \text{Con}_f(U_\alpha) \) into \( \text{Der}(\ast_\alpha)(D_\alpha(\xi_\alpha + X_f)) = D_\alpha(\xi_\alpha) + \Phi_\alpha([f]) \) such that:

\[
D_\alpha(\xi_\alpha) = \nu \partial_\nu + L_{\xi_\alpha} + \sum_{r > 0} \nu^{2r} D_{2r}^\alpha(\xi_\alpha),
\]

the \( D_{2r}^\alpha(\xi_\alpha) \) being differential operators, vanishing on constants. Of course, these assumptions hold for \( k = 1 \).

Now it is well known ([6], [7]) that for each \( \alpha < \beta \), we can find a differential operator vanishing on constants \( H_{\alpha\beta} \) such that, up to order \( 2k + 2 \),

\[
u' *_\alpha v = \exp \nu^{2k} H_{\alpha\beta}(\exp -\nu^{2k} H_{\alpha\beta} u *_\alpha \exp -\nu^{2k} H_{\alpha\beta} v)
\]

coincide with \( u *_\beta v \). Thus:

\[
(D_\alpha - D_\beta)(\xi_\alpha) =
\]

\[
= \sum_{r=0}^{k-1} \nu^{2r} \Phi_\alpha([f_{\alpha\beta}^{2r}]) + \nu^{2k} \left( L_{\xi_\alpha} H_{\alpha\beta} + 2k H_{\alpha\beta} + \Phi_\alpha([g_{\alpha\beta}(\xi_\alpha)]) \right)
\]

here \( [f_{\alpha\beta}^{2r}] \) in \( F(U_{\alpha\beta}) \) do not depend of \( \xi_\alpha \) while:

\[
[g_{\alpha\beta}(\xi_\alpha + X_f)] = [g_{\alpha\beta}(\xi_\alpha) + H_{\alpha\beta} f].
\]

For each \( \alpha < \beta \), we choose a vector field \( \xi_{\alpha\beta} \) in \( \text{Con}_f(U_{\alpha\beta}) \), a \( C^\infty \) function \( g_{\alpha\beta}(\xi_{\alpha\beta}) \) and put for any \([f] \) in \( F(U_{\alpha\beta})\):

\[
g_{\alpha\beta}(\xi_{\alpha\beta} + X_f) = g_{\alpha\beta}(\xi_{\alpha\beta}) + H_{\alpha\beta} f,
\]

\[
f_{2r}^{\alpha\alpha} = 0, \quad f_{2r}^{\beta\alpha} = -f_{2r}^{\alpha\beta},
\]

\[
g_{\alpha\alpha} = 0, \quad g_{\beta\alpha} = -g_{\alpha\beta}.
\]
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Now, on $U_{\alpha\beta\gamma}$ ($\alpha < \beta < \gamma$), $H_{\alpha\beta\gamma}$ ($= H_{\alpha\beta} + H_{\beta\gamma} + H_{\gamma\alpha}$) can be written as $\Phi([h_{\alpha\beta\gamma}])$. The problem is to choose simultaneously the $C^\infty$ functions $h_{\alpha\beta\gamma}$. As in [3], we choose the unique $C^\infty$ solution of all the equations:

$$L_{\xi_{\alpha\beta\gamma}} h_{\alpha\beta\gamma} + (2k - 1) h_{\alpha\beta\gamma} = -g_{\alpha\beta\gamma}(\xi_{\alpha\beta\gamma}),$$

for each $\xi_{\alpha\beta\gamma}$ in $\text{Con } f(U_{\alpha\beta\gamma})$. $h_{\alpha\beta\gamma}$ is totally antisymmetric in $\alpha$, $\beta$, $\gamma$ and $h_{\alpha\beta\gamma} - h_{\alpha\beta\delta} + h_{\alpha\gamma\delta} - h_{\beta\gamma\delta}$ vanishes on $U_{\alpha\beta\gamma\delta}$. We define then:

$$s_{\alpha\beta} = \sum_\gamma h_{\alpha\beta\gamma} \psi_\gamma \quad \text{in } C^\infty(U_{\alpha\beta}),$$

$$G_{\alpha\beta} = H_{\alpha\beta} - \{s_{\alpha\beta}, \cdot\},$$

$$K_\alpha = \sum_\beta G_{\alpha\beta} \psi_\beta.$$ 

$G_{\alpha\beta\gamma}$ vanishes on $C^\infty(U_{\alpha\beta\gamma})$, $K_\alpha$ is well defined and, for each $\alpha$,

$$u \star_\alpha' v = \exp \nu^{2k} K_\alpha(\exp -\nu^{2k} K_\alpha u \star_\alpha \exp -\nu^{2k} K_\alpha v),$$

$$D'_\alpha(\xi_\alpha) = \exp \nu^{2k} K_\alpha \circ D_\alpha(\xi_\alpha) \circ \exp -\nu^{2k} K_\alpha$$

satisfy the induction hypothesis at order $2k + 2$.

If the second Čech cohomology group of $M$ vanishes, there exists a global conformal vector field $\xi$ on $M$ and a global derivation $D'(\xi)$ of the $\ast$-product therefore we refund here the proof of [11]. In the general case, our proof by building directly a $\ast$-product does not need the theorem of [6] which allows to construct $\ast$-product, starting with particular deformation of the Poisson bracket.

3. Parametrization of coadjoint orbits

Let $G$ be a connected and simply connected Lie group, $g$ its Lie algebra and $g^*$ the dual of $g$. $G$ acts on $g^*$ by the coadjoint action, denoted here by:

$$\langle g \cdot x, X \rangle = \langle x, \text{Ad} g^{-1}(X) \rangle, \forall \ X \in g, \forall \ x \in g^*, \forall \ g \in G.$$

Let $x_0$ be a point of $g^*$ and $M$ its coadjoint orbit $G \cdot x_0$, endowed with the canonical 2-form:

$$\omega_\pi(X^-, Y^-) = \langle x, [X, Y] \rangle (= B_\pi(X, Y)), \ \forall \ X, Y \in g,$$

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here $X^-$ is the vector field defined on $M$ by:

$$X^- f(x) = \frac{d}{dt} f(\exp(-tX \cdot x))|_{t=0}.$$

From now on, we suppose there exists a real polarization $\mathfrak{h}$ in $x_0$. This means $\mathfrak{h}$ is a maximal isotropic subspace in $\mathfrak{g}$ for the bilinear form $B_{x_0}$, $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and, if $G(x_0)$ is the stabilizer of $x_0$, $\text{Ad} G_{x_0}(\mathfrak{h}) \subset \mathfrak{h}$.

If $H_0$ is the analytic subgroup of $G$, with Lie algebra $\mathfrak{h}$, we denote by $H$ the subgroup $G(x_0)H_0$ of $G$. Then $M$ becomes a fibre bundle over $G/H$:

$$\pi : M = G/G(x_0) \rightarrow G/H.$$ 

In this part, we recall the results of Pedersen [10].

Let $\mathcal{E}^0$ be the subspace $\pi_*\left(C^\infty(G/H)\right)$. It is an abelian subalgebra of $\left(C^\infty(M), \{\cdot, \cdot\}\right)$. Let $\mathcal{E}^1$ be the algebra:

$$\mathcal{E}^1 = \{u \in C^\infty(M) \text{ such that } \{u, \mathcal{E}^0\} \subset \mathcal{E}^0\}.$$ 

For each open subset $V$ in $G/H$, we define $\mathcal{E}^0(V)$ as $\pi_*\left(C^\infty(V)\right)$ and $\mathcal{E}^1(V)$ as

$$\left\{ u \in C^\infty(\pi^{-1}(V)) \text{ such that } \{u, \mathcal{E}^0(V)\} \subset \mathcal{E}^0(V) \right\}.$$ 

The space $\mathcal{E}^1$ is sometimes called the space of quantizable functions. It is easy to verify that the functions $\overline{X}$, for $X$ in $\mathfrak{g}$ are in $\mathcal{E}^1$. Now let $m$ be a supplementary space of $\mathfrak{h}$ in $\mathfrak{g}$ and $V$ be a sufficiently small neighborhood in $G/H$ such that $m$ is a supplementary space of $\text{Ad} g\mathfrak{h}$ for each $g$ in $G$ such that $g \cdot x_0$ belongs to $V$. Pedersen proved that, if $(X_1, \ldots, X_k)$ is a basis of $m$, then, on $\pi^{-1}(V)$, we can write each function $u$ of $\mathcal{E}^1(V)$ in the form:

$$u|_{\pi^{-1}(V)} = \left(\sum_{i=1}^n \alpha_i \overline{X_i} + \alpha_0\right)|_{\pi^{-1}(V)},$$

where the $\alpha_i$ are in $\mathcal{E}_0(V)$. Moreover, the $\alpha_i$ are uniquely determined on $\pi^{-1}(V)$ by that relation. Now we define a "local" induced representation. First there exists a local character $\chi$ of $H$: let $\mathcal{V}$ be a neighborhood of $0$ in $\mathfrak{h}$ such that $\exp$ is a diffeomorphism on $\mathcal{V}$, we put:

$$\chi(\exp X) = e^{i(x_0, X)} \text{ if } X \in \mathcal{V}.$$
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Then, if $\mathcal{U}$ is a neighborhood of 0 in $m$ and $\mathcal{V}$, $\mathcal{U}$ sufficiently small, the neighborhood $\mathcal{G} = \exp(\mathcal{U}) \exp(\mathcal{V})$ of unity in $G$ is diffeomorphic to $\mathcal{U} \times \mathcal{V}$, we choose $V$ to be $\exp(\mathcal{U})H$ and define the local representation $(E, \rho)$ by:

$$E = \{ \phi \in C^\infty \text{ such that } \phi(xh) = \chi(h)^{-1}\phi(x) \text{ if } h \in \exp(\mathcal{V}), x, xh \in \mathcal{G} \}$$

and

$$(\rho(a)\phi)(x) = \phi(a^{-1}x) \text{ if } a, x, a^{-1}x \in \mathcal{G}.$$ 

Of course, we can identify $E$ with $C^\infty(V)$ by putting, for each $f$ in $C^\infty(V)$,$$
\phi(xh) = \chi(h)^{-1}f(xH) \text{ if } x \in \exp(\mathcal{U}), h \in \exp(\mathcal{V}).$$

Generally, $\rho$ cannot be extended to a representation of $G$. But infinitesimally,

$$d\rho(X)\phi(x) = \frac{d}{dt} (\rho(\exp tX)\phi)(x)|_{t=0}$$

is a representation of $g$ on the space $C^\infty(V)$. Moreover, by construction, the $d\rho(X)$ are differential operators of order 1 on $V$, we write:

$$d\rho(X) \in \text{Diff}^1(V).$$

Finally, we call $U$ the set $\pi^{-1}(V)$ and define a map $\delta$ from $E^1(V)$ to $\text{Diff}^1(V)$ by:

$$\delta(u) = \delta \left( \sum_{i=1}^{k} \tilde{X}_i + \alpha_0 \right) = \sum_{i=1}^{k} \alpha_i d\rho(X_i) + a_0.$$

**Theorem [10].** — $\delta$ is an isomorphism of Lie algebras between $E^1(V)$ and $\text{Diff}^1(V)$.

The proof of this in [10], indeed, it is a direct consequence of the fact that $d\rho$ is a representation. Now, we define canonical coordinates on $U$: let $(y_1, \ldots, y_k)$ be a coordinate system on $V$ in $G/H$, we define:

$$q_i = \pi_* y_i, \quad p_i = \delta^{-1}(\partial_{y_i}).$$

$(p_i, q_i)$ is a canonical system of coordinates on $U$, the $q_i$ belong to $E^0(V)$ and the $p_i$ to $E^1(V)$. Then by construction, we have the following theorem.
**Theorem.** — On the intersection of two such chart \( U \) and \( U' \), the coordinates satisfy:

\[
q'_i = Q_i(q), \quad p'_i = \sum_{j=1}^{k} \alpha_{ij}(q)p_j + \alpha_{i0}(q).
\]

Endowed with that atlas, \( M \) is an open subset of an affine bundle \( L \) over \( G/H \), whose transition functions are defined by the relations \((*)\).

**Remarks**

The functions \( \tilde{X} \) being in \( \mathcal{E}^1 \), they have the following form in our coordinate system:

\[
\tilde{X} = \sum_{i=1}^{k} \alpha_i(q)p_i + \alpha_0(q).
\]

If \( \mathfrak{h} \) satisfies the Pukanszky condition, then \( M \) is exactly the bundle \( L \).

4. **Construction of covariant \( \star \)-product**

We consider now our orbit \( M \) as an open submanifold of the fibre bundle \( \pi: L \rightarrow G/H \). \( L \) is canonically polarized with the tangent spaces \( T_xL_x \) of its fibres \( L_x \). Then we build up a \( \star \)-product on \( L \) as in the second part. We still denote by \( \mathcal{E}^0 \) (resp. \( \mathcal{E}^0(V) \)) the space \( \pi_*(C^\infty(G/H)) \) (resp. \( \pi_*(C^\infty(V)) \)). Moreover, we choose our canonical charts with domain \( \pi^{-1}(V_\alpha) \) where \( V_\alpha \) is one of the local domains of chart defined in the third part and the partition of unity \( \psi_\alpha \) subordinated to \( U_\alpha \) in \( \mathcal{E}^0 \). Finally, we add to our induction hypothesis that, for each \( \alpha \), \( C_{r,\alpha} \) is vanishes on \( \mathcal{E}^1(V_\alpha) \) for \( r > 2 \) and \( C_{2,\alpha}(\mathcal{E}^1(V_\alpha), \mathcal{E}^1(V_\alpha)) \subset \mathcal{E}^0(V_\alpha) \).

If we choose the \((p,q)\) coordinates of the preceding part on our neighborhood \( U_\alpha \) and begin with Moyal product with these coordinates, then for \( k = 1 \), the induction hypothesis holds.

Now, it is not very difficult to choose \( H_{\alpha\beta} \) such that \( H_{\alpha\beta} \) vanishes on \( \mathcal{E}^1(V_{\alpha\beta}) \) (we choose first \( H'_{\alpha\beta} \) such that \( H'_{\alpha\beta} \) vanishes on \( \mathcal{E}^1(V_{\alpha\beta}) \)), then we prove the existence of a \( C^\infty \) function \( \varphi_{\alpha\beta} \) such that \( H_{\alpha\beta} = H'_{\alpha\beta} + \partial \varphi_{\alpha\beta} \) vanishes on \( \mathcal{E}^1(V_{\alpha\beta}) \)). With that choice, \( H_{\alpha\beta\gamma} \) is a Hamiltonian vector field vanishing on \( \mathcal{E}^1(V_{\alpha\beta\gamma}) \) so it is identically zero. Hence we can construct directly the family \( (K_\alpha)_{\alpha \in \Lambda^*} \).
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Now, because $K_{\alpha}$ vanishes on $\mathcal{E}^1(V_{\alpha \beta})$, our induction hypothesis is still true for $\ast_{\alpha}$. In this way, we obtain a $\ast$-product on $L$, after restriction to $M$, we have a covariant $\ast$-product on $M$, since each $\tilde{X}$ is in $\mathcal{E}^1$.

**Theorem.** — Let $x_0$ be an element in the dual $g^*$ of a Lie algebra $g$ such that there exists in $x_0$ a real polarization $\mathfrak{h}$. Then on the coadjoint orbit $M$ of $x_0$, there exists a covariant $\ast$-product.

Let us recall [2] that for each covariant $\ast$-product on $M$, there exists a representation of $G$ into the group of automorphisms of $(C^\infty(M)[[\nu]], \ast)$, which is a deformation of the geometric action of $G$ on $M$ and $C^\infty(M)$.

**References**


