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A note on the tail accuracy of the univariate saddlepoint approximation

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RÉSUMÉ. — Nous montrons que, sous certaines conditions de régularité, l'approximation de point de selle univariée (non normalisée) devient exacte à la limite lorsqu'on approche la frontière du support de la densité de probabilité. Sous ces conditions, le résultat reste stable par convolution de densités. La preuve découle des travaux récents de Balkema, Klüppelberg et Resnick (1990), et nous discutons également des relations de ces résultats avec les travaux de Daniels (1954) et Jensen (1988, 1989) sur les comportements à la frontière de l'approximation de point de selle.

ABSTRACT. — The (unnormalized) univariate saddlepoint approximation is shown to become exact in the limit as the boundary of the support of the probability density is approached, subject to certain regularity conditions. Under these conditions, the result is closed under convolution of densities. The derivation relies on recent results due to Balkema, Klüppelberg and Resnick (1990), and a discussion of the relation of those results to work by Daniels (1954) and Jensen (1988, 1989) concerning the boundary behaviour of the saddlepoint approximation, is also given.

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1. Introduction

The purpose of this note is to discuss a connection between the works of Daniels [3] and Jensen [4, 5] on the tail accuracy of the saddlepoint approximation of probability density functions and the results of a recent

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paper by Balkema, Klüppelberg and Resnick [1] (hereafter referred to as [B/K/R]) concerning the tail behaviour of convolutions of probability density functions f that are asymptotically of the form

$$f(t) \sim \gamma(t)\, e^{-\psi(t)} \quad \text{as} \quad t \to \infty \tag{1.1}$$

with $\psi(t)$ a convex function and $\gamma(t)$ "gently varying" as compared to $\psi''(t)$, in a sense that will be made precise below.

Fourier transformation and other aspects of complex analysis are essential tools in the investigations of Daniels and Jensen referred to above, whereas [B/K/R] rely on methods of convex analysis and regular variation.

Another difference is that, except for a discussion of the Gaussian autoregressive process of order 1 in Jensen [4], the papers by Daniels and Jensen are concerned entirely with independent identically distributed observations whereas [B/K/R] also provides general results concerning sums of non-identically distributed observations. However, the results of Daniels [3] and particularly Jensen [4, 5] are in other respects of a much more general nature than those of [B/K/R].

As we shall point out, it follows from results in [B/K/R] that, subject to the conditions on γ and ψ already indicated, the (unnormalized) saddlepoint approximation to the probability density $f(t)$ of (1.1) becomes exact in the limit as $t \to \infty$. (This conclusion could also be reached via theorem 3 of Jensen [5].) Combination of this with another result of [B/K/R] stating that the class of densities satisfying (1.1) is — largely speaking — closed under convolution allows one to conclude tail exactness of the saddlepoint approximation in a considerable range of cases. At the end of the paper we provide an illustration of this.

We proceed to give a more specific description of the contents of the paper.

Let $f(t)$ be a probability density function, defined and positive on an interval I that is unbounded above. The saddlepoint approximation to $f(t)$ may be expressed as

$$f^{\dagger}(t) = \frac{1}{\sqrt{2\pi}}\, K^{*\prime\prime}(t)^{1/2}\, e^{-K^{*}(t)}$$

where $K^*(t)$ denotes the convex conjugate of the cumulant function $K(\tau)$ given by $K(\tau) = \log C(\tau)$ with

$$C(\tau) = \int_I e^{\tau t} f(t)\, \mathrm{d}t\,.$$

The ratio $f^{\dagger}(t)/f(t)$ expresses the relative accuracy of the saddlepoint approximation and we are interested in the asymptotic behaviour of the relative error

$$\mathrm{RE}(t) = \left| \log \left\{ \frac{f^{\dagger}(t)}{f(t)} \right\} \right|$$

as $t \to \infty$. Specifically, we will show that if, as $t \to \infty$, the density $f(t)$ behaves asymptotically as indicated by (1.1) then $\mathrm{RE}(t) \to 0$ as $t \to \infty$. More generally it will follow that if X_0 denotes the sum $X_1 + \cdots + X_n$ of n independent, but not necessarily identically distributed, observations having densities of the form (1.1) and if $\mathrm{RE}_n(t)$ denotes the relative error of the saddlepoint approximation to the density of X_0 at a point t then, subject to a mild restriction, $\mathrm{RE}_n(t) \to 0$ for $t \to \infty$.

2. Summary of relevant results from [B/K/R]

The following theorem is the main result of [B/K/R]. To state this, recall that a function σ is said to be self-neglecting if

$$\lim_{t\to\infty} \frac{\sigma\big(t + x\sigma(t)\big)}{\sigma(t)} = 1 \quad \text{locally uniformly in } x\,.$$

Any self-neglecting function is asymptotically equivalent to an absolutely continuous function whose first derivative tends to 0; hence this provides a simple sufficient condition (Bingham, Goldie and Teugels [2]).

THEOREM 1.— [B/K/R] *Suppose $X_1, \ldots, X_n$ are independent observations with bounded densities $f_1, \ldots, f_n$ which are strictly positive on an interval I that is unbounded above. Assume further that for $i = 1, \ldots, n$ the f_i satisfy the asymptotic equality*

$$f_i(t) \sim \gamma_i(t)\, e^{-\psi_i(t)} \quad as \quad t \to \infty\,, \tag{2.1}$$

where the functions ψ_i satisfy

$$\psi_i(t) \text{ is } C^2 \quad and \quad \psi_i''(t) > 0 \text{ for large } t\,, \tag{2.2}$$

$$\sigma_i := \left(\psi_i''\right)^{-1/2} \quad is\ self\text{-}neglecting, \tag{2.3}$$

and the functions γ_i satisfy the condition

$$\lim_{t\to\infty} \frac{\gamma_i\big(t + x\sigma_i(t)\big)}{\gamma_i(t)} = 1 \quad \textit{locally uniformly in } x\,. \tag{2.4}$$

Furthermore, we assume that $\tau_\infty = \lim_{t\to\infty} \psi_i'(t)$, where $\tau_\infty \le \infty$, is independent of i. Then the density $f_0 = f_1 * \cdots * f_n$ of $X_0 := X_1 + \cdots + X_n$ has the same form

$$f_0(t) \sim \gamma_0(t)\, e^{-\psi_0(t)} \quad as \quad t \to \infty\,,$$

where ψ_0 satisfies (2.2) and (2.3) and γ_0 satisfies (2.4). Explicit formulas for γ_0 and ψ_0 can be given as follows: choose q_i such that $\psi_i'(q_i) = \tau$ and write $t = t(\tau) = q_1 + \cdots + q_n$, then t is a continuous strictly increasing function of τ and $t(\tau) \uparrow \infty$ as $\tau \uparrow \tau_\infty$. Now one may choose

$$\psi_0(t) = \psi_1(q_1) + \cdots + \psi_n(q_n)$$
$$\sigma_0^2(t) = \sigma_1^2(q_1) + \cdots + \sigma_n^2(q_n)$$

$$\sqrt{2\pi}\,\sigma_0(t)\gamma_0(t) = \prod_{1\le i\le n} \left\{\sqrt{2\pi}\,\sigma_i(q_i)\gamma_i(q_i)\right\}.$$

Then

$$\sigma_0^2 = \left(\psi_0''\right)^{-1/2} \quad and \quad \lim_{t\to\infty} \psi_0'(t) = \tau_\infty\,.$$

The conditions of the theorem have already been discussed in great detail in [B/K/R]; for selfcontainedness of this note we repeat some points important in our context.

The densities f_i need not have the same support, they need not be continuous or bounded, it suffices that the distribution functions F_i have densities on a left neighbourhood of ∞. For instance, if $F_i = 1 - e^{-\psi_i}$ on a left neighbourhood of ∞ and if ψ_i satisfies (2.2) and (2.3), then (2.1) holds and the theorem applies with $\gamma_i = \psi_i'$.

The decomposition $f_i(t) = u_i(t)\gamma_i(t)\, e^{-\psi_i(t)}$ with $u_i(t) \to 1$ as $t \to \infty$ is far from unique. If desired we may choose $\gamma_i \equiv 1$, although even in the i.i.d case this does not lead to a substantial simplification of γ_0 (see section 4 below). Moreover, it is often convenient to make use of the extra freedom given by the function γ.

We think of a self-neglecting function as a function whose first derivative goes to zero. In a similar spirit, for γ satisfying (2.4) one can even assume that γ is C^∞ and

$$\frac{\gamma^{(k)}(t)\sigma(t)}{\gamma(t)} \to 0 \quad \text{as} \quad t \to \infty \tag{2.5}$$

for every $k \in \mathbb{N}$. A simple sufficient condition for (2.4) is (2.5) for $k = 1$.

Examples of possible functions ψ are

$$\begin{aligned} &t^\alpha, \quad \alpha > 1 \\ &e^t \\ &t - t^\alpha, \quad 0 < \alpha < 1 \\ &t \log t \\ &t - (\log t)^\alpha, \quad 0 < \alpha < 1 . \end{aligned}$$

As mentioned above, a function γ which arises in a natural way is ψ' which implies that the distribution tail satisfies $1 - F(t) \sim e^{-\psi(t)}$ as $t \to \infty$. Another possible choice for γ is any normalized regularly varying function. If $\gamma \in \mathrm{RV}(\beta)$, $\beta \in \mathbb{R}$, γ normalized, then

$$\frac{\gamma'(t)\sigma(t)}{\gamma(t)} \sim \beta \frac{\sigma(t)}{t} \to 0 \quad \text{as} \quad t \to \infty$$

since σ is self-neglecting (see [2]).

3. Tail exactness of the saddlepoint approximation

For densities f satisfying conditions (2.1)-(2.4) it is possible to express the saddlepoint approximation $f^\dagger$ in terms of the functions γ and ψ. This follows from the following asymptotic representation of the moment generating function (m.g.f.) C of f.

PROPOSITION 2.— [B/K/R] *Suppose f is a density satisfying the conditions (1.1)-(1.4) (2.1)-(2.4). Then its m.g.f. C is finite on some interval $[0, \tau_\infty)$, with $\tau_\infty > 0$, and*

$$C(\tau) \sim \sqrt{2\pi}\sigma(t)\gamma(t)\, e^{\psi^*(\tau)} \quad \textit{as} \quad t \to \tau_\infty ,$$

where $t = t(\tau) = \psi'^{\leftarrow}(\tau)$ is the inverse of ψ' and ψ^* is the convex conjugate of ψ.

By assumption, the function ψ is convex on I and ψ' is continuous and strictly increasing on I. The image $\Delta = \psi'(I)$ is an interval open at its upper endpoint, which is equal to τ_∞, and the convex conjugate

$$\psi^*(\tau) = \sup_{t\in I}\big(t\tau - \psi(t)\big)$$

is finite on Δ. Each $\tau \in \Delta$ is the slope of the tangent line at a unique point $t = t(\tau) \in I$. Furthermore, $t \to \infty$ if and only if $\tau \to \tau_\infty$. Also

$$\tau t = \psi(t) + \psi^*(\tau)$$

for $\tau \in \Delta$, $t \in I$ related as above. Hence

$$t = \psi'^{\leftarrow}(\tau) = \psi^{*\prime}(\tau)$$

and this relation defines a diffeomorphism $t \leftrightarrow \tau$ between points $t \in I$ and $\tau \in \Delta = \psi'(I)$, where

$$\sigma^2(t) = \frac{1}{\psi''(t)} = \psi^{*\prime\prime}(\tau) =: \frac{1}{s^2(\tau)}\,.$$

We use this relationship to obtain the following result.

THEOREM 3.— *Suppose f satisfies the conditions (2.1)-(2.4), then its saddlepoint approximation*

$$f^\dagger(t) = \frac{1}{\sqrt{2\pi}}\,K^{*\prime\prime}(t)^{1/2}\,e^{-K^*(t)}\,,$$

where $K^*(t)$ denotes the convex conjugate of $K(\tau) = \log C(\tau)$, becomes asymptotically exact; i.e.

$$\lim_{t\to\infty}\frac{f^\dagger(t)}{f(t)} = 1\,.$$

Proof.— By proposition 2 we obtain

$$K(\tau) = \psi^*(\tau) + \log\{\gamma(t)\sigma(t)\sqrt{2\pi}\} + \mathrm{o}(1) \quad \text{as} \quad \tau \to \tau_\infty\,,$$

for $t = t(\tau) = \psi^{*\prime}(\tau)$. This implies for the convex conjugate K^* of K

$$\begin{aligned} K^*(t) &= \tau t - K(\tau) \\ &= \tau t - \psi^*(\tau) - \log\big(\gamma(t)\sigma(t)\sqrt{2\pi}\big) + \mathrm{o}(1) \\ &= -\log\big(f(t)\sigma(t)\sqrt{2\pi}\big) + \mathrm{o}(1) \quad \text{as} \quad t \to \infty\,. \end{aligned}$$

Now, for a random variable X with density f the exponentially tilted random variable X_τ is defined for each $\tau \in \Delta$ as the random variable with density

$$f_\tau(t) = e^{\tau t} \frac{f(t)}{\widehat{f}(\tau)}, \quad t \in \mathbb{R}.$$

In theorem 6.6 of [B/K/R] it was proved that as $\tau \to \tau_\infty$ the normalized random variable $(X_\tau - t)/\sigma(t)$ converges weakly to a standard normal random variable and also its moments converge to the respective moments of the standard normal distribution. In particular $t = t(\tau) \to \infty$ if and only if $\tau \to \tau_\infty$ and $\operatorname{var} X_\tau \sim \sigma^2(t)$. Then $\sigma^2(t)$ is asymptotically equal to $\left(K^{*\prime\prime}(t)\right)^{-1} = K''(\tau)$ which implies that

$$f^\dagger(t) \sim \frac{1}{\sqrt{2\pi}} \sigma^{-1}(t) f(t) \sigma(t) \sqrt{2\pi}\, e^{o(1)} \sim f(t) \quad \text{as} \quad t \to \infty. \square$$

Notice that this result holds in particular for the density f_0 of the sum $X_0 = X_1 + \cdots + X_n$ where the X_i are independent with densities f_i satisfying the conditions (2.1)-(2.4) of theorem 1.

Remark.— Our result should be compared with theorem 1 of Jensen [4]. We shall show that any density f satisfying conditions (2.1)-(2.4) above also satisfies conditions (i), (ii) and (iii) of Jensen's theorem 1 and condition (iii)' of the corollary of that theorem. An alternative proof of theorem 3 above is therefore possible via Jensen [4].

By theorem (6.6) of [B/K/R] the normalized densities of the exponentially tilted random variable X_τ are asymptotically normal with exponential tails (ANET); i.e. $\lim_{\tau \to \infty} g_\tau(u) = \sigma$, $f_\tau(g + \sigma u) = \varphi(u)$ locally uniformly in u, where φ is the standard normal density, and for given $\epsilon > 0$ there exists an index τ_0 such that for $\tau \geq \tau_0$

$$\left| g_\tau(u) - \varphi(u) \right| < \epsilon\, e^{-|u|/\epsilon}, \quad u \in \mathbb{R}. \tag{3.1}$$

Moreover, the moments of $(X_\tau - q)/\sigma$ converge to those of a standard normal variable. This immediately implies condition (i) of Jensen's theorem. Furthermore, the characteristic functions $\mu_\tau(\xi)$ satisfy for $\tau \geq \tau_0$

$$\begin{aligned} \left| \mu_\tau(\xi) - \widehat{\varphi}(\xi) \right| &\leq \int_{-\infty}^{\infty} \left| g_\tau(u) - \varphi(u) \right| \mathrm{d}u \\ &\leq \epsilon \int_{-\infty}^{\infty} e^{-|u|/\epsilon}\, \mathrm{d}u = 2\epsilon^2 \end{aligned}$$

which is condition (ii). Finally, we check condition (iii)' which implies (iii): for $p > 1$ we write

$$\int_{-\infty}^{\infty} g_\tau^p(u)\,\mathrm{d}u = \int_{-\xi}^{\xi} + \int_{-\infty}^{-\xi} + \int_{\xi}^{\infty} g_\tau^p\,\mathrm{d}u\,.$$

By proposition 6.1 of [B/K/R] the first integral converges to $\int_{-\xi}^{\xi}\varphi^p(u)\,\mathrm{d}u$ and for given $\epsilon > 0$ there exists an index τ_0 and a constant $M > 1$ such that for all $\tau \geq \tau_0$

$$\int_{-\infty}^{-\xi} + \int_{\xi}^{\infty} g_\tau^p(u)\,\mathrm{d}u \leq \int_{-\infty}^{-\xi} + \int_{\xi}^{\infty} e^{-|u|p/\epsilon}\,\mathrm{d}u = 2\frac{\epsilon}{p}\,e^{-\xi p/\epsilon}\,.$$

4. Examples

Example 4.1.— Suppose $Y_1, \ldots, Y_n$ are independent identically distributed observations with density f satisfying the conditions of theorem 1. We are interested in the tail behaviour of the density of the weighted sum

$$X_0 = \sum_{i=1}^{n} w_i Y_i$$

for certain weights w_i. The random variables $X_i := w_i Y_i$ have densities

$$f_i(t) = \frac{f(x/w_i)}{w_i}$$

and hence the f_i satisfy the conditions of theorem 1 with

$$\gamma_i(t) = \frac{\gamma(t)}{w_i}$$
$$\psi_i(t) = \psi\left(\frac{t}{w_i}\right).$$

We choose q_i according to theorem 1 as

$$q_i = q_i(\tau) = w_i \psi'^{\leftarrow}(\tau w_i)$$

and obtain

$$\sigma_i^2 = \sigma_i^2(q_i) = \frac{w_i^2}{\psi''(q_i/w_i)} = \frac{w_i^2}{\psi''\big(\psi'^{\leftarrow}(\tau w_i)\big)}\,.$$

Then the density f_0 of X_0 has the form

$$f_0(t) \sim \gamma_0(t)\, e^{-\psi_0(t)}$$

where

$$t = t(\tau) = \sum_{i=1}^{n} q_i = \sum_{i=1}^{n} w_i \psi'^{\leftarrow}(\tau w_i)$$

and

$$\psi_0(t) = \sum_{i=1}^{n} \psi\Big(\frac{q_i}{w_i}\Big) = \sum_{i=1}^{n} \psi\big(\psi'^{\leftarrow}(\tau w_i)\big)$$

$$\begin{aligned}\sigma_0^2(t) &= \sum_{i=1}^{n} \sigma_i^2(q_i) = \sum_{i=1}^{n} \frac{w_i^2}{\psi''(q_i/w_i)}\\ &= \sum_{i=1}^{n} \frac{w_i^2}{\psi''\big(\psi'^{\leftarrow}(\tau w_i)\big)}\end{aligned}$$

$$\gamma_0(t) = (2\pi)^{\frac{n-1}{2}} \prod_{i=1}^{n} \frac{\gamma_i(q_i)}{\sqrt{\psi''(q_i/w_i)}} \left\{\sum_{i=1}^{n} \frac{w_i^2}{\psi''(q_i/w_i)}\right\}^{-\frac{1}{2}}.$$

These formulas reduce considerably in the i.i.d case with all weights w_i equal to 1, since $q_i = q = {}^{t}/_{n}$ and hence $\sigma^2(q) = \sigma^2({}^{t}/_{n}) = \big(\psi''({}^{t}/_{n})\big)^{-1}$. In that case the density f_0 of $X_0 = Y_1 + \cdots + Y_n$ is of the form

$$f_0(t) \sim \gamma_0(t)\, e^{-\psi_0(t)}$$

where

$$\begin{aligned}\psi_0(t) &= n\psi\Big(\frac{t}{n}\Big)\\ \sigma_0^2(t) &= n\sigma^2\Big(\frac{t}{n}\Big) = \frac{n}{\psi''(t/n)}\\ \gamma_0(t) &= \frac{1}{\sqrt{n}}\Big(2\pi\sigma^2\big(t(n)\big)\Big)^{\frac{n-1}{2}} \gamma^n\Big(\frac{t}{n}\Big).\end{aligned}$$

Example 4.2. — The generalized inverse Gaussian distribution with parameters ν, λ and μ has probability density function given, on the positive half line $\mathbb{R}_+$, by

$$\left\{2\mu^{\nu} K_{\nu}\Big(\frac{\lambda}{\mu}\Big)\right\}^{-1} y^{\nu-1}\, e^{-\frac{1}{2}\lambda\frac{(y-\mu)^2}{\mu^2 y}}.$$

Here $\nu \in \mathbb{R}$, $\lambda \in \mathbb{R}_+$ and $\mu \in \mathbb{R}_+$. We denote this distribution by $\mathrm{GIG}(\nu, \lambda, \mu)$. Note that the parameter λ has a role as a measure of precision.

Now, let $Y_1, \ldots, Y_n$ be independent random variables with Y_i following the distribution $\mathrm{GIG}(\nu, w_i\lambda, \mu)$, the w_i's being known weights. The minimal sufficient statistic is then $(\sum \log Y_i, \sum w_i Y_i^{-1}, \sum w_i Y_i)$. Let $X_i = \log Y_i$ and suppose that interest centers on the distribution of $X_0 = \sum X_i$, that is the first component of the minimal sufficient statistic. A manageable expression for the exact distribution of X_0 does not exist, but resort may be had to the saddlepoint approximation of the distribution, which is accessible since the cumulant transform of X_0 is expressible in terms of the Bessel function K_ν.

It is straightforward to check that $X_1, \ldots, X_n$ satisfy the conditions of theorem 1, and combining this with theorem 3 we find that the saddlepoint approximation $f_0^\dagger(t)$ to the density $f_0(t)$ of X_0 is asymptotically exact for $t \to \infty$.

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