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<http://www.numdam.org/item?id=AFST_1991_5_12_3_365_0>
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1. Introduction and Main Result

Hofer and Zehnder [1] extended the result of Viterbo [2]. The aim of the present paper is to extend the result of [1] to the case of Hamiltonian inclusions.

Let $H : \mathbb{R}^{2N} \to \mathbb{R}$ be locally Lipschitz continuous, which is written as $H \in C^{1-0}(\mathbb{R}^{2N}, \mathbb{R})$. Consider the Hamiltonian inclusion.

$$\dot{x} \in J\partial H(x) \quad (1)$$

where $\partial H$ is Clarke's generalized gradient of $H$ and $J$ is the standard $2N \times 2N$ symplectic matrix (see [3]). By a solution of (1) we mean an
absolutely continuous function \( x(t) \) satisfying (1) for almost all \( t \). It is well-known that, if \( H \) is regular, then any solution of (1) is conservative, i.e. \( H(x(t)) \equiv \text{constant} \). However, in general, if \( H \) is not regular, then a solution of (1) need not be conservative.

Our main result is the following

**Theorem 1.** Let \( H \in C^{1-0}(\mathbb{R}^2, \mathbb{R}) \) and \( c \in \mathbb{R} \). Suppose that \( \Sigma_c = H^{-1}(c) \) is a nonempty compact subset of \( \mathbb{R}^2 \) and

\[
0 \notin \partial H(x) \quad \text{for} \quad x \in \Sigma_c.
\] (2)

Then for any bounded neighborhood \( \Omega \) of \( \Sigma_c \), there are positive constants \( \beta \) and \( d \) such that for any \( \delta > 0 \), (1) has a \( T = T(\delta) \)-periodic conservative solution \( x(t) \) in \( \Omega \) such that \( H(x(t)) \equiv c' \in (c - \delta, c + \delta) \) and

\[
\beta \leq \frac{1}{2} \int_0^T (-J \dot{x}, x) \, dt \leq d.
\] (3)

The following results obtained by the author [4] will be used in the proof of Theorem 1.

**Proposition 1** ([4]). Let \( \Omega \) be an open subset of \( \mathbb{R}^2 \) and \( H \in C^{1-0}(\Omega, \mathbb{R}) \). Then for any continuous function \( \epsilon : \Omega \to (0, +\infty) \) there is a \( C^\infty \)-function \( g : \Omega \to \mathbb{R} \) such that

i) \( |g(x) - H(x)| \leq \epsilon(x) \) for \( x \in \Omega \),

ii) \( \forall x \in \Omega, \exists y \in \Omega \) and \( \xi \in \partial H(y) \) such that \( |x - y| \leq \epsilon(x) \) and \( |g'(x) - \xi| \leq \epsilon(x) \).

A \( C^1 \)-function \( g : \Omega \to \mathbb{R} \) satisfying the condition i) and ii) in Proposition 1 is called an \( \epsilon(x) \)-admissible approximation for \( H \) on \( \Omega \). In particular, when \( \epsilon(x) \equiv \epsilon \), \( g \) is called an \( \epsilon \)-admissible approximation for \( H \) on \( \Omega \).

**Proposition 2** ([4]). Let \( \Omega \) be an open subset of \( \mathbb{R}^2 \), \( H \in C^{1-0}(\Omega, \mathbb{R}) \) and \( \epsilon_n \to 0 \) (\( n \to \infty \)) with \( \epsilon_n > 0 \). Suppose that for each \( n \), \( H_n \in C^1(\Omega, \mathbb{R}) \) is an \( \epsilon_n \)-admissible approximation for \( H \) on \( \Omega \) and \( x_n \) is a \( T_n \)-periodic solution of the Hamiltonian system

\[
\dot{x} = JH_n'(x).
\] (4)
If

\[ \{ T_n \mid n = 1, 2, \ldots \} \text{ is bounded}, \]
\[ \{ x_n(t) \mid t \in \mathbb{R}, n = 1, 2, \ldots \} \text{ is contained in a compact subset of } \Omega, \]

then \( \{ x_n \} \) has a subsequence \( \{ x_{n_k} \} \) which converges uniformly to a

\( T \)-periodic solution \( x \) of (1) with \( T = \lim T_{n_k} \) and

\[ H(x(t)) \equiv c = \lim H_{n_k}(x_{n_k}(t)) \]

In section 2 we give the proof of theorem 1. In section 3 we extend
the a priori bound criterion of Benci-Hofer-Rabinowitz [5] to the case of
Hamiltonian inclusions.

2. Proof of theorem 1

Without loss of generality we may assume that \( c = 1 \) and \( \Omega_1 \) is connected.

Let \( \Omega \), a bounded neighborhood of \( \Omega_1 \), be given. By the upper semi-
continuity of \( H \), the compactness of \( \Omega_1 \) and the condition (2), we may
choose a bounded neighborhood \( V \) of \( \Omega_1 \) such that \( V \subset \Omega \) and \( 0 \notin \partial H(x) \) for
\( x \in V \). Then there are positive constants \( m \) and \( M \) such that \( m < |\xi| < M \)
for \( \xi \in \partial H(V) \). Using the pseudo-gradient flow (see [6]) we can construct a
Lipschitz homeomorphism \( \psi : (-s, s) \times \Omega_1 \rightarrow V \) such that

\[ H(\psi(t, x)) = 1 + t \text{ for } (t, x) \in (-s, s) \times \Omega_1. \]

Set

\[ U = \psi((-s, s) \times \Omega_1), \quad D = \text{diam } U, \quad \Sigma_c = (H|_U)^{-1}(c). \]

We fix positive numbers \( r, b \), such that

\[ D < r < 2D, \quad \frac{3}{2} \pi r^2 < b < 2\pi r^2. \]

Take a sequence \( \epsilon_n \rightarrow 0 \) such that \( 0 < \epsilon_n < \min \{ s/3, m/3 \} \) for all \( n \). By
proposition 1, for each \( n \), there is an \( \epsilon_n \)-admissible approximation \( H_n \) for
\( H \) on \( U \) and \( H_n \in C^\infty(U, \mathbb{R}) \). Then we have

\[ \begin{cases} |H_n(x) - H(x)| \leq \frac{s}{3} & \text{for } x \in U \text{ and all } n, \\ \frac{2}{3} m < |H'_n(x)| < M + \frac{m}{3} & \text{for } x \in U \text{ and all } n, \end{cases} \]
For each $n$ let $\psi_n$ be the flow in $U$ generated by
\[ \dot{x} = -\frac{H_n'(x)}{|H_n'(x)|^2}, \quad x(0) \in U. \]

Set $\Sigma_{1,n} = H_n^{-1}(1)$. It is easy to see that $\psi_n \left( \left[-s/2, s/2\right] \times \Sigma_{1,n} \right) \subset U$ and
\[ H_n(\psi_n(t,x)) = 1 + t \quad \text{for} \quad (t,x) \in \left[-\frac{s}{2}, \frac{s}{2}\right] \times \Sigma_{1,n}. \]

**Lemma 1.** For each $n$, $\Sigma_{1,n}$ is a connected compact hypersurface in $U$.

**Proof.** It suffices to prove the connectedness of $\Sigma_{1,n}$. For fixed $n$ let $x_1, x_2 \in \Sigma_{1,n}$. Then there are $-t_1 < 0$ and $-t_2 < 0$ such that
\[ \psi_n(-t_1, x_1) = y_1 \in \Sigma_{1+s/2} \quad \text{and} \quad \psi_n(-t_2, x_2) = y_2 \in \Sigma_{1+s/2}. \]

Note that $\Sigma_{1+s/2}$ is connected since $\Sigma_{1+s/2}$ is homeomorphic to $\Sigma_1$. Let $p$ be a path in $\Sigma_{1+s/2}$ joining $y_1$ to $y_2$. It is easy to see that along the descent flow lines of $\psi_n$, $p$ can be deformed to a path in $\Sigma_{1,n}$ joining $x_1$ to $x_2$. So $\Sigma_{1,n}$ is connected and the proof of lemma 1 is complete.

Set $U_n = \psi_n \left( \left[-s/2, s/2\right] \times \Sigma_{1,n} \right)$. Then $\psi_n : \left(-s/2, s/2\right) \times \Sigma_{1,n} \to U_n \subset U$ is a diffeomorphism. We denote by $A_n$ and $B_n$ the unbounded and bounded component of $\mathbb{R}^{2N} \setminus U_n$ respectively and by $B$ the bounded component of $\mathbb{R}^{2N} \setminus U$. We may assume that $0 \in B$, then $0 \in B_n$ since $B \subset B_n$ for all $n$.

Let $\delta > 0$ be given. We may assume $\delta < s/2$.

Following [1], we pick a $C^\infty$-function $f : \left(-s/2, s/2\right) \to \mathbb{R}$ satisfying
\[ f|_{(-s/2, -\delta)} = 0, \quad f|_{[\delta, s/2]} = b \quad \text{and} \quad f'(t) > 0 \quad \text{for} \quad -\delta < t < \delta. \]

Choose a $C^\infty$-function $g : (0, \infty) \to \mathbb{R}$ such that
\[
\begin{align*}
g(t) &= b \quad \text{for} \ t \leq r, \\
g(t) &= \frac{3}{2} \pi t^2 \quad \text{for} \ t \geq r, \\
g(t) &= \frac{2}{3} \pi t^2 \quad \text{for} \ t > r, \\
0 < g'(t) &\leq 3 \pi t \quad \text{for} \ t > r.
\end{align*}
\]
For each $n$ define a $C^\infty$-function $G_n : \mathbb{R}^{2N} \to \mathbb{R}$ by

$$G_n(x) = \begin{cases} 0 & \text{if } x \in B_n \\ f(t) & \text{if } x \in \psi_n(t \times \Sigma_{1,n}), -\delta \leq t \leq \delta \\ b & \text{if } x \in A_n \text{ and } |x| \leq r \\ g(|x|) & \text{if } |x| > r. \end{cases}$$

Then, by [1], for each $n$ the Hamiltonian system

$$\dot{x} = JG'_n(x)$$

has a 1-periodic solution $x_n$ in $U_n$ such that

$$H_n(x_n(t)) = c_n \in (1 + \delta, 1 - \delta) \quad \text{for all } t$$

and

$$\beta \leq \frac{1}{2} \int_0^1 \langle -J\dot{x}_n, z_n \rangle \, dt \leq d,$$

where $\beta$ and $d = 16 \pi D^2$ are positive constants independent of $n$ and $\delta$.

By the definition of $G_n$ we have

$$G_n(x) = f(H_n(x) - 1) \quad \text{and} \quad G'_n(x) = f'(H_n(x) - 1)H'_n(x)$$

for $x \in (H_n|_{U_n})^{-1}((1 - \delta, 1 + \delta))$.

Set $z_n(t) = x_n(f'(c_n - 1)t)$. Then $z_n$ is a $T_n$-periodic solution in $U_n$ of the Hamiltonian system

$$\dot{z} = JH'_n(z)$$

with $T_n = f'(c_n - 1)$ and

$$\beta \leq \frac{1}{2} \int_0^{T_n} \langle -J\dot{z}_n, z_n \rangle \, dt \leq d.$$  \hspace{1cm} (7)

From the fact that $|c_n - 1| < \delta$ and $f'$ is bounded on $(-\delta, \delta)$ it follows that $\{T_n \mid n = 1, 2, \ldots\}$ is bounded. Noting that

$$U_n \subset \left\{ x \in U \mid 1 - \frac{5}{6}s \leq H(x) \leq 1 + \frac{5}{6}s \right\} \subset U,$$

from proposition 2 it follows that $\{z_n\}$ has a subsequence $\{z_{nK}\}$ which converges uniformly to a conservative $T$-periodic solution $z$ of (1) such that

$$T = \lim T_{nK}, \quad H(z(t)) = \bar{c} = \lim c_{nK} \in [1 - \delta, 1 + \delta] \quad \text{and} \quad z(t) \in U, \forall t.$$  \hspace{1cm} (3) follows from (7). The proof of theorem 1 is complete. □
3. A criterion for a priori bounds

For \( x \in \mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N \) set \( x = (p, q) = (\pi_1 x, \pi_2 x) \). Note that in general neither of the sets \( \partial_p H(x) \times \partial_q H(x) \) and \( \partial H(x) \) need be contained in the other, but both of them are contained in \( \pi_1 \partial H(x) \times \pi_2 \partial H(x) \) (see [3]). The following theorem is an extension of the result of Benci-Hofer-Rabinowitz [5].

**Theorem 2.** Under the assumptions of theorem 1, if there is a function \( K \in C^1(\mathbb{R}^{2N}, \mathbb{R}) \) and constants \( a, b \geq 0 \) with \( a + b > 0 \) such that

\[
\begin{align*}
  a\langle \pi_1 x, \pi_1 \xi \rangle + b\langle \pi_2 x, \pi_2 \xi \rangle + \langle K'(x), J \xi \rangle &> 0, \\
  \forall x \in \Sigma_c, \xi \in \partial H(x)
\end{align*}
\]

then (1) has a periodic solution on \( \Sigma_c \).

**Proof.** We use the notations used in the proof of theorem 1 and assume \( c = 1 \). By the upper semicontinuity of \( \partial H \) and the compactness of \( \Sigma_c \), for \( s > 0 \) small, there is a constant \( \gamma > 0 \) such that

\[
\begin{align*}
  a\langle \pi_1 x, \pi_1 \xi \rangle + b\langle \pi_2 x, \pi_2 \xi \rangle + \langle K'(x), J \xi \rangle &> \gamma, \\
  \forall x \in U, \xi \in \partial H(x)
\end{align*}
\]

where \( U = \psi((-s, s) \times \Sigma_1) \).

Let \( z \) be a conservative \( T \)-periodic solution of (1) in \( U \). Setting \( \xi(t) = -J \dot{z}(t) \), then \( \xi(t) \in \partial H(z(t)) \) a.e. and

\[
A(z) := \frac{1}{2} \int_0^T \langle -J \dot{z}, z \rangle dt = \int_0^T \langle \pi_1 z, \pi_1 \xi \rangle dt = \int_0^T \langle \pi_2 z, \pi_2 \xi \rangle dt.
\]

Noting that

\[
\int_0^T \langle K'(z), J \xi \rangle dt = \int_0^T \langle K'(z), \dot{z} \rangle dt = 0,
\]

integrating for (9) over \([0, T]\) gives

\[
(a + b)A(z) \geq \gamma T. \quad (10)
\]
We now take a sequence $\delta_n \to 0$ with $0 < \delta_n < \delta/2$. By theorem 1, for each $n$, (1) has a conservative $T_n$-periodic solution $z_n$ in $U$ such that $A(z_n) \leq d$ and $|H(z_n(t)) - 1| < \delta_n$. From (10) it follows that $\{T_n \mid n = 1, 2, 3, \ldots\}$ is bounded. It is easy to see that $\{z_n\}$ has a subsequence which converges uniformly to a conservative $T$-periodic solution $z$ of (1) and $z(t) \in \Sigma_1, \forall t$.

The proof is complete.

COROLLARY 1. — Suppose that $H \in C^{1-0}(\mathbb{R}^2, \mathbb{R})$, $c \in \mathbb{R}$ and $\Sigma_c = H^{-1}(c)$ is compact. If

$$\langle x, \xi \rangle > 0 \text{ for } x \in \Sigma_c \text{ and } \xi \in \partial H(x),$$

then (1) has a periodic solution on $\Sigma_c$.

Proof. — Note that (11) implies (2). Hence all assumptions of theorem 1 are satisfied. Taking $a = b = 1$ and $K = 0$ gives (8). Corollary 1 follows from theorem 2.

COROLLARY 2. — Suppose that $H \in C^{1-0}(\mathbb{R}^2, \mathbb{R})$, $c \in \mathbb{R}$ and $\Sigma_c = H^{-1}(c)$ is compact. If

$(p_1)$ $\langle \pi_1 x, \pi_1 \xi \rangle > 0$ for $x \in \Sigma_c$ with $\pi_1 x \neq 0$ and $\xi \in \partial H(x)$,

$(p_2)$ $0 \notin \pi_2 \partial H(x)$ for $x \in \Sigma_c$ with $\pi_1 x = 0$,

then (1) has a periodic solution on $\Sigma_c$.

Proof. — It is clear that $(p_1)$ and $(p_2)$ imply (2). By the upper semicontinuity of $\partial H$ and the compactness of $\Sigma_c$ there is a bounded neighborhood $U$ of $\Sigma_c$ such that $(p_1)$ and $(p_2)$ are also true if $\Sigma_c$ is replaced by $U$. Applying the acute angle approximation theorem (see e.g. [7]) for the multivalued map $\pi_2 \partial H : \mathbb{R}^2 \to 2^{\mathbb{R}^N}$, it is not difficult to construct a map $W \in C^1(\mathbb{R}^2, \mathbb{R}^N)$ such that

$$\langle W(x), \pi_2 \xi \rangle > 0 \text{ for } x \in U \text{ with } \pi_1 x = 0 \text{ and } \xi \in \partial H(x).$$

Set $K(x) = \langle -W(x), \pi_1 x \rangle$ for $x \in \mathbb{R}^2$. Then $K \in C^1(\mathbb{R}^2, \mathbb{R})$ and

$$\langle K'(x), J\xi \rangle = \langle -W'(x) \cdot J\xi, \pi_1 x \rangle + \langle W(x), \pi_2 \xi \rangle$$

for $x \in \mathbb{R}^2$ and $\xi \in \partial H(x)$. 

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It is easy to see that there are constants $\sigma, \gamma > 0$ such that
\[
\langle W(x), \pi_2 \xi \rangle \geq 2\gamma \quad \text{and} \quad |\langle W'(x) \cdot J \xi, \pi_1 x \rangle| \leq \gamma
\]
for $x \in U$ with $|\pi_1 x| \leq \sigma$, and $\xi \in \partial H(x)$. Let
\[
M = \sup \left\{ \langle K'(x), J \xi \rangle \mid x \in U, \xi \in \partial H(x) \right\},
\]
\[
m = \inf \left\{ \langle \pi_1 x, \pi_1 \xi \rangle \mid x \in U \text{ with } |\pi_1 x| \geq \sigma, \xi \in \partial H(x) \right\}.
\]
Set $a = (M + \gamma)/m$ and $b = 0$. Then for $x \in U$ and $\xi \in \partial H(x)$ we have
\[
a\langle \pi_1 x, \pi_1 \xi \rangle + \langle K'(x), J \xi \rangle \geq 0 + 2\gamma - \gamma = \gamma - 0 \text{ if } |\pi_1 x| \leq \sigma,
\]
\[
a\langle \pi_1 x, \pi_1 \xi \rangle + \langle K'(x), J \xi \rangle \geq M + \gamma - M = \gamma > 0 \text{ if } |\pi_1 x| \geq \sigma.
\]
Thus (8) holds and corollary 2 follows from theorem 2.

**Remark.** — When $H \in C^1$, (2) and $(p_1)$ imply $(p_2)$ (see [5]), but such conclusion is not true when $H \in C^{1-0}$.

**References**

[1] **HOFER (H.)** and **ZEHNDER (E.)**  — Periodic solutions on hypersurfaces and a result by C. Viterbo,

[2] **VITERBO (C.)**  — A proof of the Weinstein conjecture in $\mathbb{R}^{2n}$,

[3] **CLARKE (F.)**  — Optimization and Nonsmooth Analysis

[4] **FAN (X.)**  — The $C^1$-admissible approximation for Lipschitz functions and the Hamiltonian inclusions,

[5] **BENCI (V.), HOFER (H.)** and **RABINOWITZ (P.)**  — A remark on a priori bounds and existence for periodic solutions of Hamiltonian systems,


[7] **BORISOVICH (YU.), GELMAN (B.), MUSCHKIS (A.)** and **OBUHOVSKII (V.)**  — Topological methods in the fixed point theory of multivalued mappings,