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Existence, uniqueness and regularity for Kruzkov’s solutions of the Burger-Carleman’s system

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RÉSUMÉ. — Nous montrons l’existence et l’unicité d’une solution \((u(t), v(t))\) au sens Kruzkov du système de Burger-Carleman avec condition initiale \((u_0, v_0) \in L^1(\mathbb{R})_+ \times L^1(\mathbb{R})_+\). Nous montrons que pour tout \(t > 0\), \(u(t), v(t) \in L^\infty(\mathbb{R})\). Cet effet régularisant est lié à la possibilité de définir la solution au sens Kruzkov du système de Burger – Carleman.

ABSTRACT. — We prove existence and uniqueness of a Kruzkov solution \((u(t), v(t))\) of the Burger-Carleman’s system with initial data \((u_0, v_0) \in L^1(\mathbb{R})_+ \times L^1(\mathbb{R})_+\). Moreover, we show that for any \(t > 0\), \(u(t), v(t) \in L^\infty(\mathbb{R})\) with precise estimates. In fact, this regularizing effect is related to the possibility of defining Kruzkov’s solutions for the Burger-Carleman’s system.

We consider the following first order system which will be called the Burger-Carleman’s system:

\[
\begin{align*}
&u_t + \left(\frac{u^2}{2}\right)_x + u^2 - v^2 = 0 & \text{on} [0, +\infty) \times \mathbb{R} \\
&(BC) & v_t - \left(\frac{v^2}{2}\right)_x + v^2 - u^2 = 0 & \text{on} [0, +\infty) \times \mathbb{R} \\
&u(0, x) = u_0(x), v(0, x) = v_0(x)
\end{align*}
\]

with initial data \(u_0, v_0 \in L^1(\mathbb{R})_+\). We prove the existence and uniqueness of a Kruzkov’s solution of \((BC)\) (see definition 1 below) using the theory of nonlinear semigroups generated by accretive operators. We notice that

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the possibility of defining Kruzkov’s solutions for (BC) when the initial data \((u_0, v_0) \in L^1(\mathbb{R})^2_+\) depends on the \(L^1 - L^\infty\) regularizing effect for homogeneous equations proved in [2]. In fact, the estimates proved in [2] imply that for any \((u_0, v_0) \in L^1(\mathbb{R})^2_+\) and any \(t > 0, u(t), v(t) \in L^\infty(\mathbb{R})_+\) with precise estimates given below. Before stating the precise result, let us define the notion of Kruzkov’s solution for (BC):

**DEFINITION 1.** Let \(T > 0\). The pair of functions \((u, v) \in L^\infty([0, T], L^1(\mathbb{R})^2_+ \cap L^\infty(\tau, T] \times \mathbb{R})^2\) for any \(\tau > 0\) will be called a Kruzkov’s solution of (BC) in \([0, T] \times \mathbb{R}\) with initial data \((u_0, v_0) \in L^1(\mathbb{R})^2_+\) if \((u(t), v(t)) \to (u_0, v_0) \in L^1(\mathbb{R})^2\) as \(t \to 0\) and

\[
\int_0^T \int_\mathbb{R} |u - k| \xi_t + \text{sign}_0(u - k) \left[ \left( \frac{u^2}{2} - \frac{k^2}{2} \right) \xi_x + (v^2 - u^2) \xi \right] \, dx \, dt \geq 0
\]

\[
\int_0^T \int_\mathbb{R} |v - k'| \eta_t + \text{sign}_0(v - k') \left[ \left( \frac{k'^2}{2} - \frac{v^2}{2} \right) \eta_x + (u^2 - v^2) \eta \right] \, dx \, dt \geq 0
\]

holds for all \(\xi, \eta \in C^\infty_0((0, T) \times \mathbb{R}), \xi, \eta \geq 0\) and all \(k, k' \in \mathbb{R}\).

As it is customary

\[
\text{sign}_0(r) = +1 \text{ if } r > 0, 0 \text{ if } r < 0
\]

\[
\text{sign}(r) = +1 \text{ if } r > 0, [-1, 1] \text{ if } r = 0, -1 \text{ if } r < 0
\]

\[
\text{sign}^+(r) = +1 \text{ if } r > 0, [0, 1] \text{ if } r = 0, 0 \text{ if } r < 0
\]

Similarly one defines \(\text{sign}_0^+(r)\).

Then, our result says:

**THEOREM 1.** For any \((u_0, v_0) \in L^1(\mathbb{R})^2_+\), there exists a unique Kruzkov’s solution \((u, v) \in C([0, T], L^1(\mathbb{R})^2_+)\) of (BC) in \([0, T] \times \mathbb{R}\) for any \(T > 0\) with initial data \((u_0, v_0)\) such that for any \(t > 0\):

\[
(RE) \quad ||u(t)||_{L^\infty(\mathbb{R})} \leq \left( \frac{2}{t} ||u_0 + v_0||_{L^1(\mathbb{R})} + \frac{2\sqrt{2}}{\sqrt{t}} (||u_0 + v_0||_{L^1(\mathbb{R})})^{3/2} \right)^{1/2}
\]

The same estimate holds for \(||v(t)||_{L^\infty(\mathbb{R})}\). Moreover, if \((u, v), (\tilde{u}, \tilde{v})\) are two Kruzkov’s solutions of (BC) in \([0, T] \times \mathbb{R}, T > 0\), corresponding to the initial data \((u_0, v_0), (\tilde{u}_0, \tilde{v}_0) \in L^1(\mathbb{R})^2_+\) respectively, then for all \(t \in [0, T]\):

\[
|||(u(t) - \tilde{u}(t))^+||_{L^1(\mathbb{R})} + |||(v(t) - \tilde{v}(t))^+||_{L^1(\mathbb{R})} \leq |||(u_0 - \tilde{u}_0)^+||_{L^1(\mathbb{R})} + |||(v_0 - \tilde{v}_0)^+||_{L^1(\mathbb{R})}
\]
To begin with the proof, let us introduce the following operators $A, B$:

$$D(A) := \{(u, v) \in L^1(\mathbb{R})^2_+ : u^2, v^2 \in AC(\mathbb{R})\}$$

$$D(B) := \{(u, v) \in L^1(\mathbb{R})^2_+ : u^2, v^2 \in L^1(\mathbb{R})\}$$

where $AC(\mathbb{R})$ is the set of absolutely continuous functions on $\mathbb{R}$,

$$A(u, v) = \left(\left(\frac{u^2}{2}\right)_x, -\left(\frac{v^2}{2}\right)_x\right), B(u, v) = (u^2 - v^2, v^2 - u^2)$$

for $(u, v) \in D(A), (u, v) \in D(B)$ respectively. Notice that $D(A) \subset D(B)$. Thus $D(A + B) = D(A)$ and $(BC)$ can be written in the abstract form: let $U = (u, v)

$$\frac{dU}{dt} + (A + B)U = 0$$

$$U(0) = (u_0, v_0) \in L^1(\mathbb{R})^2_+$$

We show that one can use the Grandall-Liggett's theorem to solve $(BC)_a$. This is the purpose of the next two lemmas. Before stating them, let us recall the definition of $T$-accretivity. Let $E$ be a Banach lattice. A (in general, multivalued) operator $B$ on $E$ called $T$-accretive if

$$||(x - \tilde{x})^+||_E \leq ||(x - \tilde{x} + \lambda y - \lambda \tilde{y})^+||_E$$

holds for all $[x, y], [\tilde{x}, \tilde{y}] \in B$ and all $\lambda > 0$.

If $E = L^1(\mathbb{R}) \times L^1(\mathbb{R})$ endowed with the norm

$$||(u, v)||_E = \int_\mathbb{R} |u| + \int_\mathbb{R} |v|, (u, v) \in E,$$

then this is equivalent to say that for all $[(x_1, x_2), (y_1, y_2)], [(\tilde{x}_1, \tilde{x}_2), (\tilde{y}_1, \tilde{y}_2)] \in B$ there exists some $\alpha_1 \in \text{sign}^+(x_1 - \tilde{x}_1), \alpha_2 \in \text{sign}^+(x_2 - \tilde{x}_2)$ such that

$$\int_\mathbb{R} \alpha_1(y_1 - \tilde{y}_1) + \alpha_2(y_2 - \tilde{y}_2)dx \geq 0.$$ Then:

**Lemma 1.** — $A + B$ is $T$-accretive in $L^1(\mathbb{R})^2$. Moreover, for any $p \in W^{1,\infty}(\mathbb{R})$ such that $p' \geq 0$ has compact support:

$$\int_\mathbb{R} p(u)w + p(v)h \ dx \geq 0$$

holds for any $(u, v) \in D(A)$ where $(w, h) = (A + B)(u, v)$.

**Lemma 2.** — For all $\lambda > 0$, $\text{Ran} (I + \lambda(A + B)) = L^1(\mathbb{R})^2_+.$

**Proof of lemma 1.** — Let $U = (u, v), \widehat{U} = (\widehat{u}, \widehat{v}) \in D(A)$. One easily checks that

$$\int_\mathbb{R} \left[\left(\frac{u^2}{2}\right)_x \left(\frac{v^2}{2}\right)_x\right] \text{sign}^+_0(u - \widehat{u})dx = \int_\mathbb{R} \left[\left(\frac{v^2}{2}\right)_x - \left(\frac{v^2}{2}\right)_x\right] \text{sign}^+_0(v - \widehat{v})dx$$

$$= 0$$

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since \( u, \hat{u}, v, \hat{v} \geq 0 \) and \( \text{sign}_0^+ \) is an increasing function, then
\[
\int_{\mathbb{R}} B(u, v)(\text{sign}_0^+(u - \hat{u}), \text{sign}_0^+(v - \hat{v}))dx =
\]
\[
\int_{\mathbb{R}} |\text{sign}_0^+(u - \hat{u}) - \text{sign}_0^+(v - \hat{v})|[(u^2 - \hat{u}^2) - (v^2 - \hat{v}^2)]dx =
\]
\[
\int_{\mathbb{R}} |\text{sign}_0^+(u^2 - \hat{u}^2) - \text{sign}_0^+(v^2 - \hat{v}^2)|[(u^2 - \hat{u}^2) - (v^2 - \hat{v}^2)]dx \geq 0
\]
Both remarks imply that \( A + B \) is \( T \) accretive in \( L^1(\mathbb{R})_+^2 \).

Let \( \beta(r) : = r^{1/2}, r \geq 0 \). Let \( p \in W^{1,\infty}(\mathbb{R}) \) be such that \( p' \geq 0 \) has compact support.

Let \( j : \mathbb{R}_+ \rightarrow \mathbb{R} \) be \( j(r) = \int_0^r (p \circ \beta)(s)ds. \) Then, if \( z = u^2 \)
\[
\int_{\mathbb{R}} \left( \frac{u^2}{2} \right)_x p(u)dx = \int_{\mathbb{R}} \left( \frac{z}{2} \right)_x (p \circ \beta)(z)dx = \frac{1}{2} \int_{\mathbb{R}} j(z)_x dx = 0.
\]
Similarly \( \int_{\mathbb{R}} \left( \frac{v^2}{2} \right)_x p(x)dx = 0 \) and
\[
\int_{\mathbb{R}} (u^2 - v^2)p(u) + (v^2 - u^2)p(v)dx = \int_{\mathbb{R}} (u^2 - v^2)(p(u) - p(v))dx \geq 0
\]
since \( p \) is increasing and \( u, v \geq 0 \). Putting this things together we get the inequality (1).

Proof of lemma 2.—Since the proof below is independent of the value of \( \lambda > 0 \) we take \( \lambda = 1 \). We have to solve the following equations : let \( f, g \in L^1(\mathbb{R})_+ \).
\[
u + \left( \frac{u^2}{2} \right)_x + u^2 - v^2 = f \quad (2.1)
\]
\[(SP)_{f,g}
\]
\[
v - \left( \frac{v^2}{2} \right)_x + v^2 - u^2 = g \quad (2.2)
\]

1st step : We work in a \( L^2 \) - framework. Let \( I_n = [-n, n] \). Let us solve the equations \((SP)_{f,g}\) for \( f, g \in L^2(I_n)_+ \). Let \( \beta \) be as above. Then \((SP)_{f,g}\) is equivalent to
\[
\beta(w) + \left( \frac{w}{2} \right)_x + w - h = f
\]
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\[(SP)_{\beta, f, g}\]

\[\beta(h) - \left(\frac{h}{2}\right)_x + h - w = g\]

through the change of variable \(w = u^2, h = v^2\). Let \(\bar{\beta}(r) = \sqrt{r}\) if \(r \geq 0\), \(-\sqrt{|r|}\) if \(r < 0\). Let us first consider the system:

\[\bar{\beta}(w) + \left(\frac{w}{2}\right)_x + w - h = f\]

\[(SP)_{\bar{\beta}, f, g}\]

\[\bar{\beta}(h) - \left(\frac{h}{2}\right)_x + h - w = g\]

where \(f, g \in L^2(I_n)\). The existence of a solution of \((SP)_{\bar{\beta}, f, g}\) is a consequence of standard perturbation results for maximal monotone operators ([5]). Let \(T_{\bar{\beta}} : L^2(I_n)^2 \to L^2(I_n)^2\) be given by \(T_{\bar{\beta}}(w, h) = (\bar{\beta}(w), \bar{\beta}(h))\). Let \(T : L^2(I_n)^2 \to L^2(I_n)^2\) with domain:

\[\text{Dom}(T) = \{(w, h) \in H^1(I_n) \times H^1(I_n) : w(-n) = h(-n), w(n) = h(n)\}\]

be given by \(T(w, h) = \left(\frac{w_x}{2} + w - h, -\frac{h_x}{2} + h - w\right)\). Since \(T_{\bar{\beta}}, T\) are maximal monotone and \(\text{Dom}(T_{\bar{\beta}} + T) = L^2(I_n)^2\), \(T_{\bar{\beta}} + T\) is maximal monotone ([4], Corol. 2.7). Moreover, since \(\bar{\beta}\) is the subgradient of a convex function, by [5], thm. 4, \(\text{Int Ran}(T_{\bar{\beta}} + T) = \text{Int}(\text{Ran} T_{\bar{\beta}} + \text{Ran} T)\). But it is an exercise to see that \(\text{Ran} T = L^2(I_n)^2\). Therefore, \(\text{Ran}(T_{\bar{\beta}} + T) = L^2(I_n)^2\). Therefore, for \(f, g \in L^2(I_n), (SP)_{\bar{\beta}, f, g}\) has a solution \((w, h) \in H^1(I_n) \times H^1(I_n)\) with \(w(-n) = h(-n), w(n) = h(n)\). To go back to problem \((SP)_{\beta, f, g}\) it suffices to remark that \(w, h \geq 0\) if \(f, g \geq 0\). For that we multiply the first equation in \((SP)_{\bar{\beta}, f, g}\) by \(w^-\) and the second by \(h^-\). Adding both equations and integrating over \(R\), one gets:

\[\int_R gh^- + fw^- + (w^-)^{3/2} + (h^-)^{3/2} + (w^- - h^-)^2 + 2w^+h^-dx = 0\]

Since each term in the integrand is positive, \(w^- = h^- = 0, i.e., w, h \geq 0\). Thus, given \(f, g \in L^2(I_n)_+, \) there exists \(w, h \in H^1(I_n)\) with \(w(-n) = h(-n), w(n) = h(n), w, h \geq 0\) which solve \((SP)_{\beta, f, g}\). Then \(u = \sqrt{w}\) on \(I_n, 0\) in \(R - I_n, v = \sqrt{h}\) on \(I_n, 0\) in \(R - I_n\) solve \((SP)_{f, g}\).

2nd step: Let \(f, g \in L^1(R)_+. \) Let \(f_n, g_n \in L^2(I_n)_+\) be such that \(f_n \rightharpoonup f, g_n \rightharpoonup g\). Let \((u_n, v_n)\) be the solutions of \((SP)_{f_n, g_n}\) found in step 1.
Notice that the accretivity of $A+B$ implies that $u_n, v_n$ are Cauchy sequences in $L^1(R)$. Let $u, v \in L^1(R)_+$ be the limits of $u_n, v_n$ in $L^1(R)$. Now adding the corresponding equations to (2.1), (2.2) for $(SP)_{f_n, g_n}$ and using that $u_n, v_n \geq 0$ we get:

$$
\left( \frac{u_n^2 - v_n^2}{2} \right)_x \leq f_n + g_n
$$

Since $u_n(-n) = v_n(-n), u_n(n) = v_n(n)$, integrating from $-\infty$ to $x$ and from $x$ to $\infty$ we get $||u_n^2 - v_n^2||_\infty \leq 2||f_n + g_n||L^1(R)$. Since, for $a, b \geq 0$, $|a - b| \leq |a^2 - b^2|^{1/2}$, the sequence $u_n - v_n$ is bounded in $L^\infty(R)$. Then, $u_n^2 - v_n^2 = (u_n - v_n)(u_n + v_n)$ is bounded in $L^1(R)$. From $(SP)_{f_n, g_n}$ it follows that $\left( \frac{u_n^2}{2} \right)_x, \left( \frac{v_n^2}{2} \right)_x$ are bounded in $L^1(R)$. This, together with $u_n \to u, v_n \to v$ in $L^1(R)$ implies that $u_n, v_n$ are bounded in $L^\infty(R)$ and $u_n^2 \to u^2, v_n^2 \to v^2$ in $L^1(R)$. Thus

$$
\left( \frac{u_n^2}{2} \right)_x \to \left( \frac{u^2}{2} \right)_x, \left( \frac{v_n^2}{2} \right)_x \to \left( \frac{v^2}{2} \right)_x \text{ in } L^1(R), (u, v) \in D(A)
$$

and letting $n \to \infty$ in $(SP)_{f_n, g_n}$ we get a solution $(u, v) \in D(A)$ for $(SP)_{f, g}$.

Using the Crandall – Ligget’s theorem in combination with lemmas 1 and 2 above, one gets:

**Proposition 1.** For any $(u_0, v_0) \in L^1(R)_+^2$ and any $t > 0$, there exists a unique mild (or semigroup) solution $(u, v) \in C([0, T], L^1(R)_+^2)$ of (BC) with initial data $u(0) = u_0, v(0) = v_0$. If $(u, v), (\hat{u}, \hat{v})$ are two mild solutions of (BC) with initial data $(u_0, v_0), (\hat{u}_0, \hat{v}_0) \in L^1(R)_+^2$ respectively, then:

$$
||(u(t) - \hat{u}(t))^+||_{L^1(R)} + ||(v(t) - \hat{v}(t))^+||_{L^1(R)} \leq ||(u_0\hat{u}_0)^+||_{L^1(R)} + ||(v_0\hat{v}_0)^+||_{L^1(R)}
$$

Moreover, if $u_0, v_0 \in L^1(R)_+^2 \cap L^p(R)_+^2, 1 \leq \infty$, then $(u(t), v(t)) \in L^1(R)_+^2 \cap L^p(R)_+^2$ and for any $t \geq 0$

$$
||u(t)||_{L^p(R)} + ||v(t)||_{L^p(R)} \leq ||u_0||_{L^p(R)} + ||v_0||_{L^p(R)}.
$$

**Proof.** — Just remark that the last assertion is a consequence of the inequalities (1) in Lemma 1 ([1], section 2).
Before proving the regularizing estimate (RE) let us prove that the semigroup solution \((u, v)\) of \((BC)\) with initial data \((u_0, v_0) \in L^1(\mathbb{R})^2_+ \cap L^\infty(\mathbb{R})^2_+\) obtained via the Crandal-Ligget’s theorem is a Kruzkov’s solution. This is a consequence of two facts: first, if \((u(t), v(t))\) is the mild solution of \((BC)\) with initial data \((u_0, v_0) \in L^1(\mathbb{R})^2_+ \cap L^\infty(\mathbb{R})^2_+\) then \(u(t)\) and \(v(t)\) are, respectively, the mild solutions of:

\[
(*) \quad u_t + \left( \frac{u^2}{2} \right)_x = \Psi(t),
\]

\[
(**) \quad v_t - \left( \frac{v^2}{2} \right)_x = -\Psi(t),
\]

where \(\Psi(t) \equiv v^2(t) - u^2(t)\) ([10], Lemma 1.7) and second, the well known fact that mild or semigroup solutions of (*) and (**) are in fact Kruzkov’s solutions of (*), (**) respectively ([1], Prop. 2.11). Writing what means that \(u(t), v(t)\) are Kruzkov’s solutions of (*) (**) respectively we get that \((u(t), v(t))\) is a Kruzkov solution of \((BC)\) in the sense of definition 1. One can argue directly using only [1], Prop. 2.11. Recall that \((u, v)\) is obtained in the following way: let \(P_n = \{0 = a_0^n < \ldots < a_n^n = T\}\) where \(a_k^n = \frac{kT}{n}\). Let \(u_n(t), v_n(t)\) be the step functions given by \(u_n(0) = 0, v_n(0) = 0, u_n(t) = u^n_k, v_n(t) = v^n_k\) in \([a_{k-1}^n, a_k^n]\), where \((u^n_k, v^n_k)\) are constructed as solutions of the difference scheme:

\[
\frac{u^n_k - u^n_{k-1}}{a^n_k - a^n_{k-1}} + \left( \frac{(u^n_k)^2}{2} \right)_x + (u^n_k)^2 - (v^n_k)^2 = 0
\]

\[
(DS) \quad \frac{v^n_k - v^n_{k-1}}{a^n_k - a^n_{k-1}} - \left( \frac{(v^n_k)^2}{2} \right)_x + (v^n_k)^2 - (u^n_k)^2 = 0
\]

with \(u_0^n = u_0, v_0^n = v_0\). Then \(u_n(t), v_n(t) \to u(t), v(t)\) in \(L^1(\mathbb{R})\) uniformly on \([0, T]\). Let \(\Psi_n(t) = (v^n_k)^2 - (u^n_k)^2\) on \([a_{k-1}^n, a_k^n]\). Since \((u_0, v_0) \in L^1(\mathbb{R})^2_+ \cap L^\infty(\mathbb{R})^2_+\) then \(\Psi_n(t) \to \Psi(t) := v(t)^2 - u(t)^2\) in \(L^1([0, T], L^1(\mathbb{R}))\) as \(n \to \infty\). Thus \(u(t), v(t)\) are mild solutions of:

\[
\begin{cases}
  u_t + \left( \frac{u^2}{2} \right)_x = \Psi(t) \\
  v_t - \left( \frac{v^2}{2} \right)_x = \Psi(t) \\
  u(0) = u_0 \\
  v(0) = v_0
\end{cases}
\]
respectively therefore \((u(t), v(t))\) is the Kruzkov's solution of \((BC)\) in 
\([0, T] \times \mathbb{R}\) with initial data \((u_0, v_0)\) in the sense of Definition 1 ([1], Prop. 2.11).

Since \((u_0, v_0) \in L^\infty(\mathbb{R})^2\), then \(u, v \in L^\infty([0, 1] \times \mathbb{R})\).

Taking \(k > ||u(., .)||_\infty, k' > ||v(., .)||_\infty\) and then \(k < -||u(., .)||_\infty, k' < -||v(., .)||_\infty\) we see that \(u, v\) are distributional solutions of \((BC)\). We can now easily show the regularizing estimate \((RE)\) of theorem 1. Let \((u_0, v_0) \in L^1(\mathbb{R})_+\). Let \((u_{0n}, v_{0n}) \in L^1(\mathbb{R})_+ \cap L^\infty(\mathbb{R})_+\) be such that \(u_{0n} \uparrow u_0, v_{0n} \uparrow v_0\). Let \(u_n(t), v_n(t)\) be the solutions of \((BC)\) given by proposition 1. Using [2], Theorem 2, it follows that

\[
\frac{u_n(t+h) - u_n(t)}{h} \geq -\frac{1}{t+h} u_n(t)
\]

\[
\frac{v_n(t+h) - v_n(t)}{h} \geq -\frac{1}{t+h} v_n(t)
\]

for \(t, h > 0\). This implies that for any \(t > 0\) and any \(t \in [0, T]\) \(u_{nt} \geq -\frac{u_n}{t}, v_{nt} \geq -\frac{v_n}{t}\) in \(D'((0, T) \times \mathbb{R})\). It follows that

\[
\left(\frac{u_n^2 - v_n^2}{2}\right)_x \leq \frac{u_n + v_n}{t}
\]

in \(D'((0, T) \times \mathbb{R})\). Thus, for any \(\varphi \in C_0^\infty(\mathbb{R})\) with \(||\varphi||_\infty \leq 1, \varphi \geq 0\):

\[
\int_{\mathbb{R}} \frac{u_n^2(t, x) - v_n^2(t, x)}{2} \varphi'(x) dx \leq \frac{||u_n(t) + v_n(t)||_{L^1(\mathbb{R})}}{t} \leq \frac{||u_0 + v_0||_{L^1(\mathbb{R})}}{t}
\]

holds a.e. with respect to \(t\). Since \(u_n, v_n \in C([0, T], L^1(\mathbb{R})_+)\) it holds for all \(t \in [0, T]\). As we remarked above, since \((u_{0n}, v_{0n}) \in L^\infty(\mathbb{R})^2, (u_n(t), v_n(t) \in L^\infty(\mathbb{R})^2\). Then, \(u_n(t)^2 - v_n(t)^2 \in L^1(\mathbb{R})\). Now the following argument can be justified : let \(x_0\) be a Lebesgue point of \(u_n(t)^2 - v_n(t)^2\). For each \(k \in \mathbb{N}\), take \(\varphi_k(x) = 0\) if \(x < x_0, k(x-x_0)\) if \(x \in [x_0, x_0 + 1/k]\), \(1\) if \(x \geq x_0 + 1/k\). Plug \(\varphi_k\) into (4) to get :

\[
-k \int_x^{x+1/k} \frac{u_n^2(t, x) - v_n^2(t, x)}{2} dx \leq \frac{||u_n(t) + v_n(t)||_{L^1(\mathbb{R})}}{t}
\]

Since \(x_0\) is a Lebesgue point of \(u_n^2(t) - v_n^2(t)\), letting \(k \to \infty\) we get :

\[
-(u_n^2(t, x_0) - v_n^2(t, x_0)) \leq \frac{2}{t} \frac{||u_n(t) + v_n(t)||_{L^1(\mathbb{R})}}{t}
\]
Taking now \( \varphi_k(x) = 1 \) if \( x \leq x_0, 1-k(x-x_0) \) if \( x \in ]x_0, x_0+1/k[ \), \( 0 \) if \( x \geq x_0 \) and repeating the argument above, one gets:

\[
(u_n^2(t, x_0) - v_n^2(t, x_0)) \leq \frac{2}{t} ||u_n(t) + v_n(t)||_{L^1(\mathbb{R})}
\]

Therefore, \( u_n^2(t) - v_n^2(t) \in L^\infty([0, T] \times \mathbb{R}) \) and

\[
||u_n^2(t) - v_n^2(t)||_{\infty} \leq \frac{2}{t} ||u_n(t) + v_n(t)||_{L^1(\mathbb{R})}
\]

for all \( t \in ]0, T[ \). Since for \( a, b \geq 0, |a - b| \leq |a^2 - b^2|^{1/2} \), it follows that

\[
||u(t) - v(t)||_{\infty} \leq \sqrt{\frac{2}{t^{1/2}}} (||u_0 + v_0||_{L^1(\mathbb{R})})^{1/2}
\]

and

\[
||u_n^2(t) - v_n^2(t)||_{L^1(\mathbb{R})} \leq \sqrt{\frac{2}{t^{1/2}}} (||u_0 + v_0||_{L^1(\mathbb{R})})^{3/2}
\]

Since \( u_{nt} + \left( \frac{u_n^2}{2} \right)_x + u_n^2 - v_n^2 = 0 \) holds in \( D'((0, T) \times \mathbb{R}) \) then:

\[
\left( \frac{u_n^2}{2} \right)_x \leq u_n^2 - u_n^2 + \frac{u_n}{t}
\]

As before, this implies that \( u_n \in L^\infty([0, T] \times \mathbb{R}) \) and

\[
||u_n(t)||_{\infty} \leq \left\{ \frac{2}{t} ||u_0 + v_0||_{L^1(\mathbb{R})} + \frac{2\sqrt{2}}{t^{1/2}} (||u_0 + v_0||_{L^1(\mathbb{R})})^{3/2} \right\}^{1/2}
\]

for all \( n \in \mathbb{N} \) and \( t > 0 \). Letting \( n \to \infty \) we get (\( RE \)) for \( u(t) \). Similarly, (\( RE \)) holds for \( v(t) \).

Now, it is easy to show that for any \((u_0, v_0) \in L^1(\mathbb{R})^2_+\), the semigroup solution of \((BC)\) given by proposition 1 is in fact a Kruzkov's solution of \((BC)\). (\( RE \)) implies that \((u, v) \in L^\infty([0, T], L^1(\mathbb{R}))^2 \cap L^\infty([\tau, T] \times \mathbb{R})^2\) for any \( \tau > 0 \). Let \( u_{0n}, v_{0n} \in L^1(\mathbb{R})_+ \cap L^\infty(\mathbb{R})_+ \) be such that \( u_{0n} \uparrow u_0, v_{0n} \uparrow v_0 \). As has been proved above, the semigroup solutions \( u_n, v_n \) of \((BC)\) in \([0, T]\) with initial data \( u_{0n}, v_{0n} \) satisfy:

\[
\int_0^T \int_{\mathbb{R}} |u_n - k| \xi_t + \text{sign}_0(u_n - k) \left( \left( \frac{u_n^2}{2} - \frac{k^2}{2} \right) \xi_x + (u_n^2 - u_n^2)\xi \right) \, dx \, dt \geq 0
\]

(5)

\[
\int_0^T \int_{\mathbb{R}} |v_n - k'| \eta_t + \text{sign}_0(v_n - k') \left( \left( \frac{k'^2}{2} - \frac{v_n^2}{2} \right) \eta_x + (u_n^2 - v_n^2)\eta \right) \, dx \, dt \geq 0
\]
for all $\zeta, \eta \in C_0^\infty((0,T) \times \mathbb{R}), \zeta, \eta \geq 0$, all $k, k' \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Since $u_n, v_n$ satisfy the estimate $(RE)$, $u_n^2 - v_n^2 \to u^2 - v^2$ in $L^1([\tau, T] \times \mathbb{R})$ for any $\tau \in [0, T]$ and one can let $n \to \infty$ in (5) to get

$$
\int_0^T \int_\mathbb{R} \alpha(t, x, k) \left[ (u - k)\zeta_t + \left( \frac{u^2}{2} - \frac{k^2}{2} \right) \zeta_x + (v^2 - u^2)\zeta \right] \, dx \, dt \geq 0
$$

$$
\int_0^T \int_\mathbb{R} \beta(t, x, k') \left[ (v - k')\eta_t + \left( \frac{k'^2}{2} - \frac{v^2}{2} \right) \eta_x + (u^2 - v^2)\eta \right] \, dx \, dt \geq 0
$$

for all $\zeta, \eta \in C_0^\infty((0,T) \times \mathbb{R}), \zeta, \eta \geq 0$ and all $k, k' \in \mathbb{R}$ where $\alpha(t, x, k) \in \text{sign } (u(t, x) - k), \beta(t, x, k') \in \text{sign } (v(t, x) - k')$.

Using [1], Lemme 2.2, we see that $(u, v)$ is a Kruzkov’s solution of $(BC)$ on $[0, T] \times \mathbb{R}$ with initial data $(u_0, v_0)$.

The uniqueness of Kruzkov’s solutions of $(BC)$ follows easily adapting the arguments of [1], Sect. II. First of all we observe that if $(u, v), (\tilde{u}, \tilde{v})$ are Kruzkov’s solutions of $(BC)$ on $[0, T] \times \mathbb{R}$ with respective initial data $(u_0, v_0), (\tilde{u}_0, \tilde{v}_0) \in L^1(\mathbb{R})_+$ then ([1], Prop. 2.7) there exists some $\alpha(t, x) \in \text{sign } (u(t, x) - \tilde{u}(t, x)), \beta(t, x) \in \text{sign } (v(t, x) - \tilde{v}(t, x))$ such that

$$
\int_0^T \int_\mathbb{R} |u - \tilde{u}|\zeta_t + \alpha(t, x) \left[ \left( \frac{u^2 - \tilde{u}^2}{2} \right) \zeta_x + ((u^2 - u^2) - (\tilde{v}^2 - \tilde{u}^2))\zeta \right] \, dx \, dt \geq 0
$$

$$
\int_0^T \int_\mathbb{R} |v - \tilde{v}|\eta_t + \beta(t, x) \left[ \left( \frac{\tilde{v}^2 - v^2}{2} \right) \eta_x + ((u^2 - v^2) - (\tilde{u}^2 - \tilde{v}^2))\eta \right] \, dx \, dt \geq 0
$$

holds for all $\zeta, \eta \in C_0^\infty((0,T) \times \mathbb{R}), \zeta, \eta \geq 0$. Take $\zeta, \eta \in C_0^\infty((0,T) \times \mathbb{R}), \zeta \geq 0$ in both inequalities and add them. Then, observing that

$$
[(u^2 - \tilde{u}^2) - (v^2 - \tilde{v}^2)](\beta(t, x) - \alpha(t, x)) \zeta \leq 0 \text{ a.e.}
$$

one gets :

$$
\int_0^T \int_\mathbb{R} (|u - \tilde{u}| + |v - \tilde{v}|)\zeta_x + \left[ \left( \frac{u^2}{2} - \frac{\tilde{u}^2}{2} \right) \alpha + \left( \frac{v^2}{2} - \frac{\tilde{v}^2}{2} \right) \beta \right] \zeta_x \, dx \, dt \geq 0
$$

As in [1], Lemme 2.5, one obtains : for any $T \in [0, T]$ fixed

$$
\int_{|x| \leq R-C} |u(t, x) - \tilde{u}(t, x)| + |v(t, x) - \tilde{v}(t, x)| \, dx
\leq \int_{|x| \leq R-C} |u(s, x) - \tilde{u}(s, x)| + |v(s, x) - \tilde{v}(s, x)| \, dx
$$
for $0 < \tau \leq s \leq t \leq T$, where $C$ is the Lipschitz constant of the function $r \rightarrow \frac{r^2}{2}$ on $\{ |r| \leq \max(|u(t, x)|, |\bar{u}(t, x)|, |v(t, x)|, |\bar{v}(t, x)|) : t \in [\tau, T], x \in \mathbb{R} \}$ and $R > Ct$. Thus:

$$\int_{|x| \leq Ct} |u(t, x) - \bar{u}(t, x)| + |v(t, x) - \bar{v}(t, x)| dx \leq \int_{\mathbb{R}} |u(s, x) - \bar{u}(s, x)| + |v(s, x) - \bar{v}(s, x)| dx$$

(6)

for any $0 < \tau \leq s \leq t \leq T$. Since $(u(s), v(s)) \rightarrow (u_0, v_0)$ on $L^1(\mathbb{R})$ as $s \rightarrow 0$, letting $R \rightarrow \infty$ on (6) and then $\tau, s \rightarrow 0$ we get:

$$\int_{\mathbb{R}} |u(t, x) - \bar{u}(t, x)| + |v(t, x) - \bar{v}(t, x)| dx \leq \int_{\mathbb{R}} |u_0 - \bar{u}_0| + |v_0 - \bar{v}_0| dx$$

for any $t > 0$. From this estimate, the uniqueness of Kruzkov’s solutions of $(BC)$ follows. This finishes the proof of theorem 1.

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Références


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