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Pseudo-symmetry curvature conditions on hypersurfaces of Euclidean spaces and on Kahlerian manifolds

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I - Introduction

In this paper we study Riemannian manifolds satisfying the curvature condition \( R \cdot R = fQ(R) \) (this type of condition will be called a pseudo-symmetry curvature condition and will be explained in the next section). This condition arose during the study of umbilical hypersurfaces (see [AD], [DEP]) and is a generalization of the conditions \( \nabla R = 0 \) and \( R \cdot R = 0 \) (symmetric and semi-symmetric spaces [DDV]).

First, we study one simple case, namely isometric immersions into an \((N + 1)\)-dimensional Euclidean space of \( N \)-dimensional Riemannian manifolds satisfying this curvature condition or one of the related conditions \( R \cdot C = fQ(C) \) or \( R \cdot S = fQ(S) \) for the Weyl conformal curvature tensor \( C \) and the Ricci tensor \( S \). We obtain a full classification of the

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hypersurfaces satisfying one of these conditions. We show that there are many non-conformally flat Riemannian manifolds satisfying $R \cdot R = fQ(R)$ (in this respect, see [DDV, Theorem 5.1]). Furthermore, we obtain that each conformally flat hypersurface of a Euclidean space satisfies $R \cdot R = fQ(R)$. Theorems 1 and 3 show that each hypersurface of a Euclidean space satisfying $R \cdot C = fQ(C)$ satisfies $R \cdot R = fQ(R)$. This is related to a theorem of Deszcz and Grycak which states that each analytic Riemannian manifold satisfying $R \cdot C = fQ(C)$ also satisfies $R \cdot R = fQ(R)$ or $C = 0$ in case $N \geq 5$ (for a precise formulation, see [DG]; see also [G]). Concerning Kähler manifolds we obtained a stronger result: there are no Kähler manifolds that satisfy $R \cdot R = fQ(R)$ and for which $R \cdot R \neq 0$.

More precisely, we will prove the following theorems.

**Theorem 1.** Let $F : (M^n, g) \hookrightarrow E^{n+1}$ be an isometric immersion of a Riemannian manifold in a Euclidean space. Then $(M^n, g)$ satisfies $R \cdot R = fQ(R)$ if and only if for each point $p$ in $M$, $F$ has at most two distinct principal curvatures in $p$ or $R \cdot R = 0$ in $p$.

**Theorem 2.** Let $F : (M^n, g) \hookrightarrow E^{n+1}$ be an isometric immersion of a Riemannian manifold in a Euclidean space. Then $(M^n, g)$ satisfies $R \cdot S = fQ(S)$ if and only if for each point $p$ in $M$, $F$ has at most two distinct principal curvatures in $p$ or $R \cdot S = 0$ in $p$.

**Theorem 3.** Let $F : (M^n, g) \hookrightarrow E^{n+1}$ be an isometric immersion of a Riemannian manifold in a Euclidean space. Then $(M^n, g)$ satisfies $R \cdot C = fQ(C)$ if and only if for each point $p$ in $M$, $F$ has at most two distinct principal curvatures in $p$ or $R \cdot C = 0$ in $p$.

**Theorem 4.** Let $(M^n, J, g)$ be a Kähler manifold satisfying $R \cdot R = fQ(R)$. Then $(M^n, g)$ satisfies $R \cdot R = 0$.

**2 - Preliminaries**

Let $(M^n, g)$ be a (connected) $n$-dimensional Riemannian manifold ($N \geq 2$). In the following $X, Y, Z$ denote vector fields that are tangent to $M^n$. $\nabla$ is the Levi Civita connection of $(M^n, g)$ and $R$ is the Riemann-Christoffel curvature tensor of $(M^n, g)$. $\tilde{S}$ is the (1,1)-tensor related to the Ricci tensor $S$ of $(M^n, g)$ by $g(\tilde{S}X, Y) = S(X, Y)$ for all $X$ and $Y$. $\tau = tr \tilde{S}$ is the scalar curvature of $(M^n, g)$. $\nabla XY$ is the (1,1)-tensor field defined by
The Weyl conformal curvature tensor of \((M^N, g)\) (for \(N \geq 3\)) is defined by

\[
C(X, Y) := R(X, Y) - \frac{1}{N-2}(\tilde{S}X \wedge Y + X \wedge \tilde{S}Y) + \frac{\tau}{(N-1)(N-2)} X \wedge Y.
\]  

(2.1)

Let \(F : (M^N, g) \hookrightarrow \mathbb{E}^{N+1}\) be an isometric immersion of \((M^N, g)\) in an \((N+1)\)-dimensional Euclidean space. Let \(\xi\) be a local normal section of \(F\). Then the second fundamental form \(h\) and the second fundamental tensor \(A\) of \(F\) are defined by the formulas of Gauss and Weingarten:

\[
\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\xi \quad \text{and} \quad \tilde{\nabla}_X \xi = -AX \quad (\tilde{\nabla} \text{ is the standard connection of } \mathbb{E}^{N+1}).
\]

A is related to \(h\) by \(h(X, Y) = g(AX, Y)\). We will not distinguish between \(A_p\) and its matrix \((p \in M)\). The equation of Gauss is given by

\[
R(X, Y) = AX \wedge AY.
\]  

(2.2)

Let \(p \in M\). In the following \(x, y, z\) denote vectors in \(T_pM\). Let \(xAy\) denote the endomorphism \(T_pM \to T_pM : z \mapsto g(z, y)x - g(z, x)y\). Since \(A_p\) is symmetric, there exists an orthonormal \(\{e_1, \ldots, e_N\}\) consisting of eigenvectors of \(A_p\), i.e. such that

\[
A e_i = \lambda_i e_i,
\]  

(2.3)

where \(\lambda_i \in \mathbb{R}\) for each \(i \in \{1, \ldots, N\}\). \(\lambda_1, \ldots, \lambda_N\) are called the principal curvatures of \(F\) in \(p\). (2.1), (2.2) and (2.3) imply that

\[
R(e_i, e_j) = \lambda_i \lambda_j e_i \wedge e_j,
\]

\[
\tilde{S}e_i = \mu_i e_i,
\]

\[
C(e_i, e_j) = a_{ij} e_i \wedge e_j,
\]

(2.4)

where

\[
\mu_i = \lambda_i (tr A - \lambda_i),
\]

\[
a_{ij} = \lambda_i \lambda_j - \frac{1}{N-2} (\mu_i + \mu_j) + \frac{(tr A)^2 - tr A^2}{(N-1)(N-2)}
\]

for all \(i, j\) and \(k\) in \(\{1, \ldots, N\}\).

Let \(\bar{\lambda}_1, \ldots, \bar{\lambda}_r\) denote the mutually distinct eigenvalues of \(A_p\) with multiplicities \(s_1, \ldots, s_r\) respectively. Denote by \(V_\alpha\) the space of eigenvectors with eigenvalue \(\bar{\lambda}_\alpha (\alpha \in \{1, \ldots, r\})\). If \(e_i, e_k \in V_\alpha\) and \(e_j, e_\ell \in V_\beta\), then
\[ \mu_i = \mu_k \text{ and } a_{ij} = a_{kl}, \quad (i, j, k, l \in \{1, \ldots, N\} \text{ and } \alpha, \beta \in \{1, \ldots, r\}). \]

We define numbers \( \mu_\alpha := \mu_i \) and \( a_{\alpha \beta} := a_{ij} \) where \( e_i \in V_\alpha \) and \( e_j \in V_\beta \), \((i, j \in \{1, \ldots, N\} \text{ and } \alpha, \beta \in \{1, \ldots, r\})\).

Let \((M, J, g)\) be a Kähler manifold and let \( p \in M \). Then the following properties are well known:

\[ R(JX, JY) = R(X, Y) \quad (2.5) \]

and

\[ R(X, Y)J = JR(X, Y) \quad (2.6) \]

for all \( X \) and \( Y \) tangent to \( M \).

\((M^N, g)\) is called (locally) conformally flat if \((M^N, g)\) is (locally) conformally equivalent to \( E^N \). It is well known that \((M^N, g)\) is conformally flat if and only if \( C = 0 \) for \( N \geq 4 \). We recall that every surface is conformally flat and that \( C = 0 \) for every 3-dimensional Riemannian manifold. \( F \) is called quasi-umbilical if for each point \( p \) in \( M \) \( A_p \) has an eigenvalue with multiplicity at least \( N - 1 \). For \( N \geq 4 \), E.Cartan proved that \( F \) is quasi-umbilical if and only if \((M^N, g)\) is conformally flat. We remark that \( C = 0 \) in \( p \) if and only if \( A_p \) has an eigenvalue with multiplicity at least \( N - 1 \) if \( N \geq 4 \) (i.e. also the "pointwise" version of Cartan’s result holds).

Concerning the notations \( R \cdot C, R \cdot S, \ldots \) we say for example that \((M^N, g)\) satisfies \( R \cdot C = 0 \) if and only if \( R(X, Y) \cdot C = 0 \) for all vectorfields \( X \) and \( Y \) tangent to \( M \), where \( R(X, Y) \) acts as a derivation on the algebra of tensor fields on \( M \), i.e.

\[
(R(X, Y) \cdot C)(Z, U; V, W) = -C(R(X, Y)Z, U; V, W) \\
- C(Z, R(X, Y)U; V, W) - C(Z, U; R(X, Y)V, W) \\
- C(Z, U; V, R(X, Y)W)
\]

for \( X, Y, Z, U, V, W \) tangent to \( M^N \). The derivation \( R(X, Y) \cdot \) is the derivation \( \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \).

For every \((0, s)\)-tensor \( T \) on \( M \) a \((0, s + 2)\)-tensor \( Q(T) \) is defined by

\[
Q(T)(X_1, \ldots, X_s; Y, Z) = ((Y \wedge Z) \cdot T)(X_1, \ldots, X_s)
\]

(see, e.g. [T]). We say that a Riemannian manifold \((M^N, g)\) satisfies \( R \cdot T = fQ(T) \) if there exists a function \( f : M \to \mathbb{R} \) such that

\[
(R(Y, Z) \cdot T)(X_1, \ldots, X_s)(p) = f(p)Q(T)(X_1, \ldots, X_s; Y, Z)(p)
\]

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for every p in M and all X₁, . . . , Xₖ, Y, Z tangent to M.

3 - Proof of theorem 1

Suppose that F : (Mᴺ, g) \hookrightarrow E^{N+1} is an isometric immersion of a Riemannian manifold. Let p be a point in M and let \{e₁, . . . , eₙ\} be a basis for TₚM satisfying (2.3). From (2.4) it is easy to obtain that

\[(R(eᵢ, eⱼ) \cdot R)(eₖ, e₇; e₉, eₙ) - f(p)Q(R)(eₖ, e₇; e₉, eₙ; eᵢ, eⱼ) = \]

\[(= (f(p) - \lambdaᵢλⱼ)\{\delta_jk\lambda_iλₗ(δₖmδₗn - δₗmδₖn)
- δᵢkλⱼλₗ(δₗmδₖn - δₖmδₗn)
+ δₗmλᵢλₗ(δₖmδₗn - δₗmδₖn)
- δₗmλⱼλₗ(δᵢkδₖm - δₖmδᵢk)
+ δᵢkλⱼλₗ(δₗmδₖn - δₖmδₗn)
- δᵢkλₗλₗ(δₗmδₖn - δₖmδₗn)
+ δᵢkλₗλₗ(δₗmδₖn - δₖmδₗn)
- δᵢkλₗλₗ(δₖmδₗn - δₖmδₗn))\}

for all i, j, k, ℓ, m and n in {1, . . . , N}. Using this it can be verified that \(R \cdot R = fQ(R)\) in p if and only if \((R(eᵢ, eⱼ) \cdot R)(eₖ, e₇; e₉, eₙ) = f(p)Q(R)(eₖ, e₇; e₉, eₙ; eᵢ, eⱼ)\) for all mutually distinct i, j and k in \{1, . . . , N\}, i.e. if and only if

\[(f(p) - \lambdaᵢλⱼ)(\lambdaᵢ - \lambdaⱼ)λₖ = 0 \quad (3.1)\]

for all mutually distinct i, j and k in \{1, . . . , N\}.

Let \(\lambda₁, . . . , \lambdaᵣ\) be the mutually distinct eigenvalues of \(A(p)\) and denote their respective multiplicities by \(s₁, . . . , sᵣ\).

If \(r = 1\), it is clear from (3.1) that \(R \cdot R = fQ(R)\) in p.

If \(r = 2\), it is easy to see from (3.1) that \(R \cdot R = fQ(R)\) for \(f(p) = \lambda₁\lambda₂\).

Now suppose that \(r \geq 3\) and choose mutually distinct indices \(α, β\) and \(γ\) in \{1, . . . , r\}. Assume that \((M, g)\) satisfies \(R \cdot R = fQ(R)\) in p. (3.1) implies that

\[\lambda_β(f(p) - \lambda_α\lambda_γ) = 0 \quad (3.2)\]

and

\[\lambda_γ(f(p) - \lambda_α\lambda_β) = 0. \quad (3.3)\]

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Subtraction of (3.2) and (3.3) yields that \((\bar{\lambda}_\beta - \bar{\lambda}_\gamma)f(p) = 0\) from which we conclude that \(f(p) = 0\) and hence that \(R \cdot R = 0\) in \(p\). The converse is trivial (take \(f(p) = 0\)). This proves Theorem 1.

From Theorem 1 and the fact that a hypersurface of a Euclidean space is conformally flat if and only if it is quasi-umbilical it easily follows that each conformally flat hypersurface of a Euclidean space satisfies \(R \cdot R = fQ(R)\). Moreover it is now easy to give examples of non-conformally flat Riemannian manifolds satisfying \(R \cdot R = fQ(R)\): in a Euclidean space all hypersurfaces with exactly two principal curvatures with multiplicities at least two provide examples of such manifolds.

4 - Proof of theorem 2

Suppose that \(F : (M^N, g) \rightarrow E^{N+1}\) is an isometric immersion of a Riemannian manifold. Let \(p\) be a point in \(M\) and let \(\{e_1, \ldots, e_N\}\) be a basis for \(T_pM\) satisfying (2.3). From (2.4) it is easy to find that

\[
(R(e_i, e_j) \cdot S)(e_k, e_\ell) - f(p)Q(S)(e_k, e_\ell; e_i, e_j) =
(\ell = 0)
\]

for all \(i, j, k\) and \(\ell\) in \(\{1, \ldots, N\}\). It can be verified that \(R \cdot S = fQ(S)\) in \(p\) if and only if \((R(e_i, e_j) \cdot S)(e_i, e_j) = f(p)Q(S)(e_i, e_j; e_i, e_j)\) for all distinct \(i\) and \(j\) in \(\{1, \ldots, N\}\), i.e. if and only if

\[
(f(p) - \bar{\lambda}_i \bar{\lambda}_j)(\lambda_i - \lambda_j)(\text{tr} A - \lambda_i - \lambda_j) = 0 \tag{4.1}
\]

for all distinct \(i\) and \(j\) in \(\{1, \ldots, N\}\).

Denote by \(\bar{\lambda}_1, \ldots, \bar{\lambda}_r\) the mutually distinct eigenvalues of \(A(p)\) and let \(s_1, \ldots, s_r\) be their respective multiplicities. Then \(R \cdot S = fQ(S)\) in \(p\) if and only if

\[
(f(p) - \bar{\lambda}_\alpha \bar{\lambda}_\beta)(\text{tr} A - \bar{\lambda}_\alpha - \bar{\lambda}_\beta) = 0 \tag{4.2}
\]

for all distinct \(\alpha\) and \(\beta\) in \(\{1, \ldots, r\}\).

If \(r = 1\), then \(R \cdot S = fQ(S)\) in \(p\).

If \(r = 2\), then \(R \cdot S = fQ(S)\) in \(p\) for \(f(p) = \bar{\lambda}_1 \bar{\lambda}_2\).

Now assume that \(r \geq 3\). Choose mutually distinct indices \(\alpha, \beta\) and \(\gamma\) in \(\{1, \ldots, r\}\). Suppose that \((M, g)\) satisfies \(R \cdot S = fQ(S)\) in \(p\). Since \(\bar{\lambda}_\alpha, \bar{\lambda}_\beta\) and \(\bar{\lambda}_\gamma\) are mutually distinct we may assume that \(\text{tr} A - \bar{\lambda}_\alpha - \bar{\lambda}_\beta \neq 0\) and \(\text{tr} A - \bar{\lambda}_\alpha - \bar{\lambda}_\gamma \neq 0\). (4.2) now implies that \(f(p) - \bar{\lambda}_\alpha \bar{\lambda}_\beta = 0\) and \(f(p) - \bar{\lambda}_\alpha \bar{\lambda}_\gamma = 0\).
Subtraction yields that $\overline{\lambda}_\alpha = 0$ and hence that $f(p) = 0$, which means that $R \cdot S = 0$. The converse is trivial.

5 - Proof of theorem 3

Suppose that $F : (M^N, g) \hookrightarrow E^{N+1}$ is an isometric immersion of a Riemannian manifold. Let $p$ be a point in $M$ and let \{\(e_1, \ldots, e_N\}\} be a basis for $T_p M$ satisfying (2.3). From (2.4) it is easy to obtain that

\[
(R(e_i, e_j) \cdot C)(e_k, e_l; e_m, e_n) - f(p)Q(C)(e_k, e_l; e_m, e_n; e_i, e_j) = \\
= (f(p) - \lambda_i \lambda_j) \left\{ \delta_{jk} a_{il}(\delta_{in} \delta_{tm} - \delta_{im} \delta_{tn}) \\
- \delta_{ik} a_{jl}(\delta_{jn} \delta_{tm} - \delta_{jm} \delta_{tn}) \\
+ \delta_{jl} a_{ik}(\delta_{im} \delta_{kn} - \delta_{in} \delta_{km}) \\
- \delta_{il} a_{jk}(\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}) \\
+ \delta_{jm} a_{kl}(\delta_{it} \delta_{kn} - \delta_{ik} \delta_{tn}) \\
- \delta_{im} a_{kl}(\delta_{jt} \delta_{kn} - \delta_{jk} \delta_{tn}) \\
+ \delta_{jn} a_{kl}(\delta_{ik} \delta_{tm} - \delta_{it} \delta_{km}) \\
- \delta_{in} a_{kl}(\delta_{jk} \delta_{tm} - \delta_{jt} \delta_{km}) \right\}
\]

for all $i, j, k, l, m$ and $n$ in $\{1, \ldots, N\}$. Using this it can be verified that $R \cdot C = fQ(C)$ in $p$ if and only if $(R(e_i, e_j) \cdot C)(e_k, e_l; e_m, e_n) = f(p)Q(C)(e_k, e_l; e_m, e_n; e_i, e_j)$ for all mutually distinct $i, j$ and $k$ in $\{1, \ldots, N\}$, i.e. if and only if

\[
(f(p) - \lambda_i \lambda_j)(\lambda_i - \lambda_j)(\text{tr} ~ A - \lambda_i - \lambda_j - (N - 2)\lambda_k) = 0 \quad (5.1)
\]

for all mutually distinct $i, j$ and $k$ in $\{1, \ldots, N\}$. Let $\overline{\lambda}_1, \ldots, \overline{\lambda}_r$ be the mutually distinct eigenvalues of $A$ in $p$ and denote their respective multiplicities by $s_1, \ldots, s_r$.

If $r = 1$, it is clear from (5.1) that $R \cdot C = fQ(C)$ in $p$.

If $r = 2$, it is easy to see from (5.1) that $R \cdot C = fQ(C)$ in $p$ for $f(p) = \overline{\lambda}_1 \overline{\lambda}_2$.

Now suppose that $r \geq 3$ and assume that $(M, g)$ satisfies $R \cdot C = fQ(C)$ in $p$. Choose mutually distinct indices $\alpha, \beta$ and $\gamma$ in $\{1, \ldots, r\}$. Since $\overline{\lambda}_\alpha, \overline{\lambda}_\beta$ and $\overline{\lambda}_\gamma$ are mutually distinct we may suppose that $\text{tr} ~ A - \overline{\lambda}_\alpha - \overline{\lambda}_\gamma - (N - 2)\overline{\lambda}_\beta \neq 0$ and $\text{tr} ~ A - \overline{\lambda}_\beta - \overline{\lambda}_\gamma - (N - 2)\overline{\lambda}_\alpha \neq 0$. By (5.1) then, we obtain that $f(p) - \overline{\lambda}_\alpha \overline{\lambda}_\gamma = 0$ and $f(p) - \overline{\lambda}_\beta \overline{\lambda}_\gamma = 0$. It follows that $\overline{\lambda}_\gamma = 0$ and also that $f(p) = 0$ and hence $R \cdot C = 0$ in $p$. The converse is trivial.
Theorems 1 and 3 imply the following.

**COROLLARY.** — Let $F : (M^N, g) \hookrightarrow E^{N+1}$ be an isometric immersion of a Riemannian manifold in a Euclidean space. The following conditions are equivalent:

(i) $(M^N, g)$ satisfies $R \cdot R = fQ(R)$,
(ii) $(M^N, g)$ satisfies $R \cdot C = fQ(C)$.

**Proof.** — If $(M^N, g)$ satisfies $R \cdot R = fQ(R)$, then $(M^N, g)$ also satisfies $R \cdot S = fQ(S)$ since the derivations $R(X, Y)\cdot$ and $(X \wedge Y)\cdot$ commute with contractions (see Lemma 2.1 from [DDVV]). It is easy to see then that $(M^N, g)$ also satisfies $R \cdot C = fQ(C)$ (use a reasoning similar to the one in part (iii) of Lemma 2.1 in [DDVV]).

Suppose that $(M^N, g)$ satisfies $R \cdot C = fQ(C)$ and let $p$ be a point in $M$. There are two possibilities: (i) $A(p)$ has at most two distinct eigenvalues, or (ii) $A(p)$ has more than two distinct eigenvalues and $R \cdot C = 0$ in $p$. In the first case it is clear that $R \cdot R = fQ(R)$ in $p$ by Theorem 1. For the second case, it follows from Proposition 2 from [BVV] that $R \cdot R(p) = 0$ (use formula (3.1) with $f(p) = 0$).

**6 - Proof of theorem 4**

Suppose that $(M^N, J, g)$ is a Kähler manifold satisfying $R \cdot R = fQ(R)$. Suppose that $p$ is a point in $M$ for which $R \cdot R(p) \neq 0$. We will derive a contradiction.

It is clear that $f(p) \neq 0$. First, observe that

\[
Q(R)(u, v; Jz, Jw; x, y) = Q(R)(u, v; z, w; x, y)
\]

for all $x, y, u, v, z, w \in T_pM$. Indeed, using (2.5) and (2.6),

\[
Q(R)(u, v; Jz, Jw; x, y) = \frac{1}{f(p)} (R(x, y) \cdot R)(u, v; Jz, Jw)
\]

\[
= \frac{1}{f(p)} (R(x, y) \cdot R)(u, v; z, w)
\]

\[
= Q(R)(u, v; z, w; x, y).
\]

(6.1) and (2.5) imply that

\[
R(u, v; (x \wedge y)Jz, Jw) + R(u, v; Jz, (x \wedge y)Jw)
\]

\[
- R(u, v; (x \wedge y)z, w) - R(u, v; z, (x \wedge y)w) = 0.
\]

\[6.2\]
Let \( \{e_1, e_2, \ldots, e_N\} \) be an orthonormal basis for \( T_pM \). (6.2) yields that

\[
0 = \sum_{i=1}^{N} \{ R(u, v; (e_i \wedge y)Jz, Je_i) + R(u, v; Jz(e_i \wedge y)Je_i) \\
- R(u, v; (e_i \wedge y)z, e_i) - R(u, v; z, (e_i \wedge y)e_i) \} \\
= \left( \sum_{i=1}^{N} R(u, v; e_i, Je_i) \right) g(Jz, y) - (N - 2)R(u, v; z, y)
\]

for all \( u, v, z, y \in T_pM \).

Let \( x \in T_pM \setminus \{0\} \). By (6.3)

\[
\left( \sum_{i=1}^{N} R(u, v; e_i, Je_i) \right) g(Jx, Jx) = (N - 2)R(x, x; u, v)
= \left( \sum_{i=1}^{N} R(x, Jx; e_i, Je_i) \right) g(Ju, v)
\]

for all \( u, v \in T_pM \), which implies that

\[
\sum_{i=1}^{N} R(u, v; e_i, Je_i) = rg(Ju, v), \quad (6.4)
\]

for all \( u, v \in T_pM \), where

\[
r = \frac{\sum_{i=1}^{N} R(x, Jx; e_i, Je_i)}{g(Jx, Jx)}
\]

Combination of (6.3) and (6.4) gives that

\[
R(u, v; z, w) = \frac{r}{N - 2} g(Ju, v)g(Jz, w)
\]

for all \( u, v, z, w \in T_pM \). From (6.5) and (2.6) it is easy to see now that \( R \cdot R(p) = 0 \), which contradicts our initial assumption.

This proves that \( R \cdot R = 0 \) on \( M \).
References


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