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A note on elliptic functions and approximation by algebraic numbers of bounded degree


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A NOTE ON ELLIPTIC FUNCTIONS AND APPROXIMATION
BY ALGEBRAIC NUMBERS OF BOUNDED DEGREE

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Résumé : Soit $p$ une fonction elliptique de Weierstrass d’invariants $g_2$ et $g_3$ algébriques. Par un contre-exemple, on montre que pour l’obtention d’une minoration pour l’approximation simultanée de $p(a)$, $b$ et $p(ab)$ par des nombres algébriques de degré borné, une hypothèse supplémentaire sur les nombres $\beta$ qui approximent $b$ est nécessaire.

Summary : Let $p$ be a Weierstrass elliptic function with algebraic invariants $g_2$ and $g_3$. By a counterexample it is shown that lower bounds for the simultaneous approximation of $p(a)$, $b$ and $p(ab)$ by algebraic numbers of bounded degree cannot be given without an added hypothesis on the numbers $\beta$ approximating $b$.

Let $p$ be a Weierstrass elliptic function with algebraic invariants $g_2$, $g_3$ ; for $a,b \in \mathbb{C}$ such that $a$ and $ab$ are not poles of $p$, we consider the simultaneous approximation of $p(a)$, $b$ and $p(ab)$ by algebraic numbers. It was shown in [2], Theorem 2, that lower bounds for the approximation errors in terms of the heights and degrees of these algebraic numbers can only be given if the numbers $\beta$ used to approximate $b$ do not lie in the field $\mathbb{K}$ of complex multiplication of $p$. (As this condition is equivalent to the algebraic independence of $p(z)$ and $p(\beta z)$ as functions of $z$, the result proves the conjecture on admissible sets in Appendix 2 of [3].)

Now consider simultaneous approximation of the same numbers by algebraic numbers of bounded degree. The sequences of algebraic numbers constructed in [2] have rapidly rising
degrees, so they do not provide a relevant counterexample. It is the purpose of this note to show how the original example should be modified for the new problem.

Let \( \Omega = \mathbb{Z} \omega_1 + \mathbb{Z} \omega_2 \) denote the period lattice of \( \rho \), and \( \mathbb{Q} = \mathbb{Q}(g_2, g_3) \). For every \( d \in \mathbb{N} \), the set of \( z \in \mathbb{C} \setminus \Omega \) such that \( \rho(z) \) is algebraic of degree at most \( d \) is denoted by \( A_d \).

Let \( B \) be an open set in \( \mathbb{C} \) such that its closure \( \overline{B} \) is contained in the interior of the fundamental parallelogram \([0,1] \omega_1 + [0,1] \omega_2\).

**LEMMA 1.** For every \( d > 2 \), the set \( A_d \) is dense in \( \mathbb{C} \).

**Proof.** Let \( \mathcal{O} \subset \mathbb{C} \) be an arbitrary open set. Take a \( a \in \mathcal{O} \setminus \Omega \) with \( \rho'(a) \neq 0 \). According to [1], Chapter 4, Theorem 11, Corollary 2, there exist open sets \( U, V \) with \( a \in U \subset \mathcal{O}, \rho(a) \in V \), such that \( \rho \) induces a bijection from \( U \) onto \( V \). As \( \{ z \in \overline{\Omega} \mid \text{dg } z \leq d \} \) is dense in \( \mathbb{C} \), we can find \( z \in V \cap \overline{\mathcal{O}} \) with \( \text{dg } z \leq d \). For the unique \( u \in U \) with \( \rho(u) = z \), we have \( u \in \mathcal{O} \cap A_d \).

**LEMMA 2.** Assume \( d > 2 \). Then, for every \( g : \mathbb{N} \to \mathbb{R} \), there exist sequences \( (u_n)_{n=1}^\infty \), \( (\beta_n)_{n=1}^\infty \), \( (v_n)_{n=1}^\infty \), \( (\varepsilon_n)_{n=1}^\infty \), such that for all \( n \in \mathbb{N} \) the following statements are true:

1. \( u_n \in A_d \cap B, \beta_n \in [0,1] \cap \mathbb{Q}, v_n \in A_d, v_n = \beta_n u_n, \varepsilon_n \in ]0,1[ \);
2. \( e_{n+1} < \exp(-n \mid g(H_n) \mid) \), where \( H_n := \max(H(\rho(u_n)), H(\beta_n), H(\rho(v_n))) \);
3. \( e_{n+1} < e_n^2, e_{n+1} < \frac{1}{4} \text{ den}^{-4} \beta_n ; \)
4. \( 0 < |\beta_n - \beta_{n+1}| < e_{n+1}, |u_n - u_{n+1}| < e_{n+1} \).

**Proof.** Take \( u_1 \in A_d \cap B \) (the existence of such an \( u_1 \) follows from Lemma 1). Define \( v_1 := u_1, \beta_1 := 1, \varepsilon_1 := \frac{1}{2} \). Then (1) is true for \( n = 1 \). Now suppose \( u_1, \ldots, u_N, \beta_1, \ldots, \beta_N, v_1, \ldots, v_N, \varepsilon_1, \ldots, \varepsilon_N \) have been chosen in such a way that (1) holds for \( n = 1, \ldots, N \) and (2), (3), (4) hold for \( n = 1, \ldots, N - 1 \), and proceed by induction.

Choose \( \varepsilon_{n+1} \in ]0,1[ \) so small that (2) and (3) hold for \( n = N \). Take \( r > e^{-1}_{N+1} \) and consider the function \( f : \mathbb{C} \to \mathbb{C} \) defined by \( f(z) := rz \). As \( f \) is a continuous bijection, there exists an open set \( U \subset \mathbb{C} \) with \( fU \subset B \cap B(u_N, \varepsilon_{N+1}) \). Take \( w \in U \) such that \( \rho(w) \in \overline{\mathcal{O}} \) with \( \text{dg } \rho(w) \leq 2 \) (the existence of \( w \) again follows from Lemma 1). Define \( u_{N+1} := rw \). By Lemma 6.1 of [4], \( \rho(u_{N+1}) \in \overline{\mathcal{O}} \) and

\[ \text{dg } \rho(u_{N+1}) \leq [\mathbb{IF}(\rho(w)) : \mathcal{G}] \leq 2 [\mathbb{IF} : \mathbb{Q}] \leq d, \]

so \( u_{N+1} \in A_d \). Furthermore the definition of \( U \) gives \( u_{N+1} \in B \) and \( |u_N - u_{N+1}| < e_{N+1} \). Take
s \in \mathbb{N} with 0 \leq s \leq r and 0 < |\beta_N - \frac{s}{r}| < \epsilon_{N+1}; \text{ define } \beta_{N+1} := \frac{s}{r}; \text{ then } \beta_{N+1} \in [0,1] \cap \mathbb{Q} and (4) holds for n = N. Define \( v_{N+1} := \beta_{N+1} u_{N+1} = sw; \) then as above we find that \( v_{N+1} \in \mathbb{A}_d \) and (1) holds for \( n = N + 1. \)

**THEOREM.** Assume \( d \geq 2 \). Then, for every \( g : \mathbb{N} \to \mathbb{R} \), there exist \( a \in \mathbb{C}, b \in \mathbb{C} \setminus \mathbb{K}, \) such that \( a \) and \( ab \) are not poles of \( p \) and such that for every \( C \in \mathbb{R} \) there exist infinitely many tuples \((u, a, v) \in \mathbb{C}^3\) satisfying \( u, v \in \mathbb{A}_d, \beta \in \mathbb{Q} \) and

\[
\max(|p(a) - p(u)|, |b - \beta|, |p(ab) - p(v)|) < \exp(-Cg(H))
\]

while \( \max(H(p(u)), H(\beta), H(p(v))) \leq H. \)

**Proof.** According to Lemma 3 of [2], the sequences \((u_n)_{n=1}^{\infty}\) and \((\beta_n)_{n=1}^{\infty}\) constructed in Lemma 2 above are Cauchy sequences and their limits \( a, b \) satisfy

\[
\max(|a - u_n|, |b - \beta_n|) \leq e^{1/2}n+1
\]

for almost all \( n \). Thus \( a \in \overline{B} \) and therefore \( a \) cannot be a pole of \( p \). Formula (4) implies the existence of arbitrarily large \( n \) for which \( \beta_n \neq b \) ; as by (3) and (5), every \( \beta_n \) is a convergent of the continued fraction expansion of \( b \) and \( \lim \beta_n = b \), it follows that \( b \) has infinitely many convergents. Thus \( b \in \mathbb{R} \setminus \mathbb{Q} \) and therefore \( b \notin \mathbb{K}. \) On particular, \( b \neq 0 \); hence \( ab \) cannot be a pole of \( p \) either.

By the continuity of \( p \) in \( ab \), (5) implies

\[
\max(|p(a) - p(u_n)|, |b - \beta_n|, |p(ab) - p(v_n)|) \leq ce^{1/2}n+1
\]

for almost all \( n \), where \( c \) does not depend on \( n \). In the notation of (2), the right hand member of (6) satisfies

\[
ce^{1/2}n+1 < c \exp(-\frac{1}{2}n |g(H_n)|) \leq \exp(-Cg(H_n))
\]

if \( n \) is sufficiently large in terms of \( C \) and \( c. \)
REFERENCES


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