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COHOMOLOGY OF CR-SUBMANIFOLDS

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Résumé : Nous introduisons canoniquement une classe de cohomologie de Rham pour une CR-sous-variété compacte d'une variété kaehléienne. Cette classe de cohomologie est utilisée pour montrer que si un certain groupe de cohomologie de dimension paire d'une CR-sous-variété, N est trivial, alors, soit la distribution holomorphe de N n'est pas intégrable, soit la distribution totalement réelle de N n'est pas minimale.

Summary : We introduce a canonical de Rham cohomology class for a closed CR-submanifold in a Kaehler manifold. This cohomology class is used to prove that if some even-dimensional cohomology group of a CR-submanifold N is trivial, then either the holomorphic distribution of N is not integrable or the totally real distribution of N is not minimal.

1. INTRODUCTION

Let \( \tilde{M} \) be a Kaehler manifold with complex structure \( J \) and \( N \) a Riemannian manifold isometrically immersed in \( \tilde{M} \). Let \( \mathcal{D}_x \) be the maximal holomorphic subspace of the tangent space \( T_xN \), i.e., \( \mathcal{D}_x = T_xN \cap J(T_xN) \). If the dimension of \( \mathcal{D}_x \) is constant along \( N \), then \( \mathcal{D}_x \) defines a differentiable distribution \( \mathcal{D} \), called the holomorphic distribution of \( N \). A submanifold \( N \) in \( \tilde{M} \) is called a CR-submanifold \([1,2]\) if there exists on \( N \) a holomorphic distribution \( \mathcal{D} \) such that its orthogonal complement \( \mathcal{D}^\perp \) is a distribution satisfying \( \mathcal{D}_x^\perp \subset T_xN \), \( x \in N \). \( \mathcal{D}^\perp \) is called the totally real distribution of \( N \).
Let \( \mathcal{H} \) be a differentiable distribution on a Riemannian manifold \( N \) with Levi-Civita connection \( \nabla \). We put

\[
\mathcal{H}(X,Y) = (\nabla_X Y)_{\perp}
\]

for any vector fields \( X, Y \) in \( \mathcal{H} \), where \( (\nabla_X Y)_{\perp} \) denotes the component of \( \nabla_X Y \) in the orthogonal complementary distribution \( \mathcal{H}_{\perp} \) in \( N \). Let \( X_1, \ldots, X_r \) be an orthonormal basis of \( \mathcal{H} \), \( r = \text{dim}_\mathbb{R} \mathcal{H} \). If we put

\[
\bar{H} = \frac{1}{r} \sum_{i=1}^{r} \mathcal{H}(X_i, X_i).
\]

Then \( \bar{H} \) is a well-defined \( \mathcal{H}_{\perp} \)-valued vector field on \( N \) (up to sign), called the mean-curvature vector of \( \mathcal{H} \). A distribution \( \mathcal{K} \) on \( N \) is called minimal if the mean-curvature vector \( \bar{H} \) of \( \mathcal{K} \) vanishes identically.

The main purpose of this paper is to introduce a canonical cohomology class and use it to prove the following.

**Theorem 1.** Let \( N \) be a closed CR-submanifold of a Kähler manifold \( \tilde{M} \). If \( H^{2k}(N ; \mathbb{R}) = 0 \) for some \( k < \text{dim}_\mathbb{C} \mathcal{D} \), then either \( \mathcal{D} \) is not integrable or \( \mathcal{D}_{\perp} \) is not minimal.

**2. THE CANONICAL CLASS OF CR-SUBMANIFOLDS**

Let \( M \) be a Kähler manifold and \( N \) a CR-submanifold of \( \tilde{M} \). We denote by \( < , > \) the metric tensor of \( \tilde{M} \) as well as that induced on \( N \). Let \( \nabla \) and \( \tilde{\nabla} \) be the covariant differentiations on \( N \) and \( \tilde{M} \), respectively. The Gauss and Weingarten formulas are given respectively by

\[
\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y),
\]

\[
\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi
\]

for any vector fields \( X,Y \) tangent to \( N \) and any vector field \( \xi \) normal to \( N \). The second fundamental form \( \sigma \) and the second fundamental tensor \( A_\xi \) satisfy \( < A_\xi X, Y > = < \sigma(X,Y), \xi > \). We recall the following.

**Proposition 2** [2]. The totally real distribution \( \mathcal{D}_{\perp} \) of any CR-submanifold in any Kähler manifold is integrable.

For a CR-submanifold \( N \) of a Kähler manifold \( M \), we choose an orthogonal local
Let $e_1, \ldots, e_h, Je_1, \ldots, Je_h$ be the $2h$ 1-forms on $N$ satisfying
\begin{equation}
\omega^i(Z) = 0, \quad \omega^i(e_j) = \delta_{ij}, \quad i, j = 1, \ldots, 2h
\end{equation}
for any $Z \in \mathcal{D}$. Then
\begin{equation}
\omega = \omega^1 \wedge \ldots \wedge \omega^{2h}
\end{equation}
defines a $2h$-form on $N$. This form is a well-defined global $2h$-form on $N$ because $\mathcal{D}$ is orientable. We give the following.

**Theorem 3.** For any closed CR-submanifold $N$ of a Kaehler manifold $M$, the $2h$-form $\omega$ is closed which defines a canonical deRham cohomology class given by
\begin{equation}
c(N) = [\omega] \in H^{2h}(N; \mathbb{R}), \quad h = \dim_{\mathbb{C}} \mathcal{D}.
\end{equation}
Moreover, this cohomology class is nontrivial if $\mathcal{D}$ is integrable and $\mathcal{D} \perp$ is minimal.

**Proof.** First we give the following.

**Lemma 4.** If $N$ is a CR-submanifold of a Kaehler manifold $M$, then the holomorphic distribution $\mathcal{D}$ is minimal.

Let $X$ and $Z$ be vector fields in $\mathcal{D}$ and $\mathcal{D} \perp$, respectively. Then we have
\begin{equation}
< Z, \nabla_X X > = < JZ, \tilde{\nabla}_X JX > = - < \tilde{\nabla}_X JZ, X > = - < A_j Z X, JX > .
\end{equation}
Thus we find
\begin{equation}
< Z, \nabla_{JX} JX > = - < A_j Z X, X > = - < A_j Z X, JX > .
\end{equation}
Combining (2.6) and (2.7) we get $< \nabla_X X + \nabla_{JX} JX, Z > = 0$ from which we conclude that the holomorphic distribution $\mathcal{D}$ is minimal. This proves the lemma.

From (2.4) we have
\begin{equation}
d\omega = \sum_{i=1}^{2h} (-1)^i \omega^1 \wedge \ldots \wedge \omega^i \wedge \ldots \wedge \omega^{2h}.
\end{equation}
It is clear from (2.3) and (2.8) that $d\omega = 0$ if and only if
for any vectors $Z_1, Z_2 \in \mathcal{D}^\perp$ and $X_1, ..., X_{2n-1} \in \mathcal{D}$. However, it follows from straight-forward computation that (2.9) holds when and only when $\mathcal{D}^\perp$ is integrable and (2.10) holds when and only when $\mathcal{D}$ is minimal. But for a CR-submanifold in a Kaehler manifold these two conditions hold automatically (Propositition 2 and Lemma 4). Therefore, the $2h$-form $\omega$ is closed. Consequently, $\omega$ defines a deRham cohomology class $c(N)$ given by (2.5).

Let $e_1, ..., e_{2h+p}$ be an orthonormal local frame of $\mathcal{D}^\perp$ and let $\omega^{2h+1}, ..., \omega^{2h+p}$ be the $p$ 1-forms on $N$ satisfying $\omega^\alpha(X) = 0$ and $\omega^\alpha(e_p) = 0$ for any $X$ in $\mathcal{D}$, where $\alpha = 2h+1, ..., 2h+p$. Then by a similar argument for $\omega$, we may conclude that if $\mathcal{D}$ is integrable and $\mathcal{D}^\perp$ is minimal, then the $p$-form $\omega = \omega^{2h+1} \wedge ... \wedge \omega^{2h+p}$ is closed. Thus, the $2h$-form $\omega$ is coclosed, i.e., $\delta \omega = 0$. Since $N$ is a closed submanifold, $\omega$ is harmonic. Because $\omega$ is nontrivial, the cohomology class $[\omega]$ represented by $\omega$ is nontrivial in $H^{2h}(N; \mathbb{R})$. This proves the Theorem.

2. PROOF OF THEOREM 1

Let $N$ be a closed CR-submanifold of a complex $m$-dimensional Kaehler manifold $M$. Let $h = \dim_\mathbb{C} \mathcal{D}$ and $p = \dim_\mathbb{R} \mathcal{D}^\perp$. We choose an orthonormal local frame $e_1, ..., e_h, e_{h+1}, ..., e_{h+p}, e_{h+p+1}, ..., e_m$, $J_1, ..., J_m$ in $M$ in such a way that, restricted to $N$, $e_1, ..., e_h, J_1, ..., J_h$ are in $\mathcal{D}$ and $e_{h+1}, ..., e_{h+p}$ are in $\mathcal{D}^\perp$. We denote by $\omega^1, ..., \omega^m, \omega^1^*, ..., \omega^m^*$, the dual frame of $e_1, ..., e_1^*, ..., e_m^*$. We put

$$\theta^A = \omega^A + \sqrt{-1} \omega^{A*}, \overline{\theta^A} = \omega^A - \sqrt{-1} \omega^{A*}, \quad A = 1, ..., m.$$ 

Then, restrict $\theta^A$'s and $\overline{\theta^A}$'s to $N$, we have

$$\theta^\alpha = \overline{\theta^\alpha} = \omega^\alpha \quad \text{for} \quad \alpha = h+1, ..., h+p$$

$$\theta^r = \overline{\theta^r} = 0 \quad \text{for} \quad r = h+p+1, ..., m.$$ 

The Kaehler form $\widetilde{\Omega}$ of $\widetilde{M}$ is a closed 2-form on $\widetilde{M}$ given by

$$\widetilde{\Omega} = \frac{\sqrt{-1}}{2} \sum_A \theta^A \wedge \overline{\theta^A}.$$
Let $Q = i^*\widetilde{Q}$ be the 2-form on $N$ induced from $\widetilde{Q}$ via the immersion $i : N \rightarrow \tilde{M}$. Then, (3.1) and (3.2) give

$$\Omega = \frac{\sqrt{-1}}{2} \sum_{i=1}^{h} \theta^i \wedge \bar{\theta}^i.$$ 

It is clear that $\Omega$ is a closed 2-form on $N$ and it defines a cohomology class $[\Omega]$ in $H^2(N; IR)$. (2.4) and (3.3) imply that the canonical class $c(N)$ and the class $[\Omega]$ satisfy

$$[\Omega]^h = (-1)^h(h!);c(N).$$

If $\mathcal{D}$ is integrable and $\mathcal{D}^\perp$ is minimal, then Theorem 3 and (3.4) imply that $H^{2k}(N; IR) = 0$ for $k = 1, 2, \ldots, h$. (Q.E.D).

Because every hypersurface of a Kaehler manifold is a CR-hypersurface, Theorem 1 implies the following.

**COROLLARY 5.** Let $N$ be a $(2m-1)$-dimensional closed manifold with $H^{2k}(N; IR) = 0$ for some $k < m$. Then any immersion from $N$ into a (complex) $m$-dimensional Kaehler manifold $\tilde{M}$ is a CR-hypersurface such that either its holomorphic distribution is not integrable or its totally real distribution is not minimal.

*Remark.* CR-products of a Kaehler manifold are examples of CR-submanifold whose holomorphic distributions are integrable and whose totally real distributions are minimal. Therefore, the assumption on cohomology groups are necessary for Theorem 1.
REFERENCES


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