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COHOMOLOGY OF CR-SUBMANIFOLDS

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Résumé : Nous introduisons canoniquement une classe de cohomologie de Rham pour une CR-sous-variété compacte d'une variété kaehlerienne. Cette classe de cohomologie est utilisée pour montrer que si un certain groupe de cohomologie de dimension paire d'une CR-sous-variété, N est trivial, alors, soit la distribution holomorphe de N n'est pas intégrable, soit la distribution totalement réelle de N n'est pas minimale.

Summary : We introduce a canonical de Rham cohomology class for a closed CR-submanifold in a Kaehler manifold. This cohomology class is used to prove that if some even-dimensional cohomology group of a CR-submanifold N is trivial, then either the holomorphic distribution of N is not integrable or the totally real distribution of N is not minimal.

1. - INTRODUCTION

Let $\tilde{M}$ be a Kaehler manifold with complex structure $J$ and $N$ a Riemannian manifold isometrically immersed in $\tilde{M}$. Let $\mathscr{D}_x$ be the maximal holomorphic subspace of the tangent space $T_xN$, i.e., $\mathscr{D}_x = T_xN \cap J(T_xN)$. If the dimension of $\mathscr{D}_x$ is constant along $N$, then $\mathscr{D}_x$ defines a differentiable distribution $\mathcal{D}$, called the holomorphic distribution of $N$. A submanifold $N$ in $\tilde{M}$ is called a CR-submanifold [1,2] if there exists on $N$ a holomorphic distribution $\mathcal{D}$ such that its orthogonal complement $\mathcal{D}^\perp$ is a distribution satisfying $J\mathcal{D}_x^\perp \subset T_xN$, $x \in N$. $\mathcal{D}^\perp$ is called the totally real distribution of $N$. 
Let \( \mathcal{H} \) be a differentiable distribution on a Riemannian manifold \( N \) with Levi-Civita connection \( \nabla \). We put

\[
\mathcal{O}(X,Y) = (\nabla_X Y)^\perp
\]

for any vector fields \( X, Y \) in \( \mathcal{H} \), where \((\nabla_X Y)^\perp\) denotes the component of \( \nabla_X Y \) in the orthogonal complementary distribution \( \mathcal{K}^\perp \) in \( N \). Let \( X_1, \ldots, X_r \) be an orthonormal basis of \( \mathcal{H} \), \( r = \dim_{\mathbb{R}} \mathcal{H} \). If we put

\[
\vec{H} = \frac{1}{r} \sum_{i=1}^{r} \mathcal{O}(X_i, X_i).
\]

Then \( \vec{H} \) is a well-defined \( \mathcal{K}^\perp \)-valued vector field on \( N \) (up to sign), called the mean-curvature vector of \( \mathcal{H} \). A distribution \( \mathcal{K} \) on \( N \) is called minimal if the mean-curvature vector \( \vec{H} \) of \( \mathcal{K} \) vanishes identically.

The main purpose of this paper is to introduce a canonical cohomology class and use it to prove the following.

**THEOREM 1.** Let \( N \) be a closed CR-submanifold of a Kaehler manifold \( \tilde{M} \). If \( H^{2k}(N; \mathbb{R}) = 0 \) for some \( k \ll \dim_{\mathbb{C}} \mathcal{D} \), then either \( \mathcal{D} \) is not integrable or \( \mathcal{D}^\perp \) is not minimal.

2. **THE CANONICAL CLASS OF CR-SUBMANIFOLDS**

Let \( M \) be a Kaehler manifold and \( N \) a CR-submanifold of \( \tilde{M} \). We denote by \( <, > \) the metric tensor of \( \tilde{M} \) as well as that induced on \( N \). Let \( \tilde{\nabla} \) and \( \tilde{\nabla} \) be the covariant differentiations on \( N \) and \( \tilde{M} \), respectively. The Gauss and Weingarten formulas are given respectively by

\[
\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y),
\]

\[
\tilde{\nabla}_X \xi = -A_{\xi} X + D_X \xi
\]

for any vector fields \( X, Y \) tangent to \( N \) and any vector field \( \xi \) normal to \( N \). The second fundamental form \( \sigma \) and the second fundamental tensor \( A_{\xi} \) satisfy \( <A_{\xi} X, Y> = <\sigma(X,Y), \xi> \). We recall the following.

**PROPOSITION 2 [2].** The totally real distribution \( \mathcal{D}^\perp \) of any CR-submanifold in any Kaehler manifold is integrable.

For a CR-submanifold \( N \) of a Kaehler manifold \( M \), we choose an orthogonal local
frame $e_1,...,e_h,J e_1,...,J e_h$ of $D$. Let $\omega^1,...,\omega^h,\omega^{h+1},...,\omega^{2h}$ be the $2h$ 1-forms on $N$ satisfying

\begin{equation}
\omega^j(Z) = 0, \quad \omega^j(e_i) = \delta_{ij}, \quad i, j = 1,\ldots, 2h
\end{equation}

for any $Z \in D \perp$ where $e_{h+j} = J e_j$. Then

\begin{equation}
\omega = \omega^1 \wedge \ldots \wedge \omega^{2h}
\end{equation}

defines a $2h$-form on $N$. This form is a well-defined global $2h$-form on $N$ because $D$ is orientable. We give the following.

**THEOREM 3.** For any closed CR-submanifold $N$ of a Kaehler manifold $M$, the $2h$-form $\omega$ is closed which defines a canonical deRham cohomology class given by

\begin{equation}
\text{c}(N) = [\omega] \in H^{2h}(N; \mathbb{R}), \quad h = \dim_D D.
\end{equation}

Moreover, this cohomology class is nontrivial if $D$ is integrable and $D \perp$ is minimal.

**Proof.** First we give the following.

**LEMMA 4.** If $N$ is a CR-submanifold of a Kaehler manifold $M$, then the holomorphic distribution $D$ is minimal.

Let $X$ and $Z$ be vector fields in $D$ and $D \perp$, respectively. Then we have

\begin{equation}
< Z, \nabla_X X > = < JZ, \nabla_X JX > = - < \nabla_X JZ, JX > = < A JZ X, JX >.
\end{equation}

Thus we find

\begin{equation}
\end{equation}

Combining (2.6) and (2.7) we get $< \nabla_X X + \nabla_J X JX, Z > = 0$ from which we conclude that the holomorphic distribution $D$ is minimal. This proves the lemma.

From (2.4) we have

\begin{equation}
d\omega = \sum_{i=1}^{2h} (-1)^i \omega^1 \wedge \ldots \wedge \omega^i \wedge \ldots \wedge \omega^{2h}.
\end{equation}

It is clear from (2.3) and (2.8) that $d\omega = 0$ if and only if
for any vectors \( Z_1, Z_2 \in \mathcal{D} \perp \) and \( X_1, \ldots, X_{2h-1} \in \mathcal{D} \). However, it follows from straight-forward computation that (2.9) holds when and only when \( \mathcal{D} \perp \) is integrable and (2.10) holds when and only when \( \mathcal{D} \) is minimal. But for a CR-submanifold in a Kaehler manifold these two conditions hold automatically (Proposition 2 and Lemma 4). Therefore, the 2h-form \( \omega \) is closed. Consequently, \( \omega \) defines a deRham cohomology class \( c(N) \) given by (2.5).

Let \( e_{2h+1}, \ldots, e_{2h+p} \) be an orthonormal local frame of \( \mathcal{D} \perp \) and let \( \omega^{2h+1}, \ldots, \omega^{2h+p} \) be the p 1-forms on \( N \) satisfying \( \omega^\alpha(X) = 0 \) and \( \omega^\alpha(e_\beta) = 0 \) for any \( X \) in \( \mathcal{D} \), where \( \alpha, \beta = 2h+1, \ldots, 2h+p \).

Then by a similar argument for \( \omega \), we may conclude that if \( \mathcal{D} \) is integrable and \( \mathcal{D} \perp \) is minimal, then the p-form \( \omega = \omega^{2h+1} \Lambda \ldots \Lambda \omega^{2h+p} \) is closed. Thus, the 2h-form \( \omega \) is coclosed, i.e., \( \delta \omega = 0 \). Since \( N \) is a closed submanifold, \( \omega \) is harmonic. Because \( \omega \) is nontrivial, the cohomology class [\( \omega \)] represented by \( \omega \) is nontrivial in \( H^{2h}(N; \mathbb{R}) \). This proves the Theorem.

2. PROOF OF THEOREM 1

Let \( N \) be a closed CR-submanifold of a complex \( m \)-dimensional Kaehler manifold \( M \).

Let \( h = \dim_{\mathbb{R}} \mathcal{D} \) and \( p = \dim_{\mathbb{R}} \mathcal{D} \perp \). We choose an orthonormal local frame

\[
e_1, \ldots, e_h, e_{h+1}, \ldots, e_{h+p}, e_{h+p+1}, \ldots, e_m, j_1, \ldots, j_m
\]

in \( \tilde{M} \) in such a way that, restricted to \( N \), \( e_1, \ldots, e_h, j_1, \ldots, j_h \) are in \( \mathcal{D} \) and \( e_{h+1}, \ldots, e_{h+p} \) are in \( \mathcal{D} \perp \). We denote by \( \omega^1, \ldots, \omega^m, \omega^1*, \ldots, \omega^m* \), the dual frame of \( e_1, \ldots, e_m, j_1, \ldots, j_m \). We put

\[
\theta^A = \omega^A + \sqrt{-1} \omega^{A*}, \quad \bar{\theta}^A = \omega^A - \sqrt{-1} \omega^{A*}, \quad A = 1, \ldots, m.
\]

Then, restrict \( \theta^A \)'s and \( \bar{\theta}^A \)'s to \( N \), we have

\[
\theta^\alpha = \bar{\theta}^\alpha = \omega^\alpha \quad \text{for} \quad \alpha = h+1, \ldots, h+p
\]

\[
\theta^r = \bar{\theta}^r = 0 \quad \text{for} \quad r = h+p+1, \ldots, m.
\]

The Kaehler form \( \tilde{\Omega} \) of \( \tilde{M} \) is a closed 2-form on \( \tilde{M} \) given by

\[
\tilde{\Omega} = \frac{\sqrt{-1}}{2} \sum A \theta^A \Lambda \bar{\theta}^A.
\]
Let $\Omega = i^*\widetilde{\Omega}$ be the 2-form on $N$ induced from $\widetilde{\Omega}$ via the immersion $i : N \to \widetilde{M}$. Then, (3.1) and (3.2) give

$$\Omega = \frac{\sqrt{-1}}{2} \sum_{i=1}^{h} \theta^i \wedge \overline{\theta}^i.$$  

(3.3)

It is clear that $\Omega$ is a closed 2-form on $N$ and it defines a cohomology class $[\Omega]$ in $H^2(N ; \mathbb{R})$. (2.4) and (3.3) imply that the canonical class $c(N)$ and the class $[\Omega]$ satisfy

$$[\Omega]^h = (-1)^{h}(h!) c(N).$$  

(3.4)

If $\mathcal{D}$ is integrable and $\mathcal{D}^\perp$ is minimal, then Theorem 3 and (3.4) imply that $H^{2k}(N ; \mathbb{R}) = 0$ for $k = 1, 2, \ldots, h$. (Q.E.D).

Because every hypersurface of a Kaehler manifold is a CR-hypersurface, Theorem 1 implies the following.

COROLLARY 5. Let $N$ be a $(2m-1)$-dimensional closed manifold with $H^{2k}(N ; \mathbb{R}) = 0$ for some $k < m$. Then any immersion from $N$ into a (complex) $m$-dimensional Kaehler manifold $\widetilde{M}$ is a CR-hypersurface such that either its holomorphic distribution is not integrable or its totally real distribution is not minimal.

Remark. CR-products of a Kaehler manifold are examples of CR-submanifold whose holomorphic distributions are integrable and whose totally real distributions are minimal. Therefore, the assumption on cohomology groups are necessary for Theorem 1.
REFERENCES


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