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COHOMOLOGY OF CR-SUBMANIFOLDS

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Résumé : Nous introduisons canoniquement une classe de cohomologie de Rham pour une CR-sous-variété compacte d'une variété kaehlérienne. Cette classe de cohomologie est utilisée pour montrer que si un certain groupe de cohomologie de dimension paire d'une CR-sous-variété, N est trivial, alors, soit la distribution holomorphe de N n'est pas intégrable, soit la distribution totalement réelle de N n'est pas minimale.

Summary : We introduce a canonical de Rham cohomology class for a closed CR-submanifold in a Kaehler manifold. This cohomology class is used to prove that if some even-dimensional cohomology group of a CR-submanifold N is trivial, then either the holomorphic distribution of N is not integrable or the totally real distribution of N is not minimal.

1. - INTRODUCTION

Let $\widetilde{M}$ be a Kaehler manifold with complex structure J and N a Riemannian manifold isometrically immersed in $\widetilde{M}$. Let $\mathscr{D}_x$ be the maximal holomorphic subspace of the tangent space T_xN, i.e., $\mathscr{D}_x = T_xN \cap J(T_xN)$. If the dimension of $\mathscr{D}_x$ is constant along N, then $\mathscr{D}_x$ defines a differentiable distribution $\mathscr{D}$, called the *holomorphic distribution* of N. A submanifold N in $\widetilde{M}$ is called a *CR-submanifold* [1,2] if there exists on N a holomorphic distribution $\mathscr{D}$ such that its orthogonal complement $\mathscr{D}^\perp$ is a distribution satisfying $J\mathscr{D}^\perp_x \subset T^\perp_x N$, $x \in N$. $\mathscr{D}^\perp$ is called the *totally real distribution* of N.

Let $\mathcal{H}$ be a differentiable distribution on a Rieamnnian manifold N with Levi-Civita connection ∇ . We put

$$\overset{o}{\sigma}(X,Y) = (\nabla_X Y)^{\perp} \tag{1.1}$$

for any vector fields X,Y in $\mathcal{H}$, where $(\nabla_X Y)^{\perp}$ denotes the component of $\nabla_X Y$ in the orthogonal complementary distribution $\mathcal{H}^{\perp}$ in N. Let $X_1,\ldots,X_r$ be an orthonormal basis of $\mathcal{H}$, $r = \dim_{\mathbb{R}} \mathcal{H}$. If we put

$$\overset{o}{H} = \frac{1}{r}\sum_{i=1}^{r} \overset{o}{\sigma}(X_i,X_i). \tag{1.2}$$

Then $\overset{o}{H}$ is a well-defined $\mathcal{H}^{\perp}$-valued vector field on N (up to sign), called the *mean-curvature vector* of $\mathcal{H}$. A distribution $\mathcal{H}$ on N is called *minimal* if the mean-curvature vector $\overset{o}{H}$ of $\mathcal{H}$ vanishes identically.

The main purpose of this paper is to introduce a canonical cohomology class and use it to prove the following.

THEOREM 1. *Let* N *be a closed* CR-*submanifold of a Kaehler manifold* $\tilde{M}$. *If* $H^{2k}(N\,;\mathbb{R}) = 0$ *for some* $k \leqslant \dim_{\mathbb{C}} \mathcal{D}$, *then either* $\mathcal{D}$ *is not integrable or* $\mathcal{D}^{\perp}$ *is not minimal.*

2. - THE CANONICAL CLASS OF CR-SUBMANIFOLDS

Let M be a Kaehler manifold and N a CR-submanifold of $\tilde{M}$. We denote by $<\ ,\ >$ the metric tensor of $\tilde{M}$ as well as that induced on N. Let ∇ and $\tilde{\nabla}$ be the covariant differentiations on N and $\tilde{M}$, respectively. The Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = - A_{\xi} X + D_X \xi \tag{2.2}$$

for any vector fields X,Y tangent to N and any vector field ξ normal to N. The second fundamental form σ and the second fundamental tensor A_{ξ} satisfy $< A_{\xi}X,Y > = < \sigma(X,Y),\xi >$. We recall the following.

PROPOSITION 2 [2] . *The totally real distribution* $\mathcal{D}^{\perp}$ *of any* CR-*submanifold in any Kaehler manifold is integrable.*

For a CR-submanifold N of a Kaehler manifold M, we choose an orthogonal local frame $e_1,\dots,e_h,Je_1,\dots,Je_h$ of $\mathscr{D}$. Let $\omega^1,\dots,\omega^h,\omega^{h+1},\dots,\omega^{2h}$ be the 2h 1-forms on N satisfying

$$\omega^j(Z)=0,\ \omega^i(e_j)=\delta_{ij}\,,\ i,j=1,\dots,2h \tag{2.3}$$

for any $Z\in\mathscr{D}^\perp$ where $e_{h+j}=Je_j$. Then

$$\omega=\omega^1\wedge\dots\wedge\omega^{2h} \tag{2.4}$$

defines a 2h-form on N. This form is a well-defined global 2h-form on N because $\mathscr{D}$ is orientable. We give the following.

THEOREM 3. *For any closed* CR-*submanifold* N *of a Kaehler manifold* $\tilde{M}$, *the* 2h-*form* ω *is closed which defines a canonical deRham cohomology class given by*

$$c(N)=[\omega]\in H^{2h}(N\,;\mathbb{R}),\ h=\dim_{\mathbb{C}}\mathscr{D}\ . \tag{2.5}$$

Moreover, this cohomology class is nontrivial if $\mathscr{D}$ is integrable and $\mathscr{D}^\perp$ is minimal.

Proof. First we give the following.

LEMMA 4. *If* N *is a* CR-*submanifold of a Kaehler manifold* $\tilde{M}$, *then the holomorphic distribution $\mathscr{D}$ is minimal.*

Let X and Z be vector fields in $\mathscr{D}$ and $\mathscr{D}^\perp$, respectively. Then we have

$$<Z,\nabla_X X> \;=\; <JZ,\tilde{\nabla}_X JX>=-<\tilde{\nabla}_X JZ,JX> \;=\; <A_{JZ}X,JX>\ . \tag{2.6}$$

Thus we find

$$<Z,\nabla_{JX}JX> \;=-<A_{JZ}JX,X> \;=-<A_{JZ}X,JX>. \tag{2.7}$$

Combining (2.6) and (2.7) we get $<\nabla_X X+\nabla_{JX}JX,Z>\;=0$ from which we conclude that the holomorphic distribution $\mathscr{D}$ is minimal. This proves the lemma.

From (2.4) we have

$$d\omega=\sum_{i=1}^{2h}(-1)^i\omega^1\wedge\dots\wedge d\omega^i\wedge\dots\wedge\omega^{2h}. \tag{2.8}$$

It is clear from (2.3) and (2.8) that $d\omega=0$ if and only if

$$d\omega(Z_1,Z_2,X_1,\ldots,X_{2h-1}) = 0 \tag{2.9}$$

$$d\omega(Z_1,X_1,\ldots,X_{2h}) = 0 \tag{2.10}$$

for any vectors $Z_1,Z_2 \in \mathscr{D}^\perp$ and $X_1,\ldots,X_{2h-1} \in \mathscr{D}$. However, it follows from straight-forward computation that (2.9) holds when and only when $\mathscr{D}^\perp$ is integrable and (2.10) holds when and only when $\mathscr{D}$ is minimal. But for a CR-submanifold in a Kaehler manifold these two conditions hold automatically (Propositition 2 and Lemma 4). Therefore, the 2h-form ω is closed. Consequently, ω defines a deRham cohomology class c(N) given by (2.5).

Let $e_{2h+1},\ldots,e_{2h+p}$ be an orthonormal local frame of $\mathscr{D}^\perp$ and let $\omega^{2h+1},\ldots,\omega^{2h+p}$ be the p 1-forms on N satisfying $\omega^\alpha(X) = 0$ and $\omega^\alpha(e_\beta) = 0$ for any X in $\mathscr{D}$, where $\alpha,\beta = 2h+1,\ldots,2h+p$. Then by a similar argument for ω, we may conclude that if $\mathscr{D}$ is integrable and $\mathscr{D}^\perp$ is minimal, then the p-form $\omega^\perp = \omega^{2h+1} \wedge \ldots \wedge \omega^{2h+p}$ is closed. Thus, the 2h-form ω is coclosed, i.e., $\delta\omega = 0$. Since N is a closed submanifold, ω is harmonic. Because ω is nontrivial, the cohomology class $[\omega]$ represented by ω is nontrivial in $H^{2h}(N\,;\mathbb{R})$. This proves the Theorem.

2. - PROOF OF THEOREM 1

Let N be a closed CR-submanifold of a complex m-dimensional Kaehler manifold M. Let $h = \dim_{\mathbb{C}} \mathscr{D}$ and $p = \dim_{\mathbb{R}} \mathscr{D}^\perp$. We choose an orthonormal local frame

$$e_1,\ldots,e_h,e_{h+1},\ldots,e_{h+p},e_{h+p+1},\ldots,e_m,Je_1,\ldots,Je_m$$

in $\widetilde{M}$ in such a way that, restricted to N, $e_1,\ldots,e_h,Je_1,\ldots,Je_h$ are in $\mathscr{D}$ and $e_{h+1},\ldots,e_{h+p}$ are in $\mathscr{D}^\perp$. We denote by $\omega^1,\ldots,\omega^m,\omega^{1^*},\ldots,\omega^{m^*}$, the dual frame of $e_1,\ldots,e_m,Je_1,\ldots,Je_m$. We put

$$\theta^A = \omega^A + \sqrt{-1}\,\omega^{A^*},\ \bar\theta^A = \omega^A - \sqrt{-1}\,\omega^{A^*},\ A = 1,\ldots,m.$$

Then, restrict θ^A's and θ^{A^*}'s to N, we have

$$\theta^\alpha = \bar\theta^\alpha = \omega^\alpha \quad \text{for} \quad \alpha = h+1,\ldots,h+p \tag{3.1}$$

$$\theta^r = \bar\theta^r = 0 \quad \text{for} \quad r = h+p+1,\ldots,m.$$

The Kaehler form $\widetilde{\Omega}$ of $\widetilde{M}$ is a closed 2-form on $\widetilde{M}$ given by

$$\widetilde{\Omega} = \frac{\sqrt{-1}}{2} \sum_A \theta^A \wedge \bar\theta^A. \tag{3.2}$$

Let $\Omega = i^*\widetilde{\Omega}$ be the 2-form on N induced from $\widetilde{\Omega}$ via the immersion $i : N \to \widetilde{M}$. Then, (3.1) and (3.2) give

$$\Omega = \frac{\sqrt{-1}}{2} \sum_{i=1}^{h} \theta^i \wedge \bar{\theta}^i . \tag{3.3}$$

It is clear that Ω is a closed 2-form on N and it defines a cohomology class $[\Omega]$ in $H^2(N ; \mathbb{R})$. (2.4) and (3.3) imply that the canonical class c(N) and the class $[\Omega]$ satisfy

$$[\Omega]^h = (-1)^h (h!) c(N). \tag{3.4}$$

If $\mathscr{D}$ is integrable and $\mathscr{D}^\perp$ is minimal, then Theorem 3 and (3.4) imply that $H^{2k}(N ; \mathbb{R}) \neq 0$ for $k = 1,2,\ldots,h$. (Q.E.D).

Because every hypersurface of a Kaehler manifold is a CR-hypersurface, Theorem 1 implies the following.

COROLLARY 5. *Let* N *be a* (2m−1)*-dimensional closed manifold with* $H^{2k}(N ; \mathbb{R}) = 0$ *for some* $k < m$. *Then any immersion from* N *into a (complex)* m*-dimensional Kaehler manifold* $\widetilde{M}$ *is a* CR-*hypersurface such that either its holomorphic distribution is not integrable or its totally real distribution is not minimal.*

Remark. CR-products of a Kaehler manifold are examples of CR-submanifold whose holomorphic distributions are integrable and whose totally real distributions are minimal. Therefore, the assumption on cohomology groups are necessary for Theorem 1.

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