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ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF A STRONGLY NONLINEAR PARABOLIC PROBLEM

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Résumé : Nous étudions le problème d’évolution \( u_t + Au = 0 \) dans \((0,T) \times \mathbb{R}^N\), \( u(0) = u_0 \) dans \( \mathbb{R}^N \), avec \( N \geq 1 \), \( 0 < T < \infty \), \( Au = - \text{div} (|Du|^{p-2} Du) \), \( Du \) étant le gradient de \( u \), \( 1 < p < \infty \) et nous supposons que \( u_0 \) appartient à un espace de fonctions intégrables. On prouve l’existence d’un temps fini d’extinction si \( N \geq 2 \) et \( p < \frac{2N}{N+1} \). Dans le cas contraire (si \( N = 1 \) et \( p > 1 \) ou si \( N \geq 2 \) et \( p \geq \frac{2N}{N+1} \)) on prouve la loi de conservation : \( \int_{\mathbb{R}^N} u(t,x) dx = \int_{\mathbb{R}^N} u_0(x) dx \) pour tout \( t > 0 \). On estime aussi la convergence vers zéro des intégrales \( \int_{\mathbb{R}^N} |u(t,x)|^m dx \), \( m > 1 \) et on obtient certains effets régularisants.

Summary : The evolution problem \( u_t + Au = 0 \) in \((0,T) \times \mathbb{R}^N\), \( u(0) = u_0 \) in \( \mathbb{R}^N \) is considered where \( N \geq 1 \), \( 0 < T \leq \infty \), \( Au = - \text{div} (|Du|^{p-2} Du) \), with \( Du \) the gradient of \( u \), \( 1 < p < \infty \) and \( u \) is supposed to belong to some integrable space. If \( N \geq 2 \) and \( p < \frac{2N}{N+1} \) the existence of a finite extinction time is shown. On the contrary, if \( N = 1 \), \( p > 1 \) or \( N \geq 2 \), \( p \geq \frac{2N}{N+1} \) conservation of total mass holds, i.e. \( \int_{\mathbb{R}^N} u(t,x) dx = \int_{\mathbb{R}^N} u_0(x) dx \) for every \( t > 0 \). We prove also that the integrals \( \int_{\mathbb{R}^N} |u(t,x)|^m dx \), \( m > 1 \) converge to zero as \( t \) goes to infinity, and some regularizing effects are shown.
INTRODUCTION AND PRELIMINARIES

We shall consider the asymptotic behaviour in time of the solutions of

\[
\begin{cases}
  u_t + Au = 0 & \text{in } (0,T) \times \mathbb{R}^N \\
  u(0) = u_0 & \text{in } \mathbb{R}^N
\end{cases}
\]

with $N > 1$, $1 < p < \infty$ and $Au = -\sum_{i=1}^{N} \frac{\partial (|Du|^{p-2} Du)}{\partial x_i}$ where $Du = \left( \frac{\partial u}{\partial x_i} \right)$ is the gradient of $u$. The operator $A$ has been widely considered in the literature in P.D.E., and arises in several physical situations, such as one-dimensional non newtonian fluids and glaciology.

This behaviour depends strongly on $p$ and $N$: in fact, if $p \geq \frac{2N}{N+1}$ we show that the total mass $\int_{\mathbb{R}^N} u(t,x) dx$ is conserved, i.e., is independent of time. On the contrary if $p < \frac{2N}{N+1}$ we show that the solution corresponding to initial data $u_0 \in L^m(\mathbb{R}^N)$, $m = N \left( \frac{N-1}{p} \right)$ vanishes in finite time. The existence of a finite extinction time was found by Bénilan and Crandall [2] for the equation $u_t - \Delta u^m = 0$ in spatial domain $\mathbb{R}^N$ (1) if and only if $0 < m < \frac{N-2}{N}$, $N > 3$.

As it is noted in [2], equation (E) in bounded domains with homogeneous Dirichlet conditions has also that property if $0 < m < 1$. The case $N = 1$ was considered by Sabinina [8]. Several properties of solutions of (E) related to the ones we consider here can be found in Evans [5]. Finite extinction times for $(E_{\beta})$ $u_t - \Delta \beta(u) = 0$ with $\beta$ maximal monotone graph and bounded domain are discussed in terms of $\beta$ in [3].

We also consider the homogeneous Dirichlet problem

\[
\begin{cases}
  u_t - \text{div}( |Du|^{p-2} Du) = 0 & \text{in } (0,T) \times \Omega \\
  u(x,t) = 0 & \text{in } (0,T) \times \partial \Omega \\
  u(x,0) = u_0(x) & \text{in } \Omega
\end{cases}
\]

for $\Omega \subset \mathbb{R}^N$ open and bounded. We show the existence of a finite extinction time if $p < 2$, $u_0 \in L^m(\Omega)$, and $m$ as above, completing a result of Bamberger [1]: he showed that effect for $\frac{2N}{N+2} \leq p < 2$ and $u_0 \in L^2(\Omega)$. For $p \geq 2$ it is easy to see that solutions with positive initial data do not vanish.

For $p \geq \frac{2N}{N+1}$ L. Véron [11] shows a smoothing and decay effect for the solutions

\[u_0 \in L^\beta(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)\] for $\beta = \beta(m,N)$.
of \((P_\Omega)\) : in fact, if \(N\left(\frac{2}{p} - 1\right) < m_0 < m \leq \infty\) and \(u_0 \in L^{m_0}(\Omega)\), then \(u(t,.) \in L^m(\Omega)\) and in addition \(\|u\| \leq C t^{-\delta}\). \(\|u_0\|_{m_0} \leq \sigma m_0\) where \(\delta, \sigma\) depend on \(m, m_0, p\) and \(N\). We adapt his proof for \((P)\) to get similar results. We know that for \(m_0 = N\left(\frac{2}{p} - 1\right)\) solutions vanish. For \(1 < m_0 < N\left(\frac{2}{p} - 1\right)\) we prove a «backwards» effect : for \(t > 0\), \(u(t,.) \in L^1(\mathbb{R}^N)\) and \(\|u\|_1 \leq C t^{-\delta}\). \(\|u_0\|_{m_0} \leq \sigma m_0\) with \(\delta, \sigma > 0\) as before.

We shall need some facts about the operator \(A\) in \(L^2(\mathbb{R}^N)\) and in \(C^0(\mathbb{R}^N)\) bounded with homogeneous Dirichlet conditions : First, if \(J(u) = \frac{1}{p} \int_{\mathbb{R}^N} |Du|^p\) when \(u \in L^p(\mathbb{R}^N)\) and \(\|Du\| \in L^p(\mathbb{R}^N)\), \(J(u) = + \infty\) otherwise, \(J\) is a convex l.s.c. proper functional in \(L^2(\mathbb{R}^N)\) whose subdifferential \(A\) is defined as \(Au = - \text{div}(|Du|^{p-2}Du)\) in the domain \(D(A) = \left\{ u \in L^2(\mathbb{R}^N) : |Du| \in L^p(\mathbb{R}^N), \text{div}(|Du|^{p-2}Du) \in L^2(\mathbb{R}^N) \right\}\) and for every \(v \in D(A)\), \(\int_{\mathbb{R}^N} Au \cdot v = \int_{\mathbb{R}^N} |Du|^{p-2}Du \cdot Dv\). If \(p > 2\), the last condition may be omitted as it follows by density. \(A\) is accretive in \(L^1(\mathbb{R}^N)\) and \(L^\infty(\mathbb{R}^N)\), hence in every \(L^p(\mathbb{R}^N)\), \(1 \leq p \leq \infty\) : in fact for \(t > 0\) and \(u_1, u_2 \in D(A) \cap L^p(\mathbb{R}^N)\), \(\|u_1(t,.) - u_2(t,.)\|^p \leq \|u(0,.) - u_0(0,.)\|^p \) where \(u^* = \max(u,0)\). This implies a comparison principle that allows us to consider only nonnegative initial data and solutions ; for nonpositive data we consider \(-u\) instead of \(u\). Defining for \(p \neq 2\), \(A_p = A \cap (L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N))\) we may close \(A_p\) to find \(A_{p^*}\) m-accretive in \(L^p(\mathbb{R}^N)\).

As \(\mathcal{D}(\mathbb{R}^N) \subset D(A_p), D(A_{p^*}) = L^p(\mathbb{R}^N)\).

The corresponding results for \(\Omega\) bounded and homogeneous Dirichlet conditions are well known ; \(Au = - \text{div}(|Du|^{p-2}Du)\) and \(D(A) = \left\{ u \in W^{1,p}_0(\Omega) \cap L^2(\Omega) : Au \in L^2(\Omega) \right\}\). On the other hand \(A_p\) is defined as m-accretive operator in \(L^p(\mathbb{R}^N)\) by restriction if \(p > 2\) and closure if \(p < 2\).

We shall use the following inequality due to Nirenberg and Gagliardo (see [6], Th. 9.3.).

**Lemma 0.** Let \(q, r\) be any numbers satisfying \(1 \leq q, r \leq \infty\) and \(u \in C^1_0(\mathbb{R}^N)\). Then

\[
\|u\|_q \leq C \|Du\|_r \|u\|_q^{1-a}
\]

where \(\frac{1}{p} = a \frac{1}{r} + (1-a) \frac{1}{r^*}\) and \(\frac{1}{q} = \frac{1}{r} - \frac{1}{r^*}\) for all \(a\) in the interval \(0 \leq a \leq 1\), with \(C = C(N, q, r, a)\), with the following exception : \(r = N\) and \(a = 1\) (hence \(p = \infty\)).

We remark that by density the result remains true for \(u \in L^q(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)\) such that \(Du \in L^r(\mathbb{R}^N)\) if \(r, q < \infty\) and \(q \leq r^*\) if \(r^*\) is positive. To show this, approach \(u\) by \(u^1\) bounded, then convolve \(u^1\) with a regular kernel to get \(u^2 \in C^\infty(\mathbb{R}^N)\) and finally cut \(u^2\) with a smooth function \(\xi_n\) which vanishes outside \(B_{2n}(0)\) and is equal to 1 on \(B_n(0)\); let us check this last step.
Assume \( u \in C^\infty(\mathbb{R}^N) \) and put \( u_n = u \cdot \xi_n \), where \( \xi_n(x) = \xi_o \left( \frac{|x|}{n} \right) \), \( 0 < \xi_n = 1 \), \( \xi_o(x) = 1 \) if \( |x| \leq 1 \), \( \xi_o(x) = 0 \) if \( |x| > 2 \) and \( D\xi_n(x) \leq C \). It is clear that \( u_n \to u \) in \( L^q(\mathbb{R}^N) \) and \( L^p(\mathbb{R}^N) \). Also \( D\xi_n = Du \cdot \xi_n + u \cdot D\xi_n \). Therefore \( \xi_n \to u \) in \( L^q(\mathbb{R}^N) \) and we have to prove that \( u \cdot D\xi_n \to 0 \) in \( L^q(\mathbb{R}^N) \). Then, \( C \) representing different constants independent of \( n \):

\[
\| u \cdot D\xi_n \|_{L^q} ^r = \int_{\mathbb{R}^N} \| u \|_r \| D\xi_n \|_r \leq \frac{C}{n^r} \int_{n|x| \leq 2n} \| u \|_r
\]

if \( q < r \), \( \int \| u \|_r \leq \| u \|_{\infty}^{-q} \cdot \int \| u \|_q \), so \( \| u \cdot D\xi_n \|_r \leq \frac{C \| u \|_{\infty}^{-q}}{n^r} \cdot \| u \|_q \to 0 \);

if \( r > q < q^* \) and \( q^* < r \), \( \int \| u \|_r \leq \left( \| u \|_q \right)^{q/r} \cdot \left( \int 1 \right)^{1-q/r} \), so:

\[
\| u \cdot D\xi_n \|_r \leq \frac{C}{n^r} \cdot \| u \|_r \cdot L^q(n \leq |x| \leq 2n). \]

If \( r^* < 0 \) the previous proof applies as well for every \( q \), \( 1 \leq q < \infty \).

Our plan is as follows: Sections 1, 2, 3 are devoted to problem (P). Section 1 studies the existence of a finite extinction time when \( p \leq \frac{2N}{N+1} \), \( u_0 \in L^m(\mathbb{R}^N) \), \( m = N(\frac{2}{p} - 1) \). Section 2 is devoted to conservation of mass and Section 3 to the regularizing effects and decay of the integral norms \( \| u(t,.) \|_m \) as \( t \to \infty \). Finally Section 4 gathers the results on \( (P_{\Omega}) \), \( \Omega \) open and bounded.

1. **FINITE EXTINCTION TIME**

We obtain the following result.

**Theorem 1.** Let \( N \geq 2 \), \( 1 < p < \frac{2N}{N+1} \) and let \( u_0 \in L^m(\mathbb{R}^N) \) where \( m = N(\frac{2}{p} - 1) \). Then for every \( t > 0 \) \( u(t,.) \in L^\infty(\mathbb{R}^N) \) and there exists \( t_0 > 0 \) such that \( u(t,.) = 0 \) a.e. if \( t \geq t_0 \).

**Proof.** We may assume that \( u_0(x), u(t,x) \) are nonnegative. A formal proof to be justified later by discretization in time runs as follows: As \( p < \frac{2N}{N+1} \) if \( m = N(\frac{2}{p} - 1) \) we have \( m > 1 \). Let

\[
p^* = \frac{Np}{N-p} \quad \text{and} \quad q = \frac{m+p-2}{p};
\]

then \( m = p^*q \). Also for \( k \geq 0 \) we write \( (u-k)_+ = \max(u-k,0) \) and \( v = v_k = (u-k)_+^q \). Multiply \( u_t - \text{div}(|Du|^{p-2}Du) = 0 \) by \( m(u-k)_+^{m-1} \) and integrate over \( \mathbb{R}^N \) to obtain:

\[
(1.1) \quad \frac{d}{dt} \int_{\mathbb{R}^N} (u-k)_+^m = m \int_{\mathbb{R}^N} u_t(u-k)_+^{m-1} = m \int_{\mathbb{R}^N} \text{div}(|Du|^{p-2}Du)(u-k)_+^{m-1}
\]
Integration by parts and Sobolev’s inequality give

\begin{equation}
- \int_{\mathbb{R}^N} \text{div}( |Du|^p - 2Du)(u-k)^{m-1} = (m-1)q^p \int_{\mathbb{R}^N} |Du|^p \geq C_p (m-1)q^p \left( \int_{\mathbb{R}^N} v^{p^*} \right)^{p/p^*} \tag{1.2}
\end{equation}

Write \( E_{m,k}(t) = \int_{\mathbb{R}^N} (u-k)^m \, dx \). (1.1) and (1.2) give

\begin{equation}
\frac{d}{dt} E_{m,k}(t) + C_p m(m-1)q^p E_{m,k}^p(t) (t) \leq 0 \tag{1.3}
\end{equation}

Integrating (1.3) gives

\begin{equation}
\begin{cases}
E_{m,k}(t) \leq E_{m,k}(0) \left[ 1 - \frac{C_p m(m-1)p}{Nq^p (E_{m,k}(0))^{p/N}} \cdot t \right]^N_p \\ E_{m,k}(t) = 0 \quad \text{for } t \geq t_{0,k}
\end{cases} \tag{1.4}
\end{equation}

where

\[ t_{0,k} = \frac{Nq^p}{pC_p m(m-1)} E_{m,k}(0)^{p/N} \]

If we take \( k = 0 \) the existence of a finite extinction time \( t_0 = t_{0,0} \) results. Given \( T > 0 \), if we take \( k > 0 \) large enough extinction of \( E_{m,k}(t) \) in time \( t_{0,k} \leq T \) may be obtained. Hence \( u(t,.) \in L^\infty(\mathbb{R}^N) \) for \( t > 0 \), a regularizing effect.

This formal proof can be made rigorous by means of the discrete scheme and Crandall-Liggett’s results. Assume that \( u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \), let \( h > 0 \) and define a discrete approximation to the solution of (P) thus: \( u_{i+1} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) is defined implicitly in terms of \( u_i \) by

\begin{equation}
\frac{u_{i+1} - u_i}{h} + Au_{i+1} = 0 \tag{1.5}
\end{equation}

Now repeat the previous argument on (1.5) to obtain a discrete version of (1.3) and pass to the limit as \( h \to 0 \). The assumption on \( u_0 \) can be weakened by approximation for \( t_{0,k} \) depends only on \( \| u_0 \|_m \). The details repeat those in [2] for \( u_t - \Delta u^m = 0 \) and we omit them. Only the integration by parts needs some care: if \( m \geq 2, u_0 \in D(A) \cap L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \), then

\begin{equation}
- \int_{\mathbb{R}^N} Au_i u_i^{m-1} + (m-1) \int_{\mathbb{R}^N} |Du_i|^p u_i^{m-2} = 0 \tag{1.6}
\end{equation}

by the characterization of \( D(A) \). If \( m < 2 \) we have to linearize the function \( \phi(u) = u^{m-1} \) near the
origin to apply integration by parts. Passing to the limit it follows by Fatou in this case that

\[ -\int_{\mathbb{R}^N} Au u^{m-1} + m \int_{\mathbb{R}^N} |Du|^p u^{m-2} \leq 0 \]

For \( u_0 \) as in the theorem the result follows by density for \( A \) is accretive.

2. MASS CONSERVATION

We say that the mass conservation law (MCL) holds for (P) if for every \( t > 0 \)
\[ \int_{\mathbb{R}^N} u(t,x) dx = \int_{\mathbb{R}^N} u_0(x) dx. \]

In this section the validity of MCL is discussed in terms of \( p \):

**Theorem 2.** MCL holds for (P) if and only if \( N = 1, p > 1 \) or \( N \geq 2, p > \frac{2N}{N+1} \).

In order to prove Theorem 2 we need some previous results. A variant of the following Lemma has been used in [10]:

**Lemma 1.** Let \( \Omega \subset \mathbb{R}^N \) be an open set and let \( u \in L^2(\mathbb{R}^N) \) be such that \( u \in D(A) \) and \(-Au = u \) a.e. in \( \Omega \). Let \( \eta \in C^\infty(\Omega) \) be such that \( \text{supp}(\eta) \subset \subset \Omega \), \( \| \eta \|_\infty = 1 \) and let \( \chi \) be the characteristic function of \( \text{supp}(D\eta) \). Then

\[ \| \eta Du \|_p \leq p \| D\eta \|_\infty \cdot \| \chi u \|_p. \]

**Proof.** Multiply \( u = Au \) by \( u \eta^p \), integrate over \( \mathbb{R}^N \), integrate by parts \((u \in D(A))\) and apply Hölder's inequality.

**Lemma 2.** Let \( 2 \leq p \leq 2 \) and let \( u \) be a solution of \( -Au + u = f \), \( f \in L^1(\mathbb{R}^N) \). Then
\[ \int_{\mathbb{R}^N} Au = 0. \]

**Proof.** By accretivity of \( A \) in \( L^1(\mathbb{R}^N) \), we may restrict ourselves to consider \( f \in L^\infty_0(\mathbb{R}^N) \). We obtain first an estimate for \( \| Du \|_p \) over the exterior of a ball: Assume \( \text{supp}(f) \subset B_R(0) \) and take \( n > R \). Choose \( \eta_n \in C^\infty(\mathbb{R}^N) \) such that \( 0 \leq \eta_n \leq 1 \), \( \eta_n = 0 \) if \( |x| \leq n \), \( \eta_n = 1 \) if \( |x| > 2n \) and \( \| D\eta_n \|_\infty \leq \frac{C_1}{n} \), \( C_1 > 1 \). Put \( A_n = \{ x \in \mathbb{R}^N : n \leq |x| \leq 2n \} \) and \( D_n = \{ x \in \mathbb{R}^N : |x| \geq n \} \). Then (2.1) gives in \( \Omega = \mathbb{R}^N - B_R(0) \):

\[ \| Du \|_{L^p(D_n)} \leq \frac{C}{n} \| \chi u \|_{L^p(\mathbb{R}^N)} \leq \frac{C}{n} \| u \|_{L^p(A_n)}. \]
Hereafter $C$ denotes several positive constants depending only on $p$ and $N$ and not on $n$.

By virtue of [9], Corollary 2, the following estimate applies to $u(x)$, for $|x| > R$:

\[
(2.3) \quad u(x) \leq C |x| \frac{p}{2-p}
\]

Also by accretivity $\|u\|_1 \leq \|f\|_1$, so that

\[
\|u\|_{L^p(A_n)} \leq \|u\|_{L^1(A_n)} \leq \|u\|_{L^\infty(A_n)} = o(1) \quad n \rightarrow \infty
\]

\[
\|Du\|_{L^p(D_n)} = o(1) \cdot n^{-\frac{p-1}{2-p}}. \quad \text{Putting } \xi_n(x) = 1 - \eta_n(x) \text{ we have}
\]

\[
(2.4) \quad \left| \int \int \right| \leq \left| \int \int \right| \leq o(1) \cdot n^{-\frac{p-1}{2-p}} \cdot n = 0(n). \quad n \rightarrow \infty
\]

Since $\int \int Au = \lim_{n \rightarrow \infty} \int \int Au \xi_n$, the desired result follows whenever

\[
\frac{N}{p} \frac{1}{2-p} \leq 0 \quad \text{i.e.} \quad p > \frac{2N}{N+1}
\]

We say that the finite propagation property (PF) holds for (P) if for every admissible initial datum $u_0(x)$ having compact support in $\mathbb{R}^N$, the corresponding solution $u(t,x)$ is such that for every $t > 0$ $u(t,\cdot)$ has compact support in $\mathbb{R}^N$. It is known that (PF) holds for (P) if and only if $p > 2$ (see [4]). There exists a simple relation between (FP) and (MCL):

**LEMMA 3.** If $p > 2$, then (MCL) holds.

**Proof.** Let $u(t,\cdot)$ be a solution of (P) such that $u(x,0) = u_0(x)$ has compact support. If $t > 0$ we know that there exists $n$ such that $\text{supp } u(t') \subset B_n(0)$ for $0 \leq t' \leq t$. Take $\xi_n$ as before. Then for $t'$ fixed:

\[
\int \int Au = \int \int Au \cdot \xi_n = \int \int \left| \int \int Du \cdot D_\xi_n = 0
\]

Hence $\int \int u_t \, dx = 0$ and it follows that $\int \int u(t,x) = \int \int u_0(x)$. This last assertion can be justified by means of the discrete scheme as before.

If $\text{supp}(u_0)$ is not compact, approximate $u_0$ by $\left\{u_{0,n}\right\}$, a sequence of initial data with compact support #

(1) Here $o(1)$ denotes a quantity that goes to 0 as $n \rightarrow \infty$. 

Proof of Theorem 2. If \( N = 1, \ p > 1 \) or \( N > 2, \ 2 > p > \frac{2N}{N+1} \) the result follows from Lemma 2 applied to the discrete scheme

\[
\frac{u_{i+1} - u_i}{h} + Au_{i+1} = 0
\]

for then \( \int_{\mathbb{R}^N} u_i = \int_{\mathbb{R}^N} u_{i+1} \). If \( p > 2 \) it follows from Lemma 3 in the same way. The case \( p = 2 \) is classic (and it falls within the scope of [2]).

For the negative part it is sufficient to remind Theorem 1, for (MCL) is incompatible with extinction.

### 3. DECAY OF THE INTEGRAL NORMS. REGULARIZING EFFECT

Our first result is the extension to \( \mathbb{R}^N \) of the work of L. Véron [11] for the case \( \Omega \) bounded.

**Theorem 3.** Let \( p > \frac{2N}{N+m_0} \), \( u_0 \in L^{m_0}(\mathbb{R}^N) \) with \( m_0 \gg 1 \). If \( t > 0 \), \( u(t, \cdot) \in L^m(\mathbb{R}^N) \) for every \( m \) such that \( m_0 < m \leq \infty \). In addition, the following estimate holds:

\[
\| u(t, \cdot) \|_m \leq \frac{C}{t^\delta} \| u_0 \|_{m_0} \quad \text{for some constant } C = C(m, m_0, N, p), \quad \delta = \frac{N(m - m_0)}{m_0 p + N(p-2)} \text{ if } m < +\infty, \quad \delta = \frac{N}{m_0 p + N(p-2)} \text{ if } m = +\infty.
\]

\[
\sigma = \begin{cases} 
\frac{m_0(mp + N(p-2))}{m_0 p + N(p-2)} & \text{if } m < +\infty, \\
\frac{m_0 p}{m_0 p + N(p-2)} & \text{if } m = +\infty.
\end{cases}
\]

**Proof.** The case \( m = m_0 \) follows from the accretivity property; it suffices to show the case \( m = +\infty \), the intermediate cases being obtained from these by interpolation. Assume (for simplicity) that \( u \geq 0 \); for \( p \leq N \) we adapt the iterative procedure of L. Véron [11] as follows. Define the sequences \( m_n, r_n \) by:

\[
m_n = \gamma^n \cdot m_0 \quad \text{with } 1 < \gamma < \frac{N}{N-1}, \quad m_0 \left( \frac{\gamma p}{N(\gamma-1)} - 1 \right) > \frac{1}{\gamma-1}
\]

\[
r_n + \frac{p-2}{m_n} = \frac{r_n}{m_{n-1}} - \frac{p}{N}.
\]

Note that from (3.3) and (3.4) it follows:
Now we claim that, if we write \( v = u \) with \( q_n = \frac{m_{n-1} + p - 2}{p} \), Nirenberg-Gagliardo's inequality applies to \( v \). Namely one has:

\[
\frac{r_n^{p-2}}{m_{n-1}} \ll v \ll \frac{r_n^{-m_{n-1}}}{q_{n-1}}
\]

That is a consequence of the following facts: i) As it was pointed out in Theorem 1, we can suppose \( u \in D(A) \cap L^1(\Omega_N) \cap L^{\infty}(\Omega_N) \) hence \( v \in L^1(\Omega_N) \cap L^{q_{n-1}}(\Omega_N) \) for each \( m_{n-1} > 1 \), for then \( m_{n-1} \) is always greater than one, ii) \( Dv \in L^p(\Omega_N) \) as a consequence of the validity of formula (1.7), iii) Nirenberg-Gagliardo's inequality (Lemma 0) applies with the present regularity, as it was observed at the introduction.

We shall give a formal proof, just as at the first part of Theorem 1 (rigorous justification by means of the discrete schema approximation is made in the same way as there). Assume first \( p < N \). Multiply the equation \( Au = 0 \) by \( u^{-m_{n-1}} \) and integrate over \( \Omega_N \) to get

\[
\frac{d}{dt} \left( \int_{\Omega_N} v^{m_{n-1}} \right) + C_{m,n} \left( \int_{\Omega_N} |Dv|^p \right) \leq 0
\]

Next multiply (3.7) by \( \|u\|^{r_n - m_{n-1}} \) and use (3.6). It follows that

\[
\|u\|^{r_n - m_{n-1}} \cdot \frac{d}{dt} \left( \|u\|^{m_{n-1}} \right) + C \|u\|^{r_n + p - 2} \leq 0
\]

where \( C \) involves \( C_{m,n} \) and the constant in (3.6), which depends only on \( N \) and \( p \). Take \( t_n = t(1 - \frac{1}{2^n}) \) and integrate (3.8) in \([t_{n-1}, t_n]\). In this way we obtain:

\[
\left| u(t_n) \right|^{r_n + p - 2} \leq \frac{2^n}{C_n} \left| u(t_{n-1}) \right|^{r_n}
\]

The previous argument remains true if we replace \( u \) by \( u_k = (u-k)_+ \) for some \( k > 0 \). But then \( |\Omega_{k,t}| = \text{meas} \{ x : u_k(t) > 0 \} \) is finite and

\[
\|u_k(t)\|_\infty = \lim_{m_n \to \infty} \sup_{u_k(t)} \|u_k(t)\|^{m_n} \leq \lim_{m_n \to \infty} \sup_{u_k(t_n)} \|u_k(t_n)\|^{m_n}.
\]
Now (3.1), (3.2) follow from two facts: a) \( \lim_{m_n \to \infty} \sup u(t_n) \| m_n \) can be evaluated now just in the same way as in [11], which implies estimates (3.1) (3.2) for \( u_k \). b) These estimates do not depend on \( k \), and consequently we can pass to the limit and obtain the desired results for \( k = 0 \).

When \( p = N \), choose \( \{ \beta_n \} \) such that

\[
(3.10) \quad \beta_n = q_n + m_n \left( 1 - \frac{1}{N} \right)
\]

Write \( w_{n-1} = u^{\beta_n} \). Then \( D \left( u^{q_{n-1}} \right) \frac{q_{n-1} - \beta_n}{\beta_n} \cdot D w_{n-1} \), i.e.,

\[
Dw_{n-1} = \frac{\beta_{n-1} - q_{n-1}}{q_{n-1}} \cdot D \left( u^{q_{n-1}} \right) w \cdot \frac{\beta_{n-1}}{\beta_n}.
\]

Now by Hölder

\[
(3.11) \quad \left( \int_{\mathbb{R}^N} |Dw_{n-1}| \right)^N \leq \left( \int_{\mathbb{R}^N} |D \left( u^{q_{n-1}} \right)| \right)^N \cdot \left( \int_{\mathbb{R}^N} u^{m_{n-1}} \right)^{N-1}
\]

On the other hand, by Sobolev

\[
(3.12) \quad \left( \int_{\mathbb{R}^N} |Dw_{n-1}| \right)^N \geq C_N \left( \int_{\mathbb{R}^N} u^{N\beta_{n-1}} \right)^{N-1}
\]

Now multiply (3.7) by \( u \| m_{n-1} \), use (3.11), (3.12) and a standard interpolation argument to get:

\[
(3.13) \quad \| u \| m_{n-1} \frac{d}{dt} \left( \| u \| m_{n-1} \right) + C \cdot \| u \| m_{n}^{N-2} \leq 0
\]

where \( C = C_{m,n} \cdot \left( \frac{\beta_{n-1}}{q_{n-1}} \right)^N \cdot C_N \). (3.13) is the analogous of (3.8) and we can now argue as in the previous case.

When \( p > N \) we do not need to use the iterative procedure. For note that Nirenberg-Gagliardo’s inequality reads:

\[
(3.14) \quad \| v \|_\infty \leq C \| Dv \|_p \cdot \| v \|_m^{1-a} \quad \text{where} \quad a = \frac{N(m+p-2)}{mp+N(p-2)}, \quad m > 1
\]

(3.14) and (3.7) give

\[
(3.15) \quad \| u \|_m^{a} \frac{d}{dt} \left( \| u \|_m \right) + C_m \left( \frac{1}{c} \right)^p \| u \|_\infty^{p} \leq 0, \quad q = \frac{m+p-2}{p}
\]
Now note that from the inequality
\[ \phi(t)^a \frac{d}{dt} \phi(t) + k \psi(t)^\theta \leq 0 \]
it follows, integrating between 0 and t
\[ \psi(t) \leq \left( \frac{1}{kt^\theta} \frac{\phi(0)}{\phi(t)^\theta} \right)^{\omega+1} \]
(3.16)

Use (3.16) with \( \psi(t) = \| u(t) \|_\infty \), \( \phi(t) = \| u \|_m^m \), \( \omega = \frac{(1-\alpha)pq}{a} \), \( \theta = \frac{p}{a} \) and (3.1), (3.2) follow. Note that this argument includes the case \( N = 1 \) which was discarded in [11] #

When \( 1 < m_0 < N \left( \frac{2}{p} - 1 \right) \) we have the following result, concerning a «backwards regularizing effect.

THEOREM 4. Let \( 1 < m_0 < N \left( \frac{2}{p} - 1 \right) \), \( u_0 \in L^m(moN) \). If \( t > 0 \), \( u(t,.) \in L^m(moN) \) for every \( m \) such that \( 1 < m < m_0 \). In addition the following estimate holds :

\[ \| u(t,.) \|_m \leq \frac{c}{t^\delta} \| u_0 \|_m \quad \text{for some constant} \quad C = C(m,m_0,N,p), \quad \text{where} \]
\[
\delta = \frac{N(m_0-m)}{m[N(2-p)-m_0p]} \quad \text{and} \quad \sigma = \frac{m[N(2-p)-m_0p]}{m[N(2-p)-m_0p]} \]
(3.17)

Proof. Let us see first that \( u(t,.) \in L^m(moN) \) for each \( m \) such that \( 1 < m < m_0 \) (the case \( m = m_0 \) follows by accretivity). Remark that

\[ \| v \|_{m/q} \leq \frac{C}{t^\sigma} \| Dv \|_{p} \cdot \| v \|_{m_0/q} \]
(3.18)

where \( v, q \) are as in the last part of Theorem 3, the validity of (3.18) is justified as there, and

\[ a = \frac{pq}{m[m_0(N-p)-m(m+p-2)]} \]. Arguing as in Theorem 1 (with \( k = 0 \)), we arrive at

\[ \frac{d}{dt} E_m(t) + k E_m(t)^{\alpha m} \leq 0, \quad E_m(t) = \int_{\mathbb{R}^N} u_m(t,x)dx \]
(3.19)

Now notice that solutions of the inequality \( f' + af^\gamma \leq 0 \) with \( \gamma > 1 \) satisfy \( f \leq \frac{1}{((\gamma-1)\alpha t)^{\gamma-1}} \). This gives (3.17).

The case \( u(t,.) \in L^1(moN) \) is obtained by modifying slightly the previous argument :
instead of (3.18) write

\[ \|v\|_{1/q} \leq C \|Dv\|_p^a \cdot \|v\|_{1-a}^{m/q} \quad \text{with} \quad 1 < m < 3 - \frac{1}{N} \). \]

\[ a = \frac{N(m-1)(p-1)}{m(N-p)-N(p-1)} \]. Corresponding to (3.19) we have

\[ \|u\|_{m}^{(1-a)\frac{pq}{a}} \cdot \frac{d}{dt}(\|u\|_{n}^{m}) + C_m \left(\frac{1}{\gamma}\right)^{\frac{p}{c}} \|u\|_1^{a} \leq 0. \]

Now integrate (3.21) between 0 and t and use the fact that \( \|u(t)\|_m \) is not increasing in t to get the result #

4. - BOUNDED DOMAINS

Concerning \((P_{\Omega})\) with \(\Omega\) bounded, it is known that there is a finite extinction time if \(u_0 \in L^2(\Omega)\) and \(\frac{2N}{N+2} \leq p < 2\) ([1]). In that paper, extinction of the \(L^2\) norm of the solution implies this result. The method of the proof of Theorem 1, based on the extinction of the \(L^m\) norm of solutions for some \(m > 1\), enables us to extend the above mentioned result to get the following complete picture.

**THEOREM 5.** Assume that \(\Omega\) is bounded and regular. Let \(u_0 \in L^m(\Omega)\) where \(m > \max\left\{N\left(\frac{2}{p} - 1\right), 1\right\}\) and \(p < 2\). The corresponding solution of \((P_{\Omega})\) vanishes in a finite time \(t_0\). If \(p \geq 2\) there are, for \(u_0 \in C^\infty(\Omega)\) and \(u_0 > 0\), solutions which are strictly positive for every \(t > 0\).

**Proof.** Let \(m > N\left(\frac{2}{p} - 1\right)\) (the case \(m = N\left(\frac{2}{p} - 1\right)\)) is an easy modification of the proof in Theorem 1. We write again \(q = \frac{m+p-2}{p}, v = u^q\). By Hölder

\[ \left(\int_{\Omega} u^m\right)^{\frac{m}{p}} \leq \left(\int_{\Omega} u^{p^*q}\right)^{\frac{m}{p^*q}} \cdot \|\Omega\| \left(\int_{\Omega} |Dv|^{p}\right)^{\frac{p^*q-m}{p^*q}}, \text{where } p^* = \frac{Np}{N-p}, \|\Omega\| = \text{meas}(\Omega). \]

Starting as in Theorem 1 (with \(k = 0\)) we arrive at

\[ \frac{d}{dt} \left(\int_{\Omega} u^m\right) + \frac{m(m-1)}{q^p} \left(\int_{\Omega} |Dv|^{p}\right) \leq 0 \]

Next use Sobolev \(( \|Dv\|_p \geq c \|v\|_{p^*})\) and (4.1) to obtain
From (4.3) we conclude that $u$ vanishes at most at $t_0$, where

$$t_0 = \frac{(2-p)q^p}{cm^2(m-1)}. \quad \Omega \mid \omega \cdot \|u_0\|^{2-p} \cdot \frac{\omega}{m}$$

Assume now that $\Omega$ is connected (1). When $p = 2$ the fact that for $u_0 > 0$, $u(t,.) > 0$ and $t > 0$, $u(t,.) > 0$ follows from the strong maximum principle of L. Nirenberg (see [7]). If $p > 2$ take $S_\lambda = \phi$ and $g$ a positive eigenfunction corresponding to the first eigenvalue $\lambda$ of $-\Delta$ in $B_\lambda(0)$ with homogeneous Dirichlet conditions; $g$ is radially symmetric, $C^\infty$ and $Ag \leq Cg$ for some $C > 0$. To check this last assertion, note that

$$-g'' - \frac{N-1}{r} g' = g \quad \text{and hence} \quad Ag = -\lambda(p-1) \int g' |g|^{p-2} g'' - \frac{N-1}{r} |g'|^{p-2} =$$

$$= \lambda(p-1) |g'|^{p-2} + (p-1) \frac{N-1}{r} |g'|^{p-2} - \frac{N-1}{r} |g'|^{p-2} \leq \lambda(p-1) |g'|^{p-2} \leq Cg.$$

Now try as a subsolution $\tilde{v}(t,x) = T(t) g(x)$, where $T(t) = \frac{T_0}{(1+c(p-2)T_0^{p-2} t)^{1/p-2}}$ solves $T'(t) + CT(t)^{p-1} = 0$. It follows from the maximum principle that if $u_0(x) > T_0 g(x)$, the corresponding $u(t,x)$ is greater or equal than $\tilde{v}(t,x)$ for each $t > 0$.

Remark. Observe that as a consequence of the decay of some $m$-norm, $m > 1$ and $\Omega$ being bounded, MCL never holds. When $p \neq 2$ we have shown that for smooth initial data there is a retardation property: if $u_0(x) > T_0 g(x)$, the corresponding $u(t,x)$ is greater or equal than $\tilde{v}(t,x)$ for each $t > 0$.

We conclude by noting that the results of this paper are valid when $Au$ is replaced by other similar nonlinear.

$$\Delta_p u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left( |\frac{\partial u}{\partial x_i}|^{p-2} \frac{\partial u}{\partial x_i} \right)$$

As a natural generalization we may consider operators like

$$Bu = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \beta_i \left( \frac{\partial u}{\partial x_i} \right)$$

(1) For general $\Omega$ argue on each connected component.
where \( \sum_{i=1}^{N} \beta_i(s_i) \geq c \| s \|_P^p \) with \( s = (s_1, \ldots, s_N) \).

Some of the previous results have immediate counterparts. In particular Theorem 1 remains valid unchanged.

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