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SINGULAR PERTURBATIONS FOR A CLASS OF QUASI-LINEAR HYPERBOLIC EQUATIONS

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Résumé : Nous étudions le comportement pour $\epsilon \to 0_+$ de la solution d’un problème aux limites relatif à $\epsilon L_2 u_\epsilon + L_1 u_\epsilon + G(u_\epsilon) = f$ où $L_j$ $(j = 1, 2)$ est un opérateur linéaire hyperbolique d’ordre $j$ et $G$ une fonction lipschitzienne.
Dans le cas «temporel» nous obtenons la convergence de $u_\epsilon$ vers $u$ et des dérivées de $u_\epsilon$ dans des espaces de Sobolev locaux où $u$ est la solution d’un problème aux limites relatif à $L_1 u + G(u) = f$.

Summary : We study the behavior for $\epsilon \to 0_+$ of the solution of a boundary value problem relative to $\epsilon L_2 u_\epsilon + L_1 u_\epsilon + G(u_\epsilon) = f$ where $L_j$ $(j = 1, 2)$ is a linear hyperbolic operator of order $j$ and $G$ a lipschitzian function.
In the «time like» case, we obtain the convergence of $u_\epsilon$ to $u$ and of the derivatives of $u_\epsilon$ in local Sobolev spaces where $u$ is the solution of a boundary value problem relative to $L_1 u + G(u) = f$.

We study a problem of singular perturbations for a class of hyperbolic quasi-linear partial differential equations which are of the type:

$$\epsilon L_2 u_\epsilon + L_1 u_\epsilon + G(u_\epsilon) = f$$

where $L_2 = \frac{\partial^2}{\partial t^2} - \Delta$ , $L_1 = a \frac{\partial}{\partial t} + \sum_{k=1}^{n} b_k \frac{\partial}{\partial x_k}$ and $G : \mathbb{R} \to \mathbb{R}$ is a lipschitzian function.

In particular, this type of equation includes the Gordon’s equation with damping. A similar non
linear problem has been studied by R. Geel [3] with a function $G(x,t,v)$ whose derivative, with respect to $v$, satisfies a Hölder condition with exponent $\alpha > 0$, the solutions being taken in the classical sense.

We consider the problem in the «time-like» case, that is: when operator $L_1$ divides operator $L_2$ in the sense of J. Leray [5], L. Garding [2]. The results of convergence are obtained in Sobolev spaces of local type and are analogous, with some supplementary results, to those established in the case when the non-linear term is $G(v) = |v|^p v$ [4]. Moreover the theory of non-linear interpolation has the interest to give here a theorem of convergence with weakened assumptions.

The following is an outline of this work:

1. Notations hypotheses and two examples
2. Convergence of $u_\epsilon$ and $L_1 u_\epsilon$
3. Convergence of the derivatives of $u_\epsilon$
4. Application of the non-linear interpolation
5. Some remarks about correctors.

1. NOTATIONS HYPOTHESES AND TWO EXAMPLES

$\Omega$ is a bounded open set in $\mathbb{R}^n$ of class $C^1$ (J. Necas [9]) with boundary $\Gamma = \partial \Omega$.

We set $Q = \Omega \times ]0,T[$, $T$ real $> 0$, $\Sigma = \Gamma \times [0,T]$ and for every $t \in [0,T]$, $Q_t = \Omega \times ]0,t[$, $\Sigma_t = \Gamma \times [0,t]$.

We represent the norm of the usual Sobolev spaces, by:

$$
\| \cdot \|_{L^p(\Omega)} = \| \cdot \|_p \quad \quad \| \cdot \|_{H^1(\Omega)} = \| \cdot \|_2
$$

$$
\| \cdot \|_{L^p(Q)} = \| \cdot \|_p \quad \quad \| \cdot \|_{L^2(0,T;H^1(\Omega))} = \| \cdot \|_2
$$

and the inner product in $L^2(\Omega)$ by $(\cdot, \cdot)$. We keep the same notation $(\cdot, \cdot)$ for the duality between $L^p(\Omega)$, $L^p(\Omega) = \left( \frac{1}{p} + \frac{1}{p'} = 1 \right)$ and $H^{-1}(\Omega), H^1_0(\Omega)$.

We note $u'$, $u'', \ldots$ the derivatives of $u$ in the sense of vector-value distributions on $]0,T[$ and $\alpha(u,v)$ the bilinear form $\int_\Omega \nabla u \cdot \nabla v \, dx$.

We consider the following initial boundary value problem:
The condition $H_1$ (iii) implies (see M. Marcus and V.J. Mizel [7]) the

LEMMA 1.1, $G'((R)$ and for every $v \in H^1(Q)$, we have :

$$\forall v \in H^1_0(\Omega), a.e \text{ in } ]0,T[,$$

$$u_\epsilon \in L^\infty(0,T;H^1_0(\Omega)), u'_\epsilon \in L^\infty(0,T;L^2(\Omega)),$$

$$u_\epsilon(x,0) = u_0, u'_\epsilon(x,0) = u_1,$$

$$u_0, u_1, f \text{ given such that :}$$

$$u_\epsilon \in H^1_0(\Omega), u_1 \in L^2(\Omega), f \in L^2(Q)$$

The variable coefficients $a$, $b_k$ and the function $G$ satisfy the hypothesis

$$H_1 \begin{cases} 
(i) & a, b_k \in W^{1,\infty}(Q) \cap C^0(\overline{Q}) \\
(ii) & \inf_{\overline{Q}} a(x,t) = \delta > 0 \\
(iii) & G: \mathbb{R} \to \mathbb{R} \text{ is a lipschitzian function i.e.:} \\
& \forall (\lambda, \mu) \in \mathbb{R}^2, |G(\lambda) - G(\mu)| \leq \ell |\lambda - \mu|, \ell \text{ positive constant.}
\end{cases}$$

The condition $H_1$ (iii) implies (see M. Marcus and V.J. Mizel [7]) the :

LEMMA 1.1. $G' \in L^\infty(\mathbb{R})$ and for every $v \in H^1(Q)$, we have :

$$\frac{\partial}{\partial t} G(v) = G'(v)v' \in L^2(Q) \text{ and } \|G'(v)v'\|_2 \leq \ell \|v\|_2$$

$$\frac{\partial}{\partial x_k} G(v) = G'(v) \frac{\partial v}{\partial x_k} \in L^2(Q) \text{ and } \|G'(v) \frac{\partial v}{\partial x_k}\|_2 \leq \ell \|\frac{\partial v}{\partial x_k}\|_2 \quad (k = 1, 2, ..., n)$$

EXISTENCE AND REGULARITY OF THE SOLUTION $U_\epsilon$ OF $\mathcal{P}_\epsilon$

Taking into account hypothesis about $a$, $b_k$ and $f$, and lemma 1.1, one can show thanks to Galerkin’s method (in the case $a, b_k = 0$ see J.C. Saut [10]), the

THEOREM 1.2. The problem $\mathcal{P}_\epsilon$ has a unique solution, for each $\epsilon > 0$. 

\begin{align*}
P_\epsilon \\
\epsilon L_2 u_\epsilon + L_1 u_\epsilon + G(u_\epsilon) = f \\
u_\epsilon(x,0) = u_0, u'_\epsilon(x,0) = u_1 \\
u_\epsilon |_\Sigma = 0
\end{align*}
THEOREM 1.3. Under hypothesis

\[ H_2 : H_1 \text{ with } u_0 \in H^1_0(\Omega) \cap H^2(\Omega), u_1 \in H^1_0(\Omega), f \in L^2(\Omega) \]

for each \( \varepsilon > 0 \), there exists a unique solution to the problem \( \mathcal{P}_\varepsilon \) such that:

\[ u_\varepsilon \in L^\infty(0,T;H^1_0(\Omega)) , \quad u_\varepsilon' \in L^\infty(0,T;H^2(\Omega)) , \quad u_\varepsilon'' \in L^\infty(0,T;L^2(\Omega)). \]

In order to study the convergence, we have to introduce:

1. The fundamental hypothesis:

   - The results of convergence are obtained in the «time-like» case that is with the condition:
     \[ \sum_{k=1}^{n} b_k^2(x,t) < a^2(x,t) \quad \forall (x,t) \in \overline{Q}. \]
   - One can deduce from (A) the two properties:

     \[ \Phi(\xi_1, \xi_2, \ldots, \xi_n, \xi_0) = \frac{\omega}{2} \sum_{k=0}^{n} \xi_k^2, \quad \text{where } \omega = \inf \left( 1 - \sum_{k=1}^{n} a_k^2 b_k^2 \right) \]

     \[ (1.5) \]

   - For every functions \( \theta \in L^2(Q), \quad \Theta \in C^0(\overline{Q}), \quad \Theta \geq 0 \), such that \( \theta \mapsto \text{grad } \theta \) and \( \theta \, \nu' \in L^2(Q) \), we have:

     \[ \int_0^T \left( |\theta| \, |\text{grad } \theta| \frac{\omega}{2} - |\theta \, \nu'| \frac{\omega}{2} \right) ds \geq -\omega_1 \int_0^T |\theta| \, L_1 v_1 \frac{\omega}{2} ds + \frac{3}{4} \omega_1 \int_0^T |\theta| \, \text{grad } \nu \frac{\omega}{2} ds \]  

     \[ (1.6) \]

   - where the positive constant \( \omega_1 \) depends only on the coefficients.

2. Weight functions:

   - Let \( \nu = (\nu_1, \nu_2, \ldots, \nu_n, 0) \) the unit normal outward vector to \( \Sigma \) when it exists.

   - We represent by \( \Lambda \) the null-subset of \( \Sigma \) where \( \nu \) is not defined, and by \( \Sigma_-, \Sigma_+, \Sigma_o \), the subsets of \( \Sigma - \Lambda \) corresponding respectively to:

     \[ \sum_{k=1}^{n} b_k \nu_k < 0, \quad \sum_{k=1}^{n} b_k \nu_k > 0, \quad \sum_{k=1}^{n} b_k \nu_k = 0. \]

Under the hypothesis \( H_1 \) (ii) and the assumption:
H'1 : $L_1$ is a vector-field of class $C^1$ on an open set of $\mathbb{R}^{n+1}$ which contains $Q$, we may use functions $\varphi$ (F. Mignot and J.P. Puel [8]) satisfying the condition:

\begin{equation}
\begin{cases}
(i) & \varphi \in C^0(Q) \cap W^{1,\infty}(Q), \ 0 \leq \varphi \leq 1 \\
(ii) & \varphi = 0 \text{ in a neighbourhood of } \Sigma_+ \text{ in } \Sigma \\
(iii) & L_1 \varphi \leq 0 \text{ on } Q.
\end{cases}
\end{equation}

These functions are such that:

There is a null-set $Z \subset Q$ such that $V(x,t) \in (Q-Z) \cup E_-$, there existe a function $\varphi$ satisfying $\mathcal{A}_1$ such that $\varphi(x,t) \neq 0$.

For each compact $K \subset \overline{Q}$ with $K \cap \mathcal{Y}(\Sigma_+) = \varnothing$, where $\mathcal{Y}(\Sigma_+)$ denotes a neigbourhood of $\Sigma_+$ in $\Sigma$, there exists a function $\varphi$ satisfying $\mathcal{A}_1$ such that $\varphi(x,t) \geq m > 0$ on $K$.

**GREEN'S FORMULA FOR OPERATOR $L_1$**:

Under the hypothesis $H'_1$ and the condition

(B) : $\partial \Sigma_-$ is a finite reunion of $(n-1)$ dimensional $C^1$ submanifolds

$\forall w \in L^2(Q_t)$ such that $L_1 w \in L^2(Q_t)$, $w \mid_{(\Sigma_+^-)} = 0$, $w(x,0) = u_0$, we have:

\begin{align}
\int_0^t (L_1 w, w) ds &= \frac{1}{2} \int_\Omega aw^2 dx - \frac{1}{2} \int_\Omega a(x,0)u_0^2 dx + \frac{1}{2} \int_{\Sigma_+} \left( \sum_{k=1}^n b_k \nabla_k w \right)^2 d\Gamma ds \\
&\quad - \frac{1}{2} \int_{Q_t} \left( a' + \sum_{k=1}^n b_k \right) w^2 dx ds \quad (1.7)
\end{align}

We will start the study of the general case investigated below with an illustration through two simple examples which are Klein-Gordon equation with $G(u) = \sin u$.

**EXEMPLE 1.** We consider the problem $\partial^\varrho_\varrho$ where $\varrho$ is the square $\int_0^1 \times \int_0^1$ in $\mathbb{R}^2$ (fig. 1)

![Fig. 1](image-url)
We note $L_1$ the operator of first order $u \rightarrow L_1 u = u' + b \frac{\partial u}{\partial x}$.

We have seen in the general case that if $u_0 \in H^1_0(\Omega)$, $u_1 \in L^2(\Omega)$ and $f \in L^2(Q)$ the problem $\mathcal{P}_\varepsilon$ has a unique solution $u_\varepsilon$ for all $\varepsilon > 0$, such that $L_0^\infty(0,T;H^1(\Sigma_0)) \subset L_0^\infty(0,T;L^2(\Omega))$.

For the weight functions and the limit problem the subsets of the boundary taken into account are:

$$
\Gamma_- = \{(x,y) ; 0 < x < 1, y = 0, 0 < y < 1\},
\Gamma_+ = \{(x,y) ; x = 1, 0 < y < 1\},
\Gamma_0 = \{(x,y) ; 0 < x < 1, y = 0\} \cup \{(x,y) ; 0 < x < 1, y = 1\}.
$$

and $\Sigma_- = \Gamma_- \times [0,T]$, $\Sigma_+ = \Gamma_+ \times [0,T]$, $\Sigma_0 = \Gamma_0 \times [0,T]$.

(We remark that the subset $\Lambda$ of $\Sigma$ where the outward normal is not defined is composed of the four edges, 00', AA', BB' and CC').

The weight functions $\varphi$ satisfy:

$$
\begin{align*}
\varphi &\in C^0(\bar{\Omega}) \cap W^{1,\infty}(\Omega), 0 \leq \varphi \leq 1 & \text{on } \Omega \\
\mathcal{A}_1 &\implies \varphi(1,y) = 0, 0 \leq y \leq 1 \\
\frac{\partial \varphi}{\partial y} &\leq 0 & \text{on } \Omega
\end{align*}
$$

Let $\varphi(x,y) = 1 - x$.

Obviously $\varphi$ satisfies the condition $\mathcal{A}_1$ and we have here the fact $\varphi(x,y) > 0$ for $(x,y) \in \Omega \cup \Gamma_-$. Moreover, for each $\gamma, 0 < \gamma < 1$, $\varphi(x,y) > \gamma$ on $\Omega_\gamma$ where $\Omega_\gamma = \{0,1-\gamma\} \times [0,1]$.

The limit problem is here given by:

$$
\begin{align*}
\begin{cases}
&u' + b \frac{\partial u}{\partial x} + \sin u = f \\
&u(x,y,0) = u_0 \\
&u(x,y,t) = 0 & \text{on } \Sigma_-
\end{cases}
\end{align*}
$$

$\mathcal{P}$ has a unique solution such that $u \in L^\infty(0,T;L^2(\Omega))$ and $L_1 u \in L^2(Q)$. 

$\quad$
Then, with the use of the function $\varphi$ the results of convergence are

(i) For $u_0 \in H^1_0(\Omega)$, $u_1 \in L^2(\Omega)$, $f \in L^2(Q)$, the solution $u_\varepsilon$ converges to $u$ in $L^\infty(0,T;L^2(\Omega))$ weak-star and in $L^q(Q), \forall q < 2$. Moreover $u_\varepsilon$ converges to $u$ in $L^\infty(0,T;L^2(\Omega_\gamma))$ and $L^1 u_\varepsilon$ converges to $L^1 u$ in $L^2(0,T;L^2(\Omega_\gamma))$, $\forall \gamma \in ]0,1[.$

Besides for $u'_\varepsilon$ we have : $u'_\varepsilon$ converges to $u'$ in $L^\infty(0,T;L^2(\Omega))$ weak star

(ii) If we take $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, $u_1 \in H^1_0(\Omega)$, $f \in L^2(0,T;H^1(\Omega))$ such that $f(0,y,t) = f(1,y,t) = 0$ and $f' \in L^2(Q)$ we can state that $u_\varepsilon$ converges to $u$ in $H^1(\Omega_\gamma)$ where $Q_\gamma = \Omega_\gamma \times ]0,T[$, and we have the estimation :

$$|u_\varepsilon - u|_{L^2(\Omega_\gamma)} \leq K_\gamma \varepsilon^{1/2}$$

where the constant $K_\gamma$ can be written $K_\gamma = C \gamma^{-3}$ with $C$ constant independent of $\varepsilon$ and $\gamma$.

Let now, $\varphi(x,y) = (1-x)(1-y)y$.

This new function $\varphi$ satisfies the condition $\mathcal{D}_1$ and is such that :

$\varphi(x,y) > 0$ for $(x,y) \in \Omega \cup \Gamma_-$ and for each $\gamma, 0 < \gamma < 1$ ; or each $\gamma, 0 < \gamma < 1$ ;

$\varphi(x,y) > \gamma^2(1-\gamma)$ on the open subset of $\Omega : ]0,1-\gamma[ \times ]\gamma,1-\gamma[$.

Moreover this function has the supplementary property $\varphi(x,y) = 0$ on $\Gamma_\partial$.

Then by the use of this more particular function, we obtain the result of convergence and the estimation of the point (ii), without the condition $f(x,y,t) = 0$ on $\Sigma_\varepsilon$ but with $\Omega_\gamma$ replaced by $]0,1-\gamma[ \times ]\gamma,1-\gamma[$, $Q_\gamma$ by $]0,1-\gamma[ \times ]\gamma,1-\gamma[ \times ]0,T[$.

**EXEMPLE 2.** We take the same problems $P_\varepsilon$ and $P$ but we consider here the open set $\Omega = \{(x,y) \in \mathbb{R}^2 ; (x-1)^2 + y^2 < 1\}$ (fig. 2).

![Fig. 2](image-url)
Then, $\Gamma_-=\{(x,y); x = 1-(1-\gamma^2)^{1/2}, -1 < \gamma < 1\}$, $\Sigma_- = \Gamma_- \times [0,T]
\Gamma_+=\{(x,y); x = 1+(1-\gamma^2)^{1/2}, -1 < \gamma < 1\}$, $\Sigma_+ = \Gamma_+ \times [0,T]$

and $\Sigma_0$ is composed of the two generating lines $AA'$ and $BB'$.

The weight function we will use, is:

$$\varphi(x,y) = \begin{cases} 1 - y^2 & \text{if } x \leq 1 \\ 1 - (x-1)^2 - y^2 & \text{if } 1 \leq x \leq 2 \end{cases}$$

It is such that: $\varphi(x,y) = 0$ on $\Gamma_-$, $\varphi(x,y) > 0$ on $\Omega \cup \Gamma_-$ and for each $\gamma$, $0 < \gamma < 1$, $\varphi(x,y) \leq \frac{\gamma^2}{4}$ on $\Omega_{\gamma}$ where $\Omega_{\gamma} = \Omega \cap \{(x,y); (x-1+\gamma)^2 + y^2 < 1\}$ (fig. 3)

We have the same results as in example 1, point (i). Because of the fact: $\varphi(x,y) = 0$ on $\Gamma_-$, we have

if we take $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$, $f \in L^2(0,T;H^1(\Omega))$ and $f \in L^2(Q)$, $u_\varepsilon$ converges to $u$ in $H^1(\Omega)$, $\forall \gamma \in ]0,1[$, where $Q_\gamma = \Omega \times ]0,T[.$

Moreover the use of the interpolation theory can improve the results in the following way:

if $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $f \in L^2(0,T;H^1(\Omega))$, $0 < s < 1$, we have the estimation
I \, u_\varepsilon - u \|_{L^2(\Omega)} \leq K_\gamma \varepsilon^{s/2} \text{ for each } \gamma \in ]0,1[, \text{ where the constant } K_\gamma \text{ can be written } K_\gamma = C \gamma^{-3} \text{ with } C \text{ constant independent of } \varepsilon \text{ and } \gamma.

To avoid a too long text the remarks about the use of function \( \varphi \) such that \( \varphi = 0 \) on \( \Sigma_0 \cup \Sigma_- \)
will not be detailed in the general case.
Throughout this paper, \( C_j \) and \( k_j \) will denote positive constants which are independent of \( f, u_0, u_1 \) and \( \varepsilon \).

2. CONVERGENCE OF \( u_\varepsilon \) AND \( L^1 u_\varepsilon \)

In this section we obtain under the hypothesis \( H_1 \) and the condition (A) the convergence of the solution \( u_\varepsilon \) of the problem \( P_\varepsilon \) to \( u \) solution of the problem,

\[
P \left\{ \begin{array}{l}
L_1 u + G(u) = f \\
u \in L^\infty(0,T;L^2(\Omega)); u(x,0) = u_0; u|_{\Sigma_-} = 0
\end{array} \right.
\]
as also the convergence of \( L_1 u_\varepsilon \) to \( L_1 u \) in a local space. We use a method of regularization and monotonicity-compactness arguments. The first subsection is devoted to the study of a priori estimates and the second one to the convergence.

2.1 - A PRIORI ESTIMATES

**THEOREM 2.1.1.** We assume condition (A); then \( \exists \varepsilon_0 > 0 \text{ such that } \forall \varepsilon < \varepsilon_0 \) the solution \( u_\varepsilon \) of the problem \( P_\varepsilon \) satisfies:

\[
|u_\varepsilon|^2 + \sqrt{\varepsilon} |u'_\varepsilon|^2 + \sqrt{\varepsilon} \|u_\varepsilon\|_2 \leq C_1 K_1(f,u_0,u_1,\varepsilon)
\]

with

\[
K_1^2(f,u_0,u_1,\varepsilon) = f^2 + |u_0|^2 + \varepsilon^2 |u_0|^2 + \varepsilon^2 |u_1|^2 + (G(0))^2
\]

**Preuve.** We take off the method used in [4] theorem 2.1.

With assumption \( H_2 \):

Then we can make \( v = u_\varepsilon + 2 \varepsilon a^{-1} u'_\varepsilon \) in (1.4). With the same transformations as in [4] for the linear terms and taking into account that the nonlinear terms are bounded as follows

\[
|G(u_\varepsilon)| u_\varepsilon | \leq (\varepsilon + 1) |u_\varepsilon|^2 + |G(0)|^2 \text{ mes } (\Omega)
\]
\[
|\epsilon \int_0^t (a^{-1}G(u_\epsilon), u'_\epsilon) ds| \leq \frac{\epsilon \omega}{16} \int_0^t |u'_\epsilon|_2^2 ds + \epsilon k_1 \epsilon^2 \int_0^t |u_\epsilon|_2^2 ds + \epsilon k_2 (G(0))^2 \text{mes } \Omega
\]

we obtain the statement.

With assumption H1:

We use a method of approximation. We consider a family satisfying hypothesis H2, such that

\[
(\epsilon \mu; u_\epsilon, \mu; u_1, \mu) \to (f, u_0, u_1) \text{ in } L^2(Q) \times H_0^1(\Omega) \times L^2(\Omega)
\]

Then \(|u_{\epsilon, \mu}|_2 + \sqrt{\epsilon} \|u_{\epsilon, \mu}\|_2 + \sqrt{\epsilon} \|u'_{\epsilon, \mu}\|_2 \leq K_1 \mu \) and \(K_1, \mu\) is bounded independently of \(\mu\). So we can extract a subsequence still noted \(u_{\epsilon, \mu}\) such that:

- \(u_{\epsilon, \mu}\) converges to \(v_\epsilon\) in \(L^\infty(0,T; H_0^1(\Omega))\) weak star,
- \(u'_{\epsilon, \mu}\) converges weakly to \(v'_\epsilon\) in \(L^2(Q)\),
- \(u''_{\epsilon, \mu}\) converges weakly to \(v''_\epsilon\) in \(L^2(0,T; H^{-1}(\Omega))\).

As \(u_{\epsilon, \mu}\) converges to \(v_\epsilon\) in \(L^2(Q)\), \(G(u_{\epsilon, \mu})\) converges to \(G(v_\epsilon)\) in \(L^2(Q)\).

Hence we can take the limit with respect to \(\mu\) in the equation satisfied by \(u_{\epsilon, \mu}\) and in boundary conditions and initial data.

We deduce that \(v_\epsilon = u_\epsilon\) which gives us the estimates of the theorem.

The estimates on the derivatives of \(u_\epsilon\) are not sufficient to conclude about the behavior of \(u_\epsilon\) as \(\epsilon \to 0_+\). Under the assumptions of this section, they may be improved by an estimate of \(\sqrt{\varphi} L_1 u_\epsilon\) independent of \(\epsilon\), the weight function \(\varphi\) being introduced in order to compensate the behavior of the derivatives of \(u_\epsilon\), in a neighborhood of the surface defining the boundary layer.

THEOREM 2.1.2. Under assumption H'1 and condition (A), for each function \(\varphi\) satisfying \(A_1(i)\), the solution \(u_\epsilon\) of problem \(P_\epsilon\) verifies:

\[
\forall \epsilon \in ]0, \epsilon_0[, \quad \|\varphi L_1 u_\epsilon\|_2 + \sqrt{\epsilon} \sqrt{\varphi} |u'_\epsilon|_2 \leq C_2 K_1 (f, u_0, u_1, \sqrt{\epsilon})
\]

Proof: One can easily check as for theorem 2.1.1 that it is sufficient to show theorem 2.1.2. under hypothesis H2.

Then we take the inner product of the two members of (1.1) with \(\varphi \ L_1 u_\epsilon\).

We transform the linear terms as in [4] theorem 2.3, the nonlinear term is bounded by:

\[
\int_0^t (G(u_\epsilon), \varphi L_1 u_\epsilon) ds \leq \int_0^t \|\sqrt{\varphi} u_\epsilon\|_2 \sqrt{\varphi} L_1 u_\epsilon ds + \int_0^t |G(0)|_2 |\sqrt{\varphi} L_1 u_\epsilon|_2 ds
\]

\[
\leq k_3 K_1^2 (f, u_0, u_1, \epsilon) + \frac{1}{4} \int_0^t |\sqrt{\varphi} L_1 u_\epsilon|_2^2 ds
\]
and theorem 2.1.2 follows.

At last, with the additional hypothesis $H_2$, we can obtain an estimate of $u'_e$ in $L^\infty(0,T;L^2(\Omega))$ which is independent of $\varepsilon$, by the method of differential ratios.

**THEOREM 2.1.3.** With assumptions $H_1$, $H_2$, condition (A) and the coefficients $b_k$ independent of $t$; for each $\varepsilon > 0$, $\varepsilon < \varepsilon_0$, the solution $u_\varepsilon$ of $\mathcal{P}_\varepsilon$ verifies

$$
|u'_e|^2 + \varepsilon  \|u'_e\|_{L^2} + \varepsilon^2 \|u''_e\|_{L^2} \leq C_3 K_3(f, u_0, u_0', \varepsilon)
$$

where $K^2_3(f, u_0, u_0', \varepsilon) = \|f\|_{L^2}^2 + \|u_1\|_{H^2(\Omega)}^2 + \|u_0\|_{H^2(\Omega)}^2 + (G(0))^2 + |f(0)|^2$.

**Proof.** We use a method of differential ratios. We consider equality (1.4) with $v \in H_0^1(\Omega)$, at time $s$ and $s + h$ ($h > 0$).

We set $w_\varepsilon(s, h) = \frac{1}{h} [u_\varepsilon(s + h) - u_\varepsilon(s)]$ and throughout the proof the constants $k_j$ are moreover independent of $h$.

By subtracting the two equalities, we have:

$$
\varepsilon (w''_\varepsilon v) + \varepsilon a(w_\varepsilon h)v + (a(s + h) w''_\varepsilon v) + \frac{a(s + h) - a(s)}{h} u'_e(s, v) + \sum_{k=1}^{n} (b_k \frac{\partial w_\varepsilon}{\partial x_k}, v)
$$

$$
= \frac{1}{h} (G(u_\varepsilon(s + h)) - G(u_\varepsilon(s)), v) = \frac{1}{h} (f(s + h) - f(s), v)
$$

By taking $v = w'_\varepsilon h + 2\varepsilon a^{-1}(s + h)w'_\varepsilon h$ and integrating from $0$ to $t$, we obtain as in the first part of theorem 2.1.1:

$$
\frac{\varepsilon^2}{4} \delta_0 \|w'_e\|_{H^1}^2 + \varepsilon^2 \delta_0 a(w_\varepsilon h, w_\varepsilon h) + \frac{\varepsilon \omega}{8} \int_0^t (\|w'_e, h\|_{L^2}^2 + \alpha(w_\varepsilon h, w_\varepsilon h)) ds + \frac{\delta}{6} \|w_\varepsilon h\|_{L^2}^2
$$

$$
\leq K_e(h) + k_0 \int_0^t \|u'_e(s)\|_{L^2}^2 ds + k_1 \int_0^t \|w'_e h\|_{L^2}^2 ds + \frac{1}{2} \|\sqrt{a(x,0)} w_\varepsilon h(0)\|_{L^2}^2 + \frac{1}{2} \|\sqrt{a^{-1}(x,0) w'_\varepsilon h(0)}\|_{L^2}^2
$$

$$
+ \varepsilon^2 \|\sqrt{a^{-1}(x,0)} \nabla \nabla w_\varepsilon h(0)\|_{L^2}^2 + \frac{1}{h} \int_0^T |f(s + h) - f(s)|^2 ds,
$$

where $K_e(h) = \varepsilon (w''_\varepsilon h(0), w_\varepsilon h(0)) + \frac{1}{2} \|\sqrt{a(x,0)} w_\varepsilon h(0)\|_{L^2}^2 + \frac{1}{2} \|\sqrt{a^{-1}(x,0) w'_\varepsilon h(0)}\|_{L^2}^2$

$$
and \delta_0 = \inf_{Q} a^{-1}(x, t)
$$

and where the nonlinear term has been bounded as follows:

$$
\frac{1}{h} \int_0^t (G(u_\varepsilon(s + h)) - G(u_\varepsilon(s)), v) ds \leq k_2 \int_0^t \|w'_e h(s)\|_{L^2}^2 ds + k_3 \varepsilon^2 \int_0^t \|w'_e h(s)\|_{L^2}^2 ds.
$$
Thanks to (1.1), one can see that \( e u^e(0) \) is bounded in \( L^2(\Omega) \) independently of \( e \) and so that:

\[
K_e(h) \leq k_4 K_3^2(f, u_0, u_1, \epsilon) \text{ for small } h.
\]

Then (2.1) implies:

\[
\frac{\delta}{6} \| w_{e,h} \|_2^2 \leq k_4 K_3^2(f, u_0, u_1, \epsilon) + k_0 \int_0^T \| u'_e(s) \|_2^2 \, ds + k_1 \int_0^T \| w_{e,h} \|_2^2 \, ds
\]

from which we deduce, by Gronwall's lemma:

\[
\int_0^T \| w_{e,h} \|_2^2 \, ds \leq k_5 (K_3^2(f, u_0, u_1, \epsilon) + \int_0^T \| u'_e(s) \|_2^2 \, ds).
\]

(2.2)

It results from (2.1) and (2.2) that a subsequence of \( w_{e,h} \) is such that:

\[
w_{e,h} \text{ converges to } u'_e \text{ in } L^\infty(0,T; L^2(\Omega)) \text{ weak-star, weakly in } L^2(0,T; H^1_0(\Omega)) \text{ and strongly in } L^2(\Omega),
\]

\[
w'_{e,h} \text{ converges to } u''_e \text{ weakly in } L^2(\Omega) \text{ and consequently by taking the limit with respect to } h \text{ in (2.1), we have}:
\]

\[
\frac{\epsilon \omega}{8} \int_0^t (\| u''_e \|_2^2 + \| u'_e \|_2^2) \, ds + \frac{\delta}{6} \| u'_e \|_2^2 \leq k_6 K_3^2(f, u_0, u_1, \epsilon) + k_7 \int_0^t \| u'_e(s) \|_2^2 \, ds.
\]

Theorem 2.1.3 follows thanks to Gronwall's lemma:

2.2 - CONVERGENCE

2.2.1 - FIRST RESULTS OF CONVERGENCE:

We assume in all this subsection hypotheses \( H_1, H_2 \), conditions (A) et (B). The solution \( u_e \) of \( \mathcal{P}_e \) satisfies the estimates of theorems 2.1.1, 2.1.2. Moreover we deduce from (1.1) that for each functions \( \varphi \) satisfying conditions \( A_1(i) \) and for \( \epsilon < \epsilon_0 \):

\[
\epsilon \| \sqrt{\varphi} \|_{L^2(\Omega)}^2 \leq k_0 K_1(f, u_0, u_1, \sqrt{\epsilon})
\]

Then, we can extract a subsequence, still written \( u_e \), such that:

\[
\begin{align*}
 u_e & \rightharpoonup u & \text{in } L^\infty(0,T; L^2(\Omega)) \text{ weak-star} \\
 \sqrt{\varphi} \, L_1 u_e & \rightharpoonup \sqrt{\varphi} \, L_1 u & \text{weakly in } L^2(\Omega) \\
 \epsilon \sqrt{\varphi} \, L_2 u_e & \rightharpoonup 0 & \text{weakly in } L^2(\Omega) \\
 G(u_e) & \rightharpoonup \chi & \text{in } L^\infty(0,T; L^2(\Omega)) \text{ weak-star}
\end{align*}
\]

(2.3)
Where $u$ verifies ([4], section 3)

\[
\begin{cases}
L_1 u + \chi = f \\
u(x,0) = u_0, \quad u \mid_{\Sigma} = 0.
\end{cases}
\]

Its remains to prove that $X = G(u)$, which can be established by a monotonicity method ([4], section 3), by noting that we can write $G(u) = -(\ell+1)u + Mu$ where $M$ is a strictly monotone and hemicontinuous operator (the monotonicity method is used in [4] when the operator $L_1 - (\ell+1)I$ is positive; we are brought back to this case by the change of variable $U_\varepsilon = u_\varepsilon e^{\lambda t}$, the constant $\lambda$ being chosen such that the new first order linear operator is positive. We remark that the new nonlinear function is defined by $\tilde{G}(U_\varepsilon) = e^{\lambda t} G(U_\varepsilon e^{\lambda t})$ and verifies

\[
|\tilde{G}(U) - \tilde{G}(V)| \leq \ell |U - V|
\]

We can then apply the monotonicity method to the function $U_\varepsilon$ which satisfies the same properties of regularity and the same estimates as $u_\varepsilon$, because:

\[
U_\varepsilon = u_\varepsilon e^{\lambda t}
\]

So $u$ is solution of the problem

\[
P \quad \begin{cases}
L_1 u + G(u) = f \\
u \in L^\infty(0,T;L^2(\Omega)) \\
u(x,0) = u_0, \quad u \mid_{\Sigma} = 0
\end{cases}
\tag{2.4}
\]

Remark. It results from (2.4) that $L_1 u \in L^\infty(0,T;L^2(\Omega))$. Moreover it is easy to see that $u$ is unique, thanks to Green's formula (1.7).

Hence, we have the

**LEMMA 2.2.1. (weak convergence)** With assumptions $H_1$, $H_2$ conditions (A), (B), the solution $u_\varepsilon$ of $P_\varepsilon$ converges to the solution of problem $P$ in $L^\infty(0,T;L^2(\Omega))$ weak star.

Moreover $\sqrt{\varphi} L_1 u_\varepsilon$ converges to $\sqrt{\varphi} L_1 u$ weakly in $L^2(\Omega)$ and if $b'k = 0$ ($k=1,2,\ldots,n$) $u_\varepsilon'$ converges to $u'$ in $L^\infty(0,T;L^2(\Omega))$ weak star.

Our aim is now to obtain some results of strong convergence.

**LEMMA 2.2.2.** With the hypotheses of lemma 2.2.1, the solution $u_\varepsilon$ of problem $P_\varepsilon$ verifies $\sqrt{\varphi} u_\varepsilon$ converges to $\sqrt{\varphi} u$ in $L^\infty(0,T;L^2(\Omega))$ for each function $\varphi$ satisfying $A_T$. 
Proof. We consider \( w_\varepsilon = u_\varepsilon - u \); \( w_\varepsilon \) satisfies

\[
\begin{align*}
    e \left(L_2 u_\varepsilon + L_1 w_\varepsilon + G(u_\varepsilon) - G(u)\right) &= 0 \quad \text{a.e. in } L^2(\Omega) \\
    \varphi w_\varepsilon &= 0, \quad w_\varepsilon(x,0) = 0.
\end{align*}
\]

(2.5)

We can take the inner product of the two members of (2.5) with \( \varphi \in L^\infty(0,T; L^2(\Omega)) \). After integration from 0 to \( t \), it comes:

\[
e \int_0^t \left( (u_\varepsilon') \varphi u_\varepsilon + \alpha(u_\varepsilon, \varphi u_\varepsilon) \right) ds + \frac{1}{2} \left| \sqrt{\varepsilon} \varphi \right| L^2_2 w_\varepsilon ds - \frac{1}{2} \int_0^t \left( L_1 \varphi \right) w_\varepsilon^2 dx ds
\]

\[
= \int_0^t \left( e \sqrt{\varepsilon} L_2 u_\varepsilon', \sqrt{\varepsilon} \varphi u_\varepsilon \right) ds + \frac{1}{2} \int_0^t \left( a' + \sum_{k=1}^n \frac{\partial b_k}{\partial x_k} \right) \varphi w_\varepsilon^2 dx ds - \int_0^t \left( G(u_\varepsilon) - G(u) \right) \varphi w_\varepsilon ds
\]

from where, we deduce, by integrating by parts the term

\[
\int_0^t \left\{ (u_\varepsilon', \varphi u_\varepsilon) + \alpha(u_\varepsilon, \varphi u_\varepsilon) \right\} ds, \quad \text{and taking into account (1.6) with } \theta = \sqrt{\varepsilon}, \quad L_1 \varphi \leq 0,
\]

\[
\left| G(u_\varepsilon) - G(u) \right| \leq \| \varphi \| w_\varepsilon
\]

\[
e \frac{3 \omega}{4} \int_0^t \left| \sqrt{\varepsilon} \varphi \right| \left| \text{grad } u_\varepsilon \right|_2^2 ds + \frac{\delta}{2} \left| \sqrt{\varepsilon} \varphi \right| L^2_2 w_\varepsilon^2 ds \leq H_\varepsilon(t) + k_1 \int_0^t \left| \sqrt{\varepsilon} \right| \varphi w_\varepsilon^2 ds
\]

(2.6)

with

\[
H_\varepsilon(t) \leq \int_0^t \left( e \sqrt{\varepsilon} L_2 u_\varepsilon', \sqrt{\varepsilon} \varphi u_\varepsilon \right) ds + \sqrt{\varepsilon} \left\{ \left| \left| e \varphi u_\varepsilon \right|_2 \right| \left| \left| \left| e \varphi \right|_2 \right| \right| \right\}
\]

\[
+ k_2 \left| e \varepsilon \right| \left| \left| u_\varepsilon \right|_2 \right| \left| \left| u_\varepsilon \right|_2 \right| + e \left\{ (u_1, \varphi, (x,0)u_0) + \omega \left| \sqrt{\varepsilon} \right| L^2_1 u_\varepsilon^2 \right\}
\]

\[
\leq \int_0^t \left( e \sqrt{\varepsilon} L_2 u_\varepsilon', \sqrt{\varepsilon} \varphi u_\varepsilon \right) ds + \sqrt{\varepsilon} \left| k_3 \right| \left| \left| f, u_0, u_1, \sqrt{\varepsilon} \right|_2 \right|
\]

thanks to the theorems 2.1.1, 2.1.2.

From (2.6) we deduce that:

\[
\int_0^t \left| \sqrt{\varepsilon} \right| \varphi w_\varepsilon^2 ds \leq k_4 \int_0^t H_\varepsilon(s) ds.
\]

As (2.3) implies: \( H_\varepsilon(s) \) bounded by a constant independent of \( \varepsilon \) and \( \lim_{\varepsilon \to 0} H_\varepsilon(s) = 0 \), lemma 2.2.2 follows thanks to Lebesgue's theorem.

The following lemma gives a result of convergence for \( \varphi L_1 u_\varepsilon \).

**Lemma 2.2.3.** We assume hypotheses \( H_1, H_2 \) and conditions (A), (B). Then, for each function \( \varphi \) satisfying condition \( \mathcal{A}_1 \), the solution \( u_\varepsilon \) of \( \mathcal{P}_\varepsilon \) verifies:

\[
\varphi L_1 u_\varepsilon \to \varphi L_1 u \quad \text{in} \quad L^2(Q)
\]

**Proof.** We consider the inner product of the two members of (2.5) with \( \varphi \varepsilon^2 L_1 w_\varepsilon \in L^2(Q) \) and we integrate...
from 0 to t. We obtain:

\[ \int_0^t \left( \varepsilon L_2 u_\varepsilon \phi^2 L_1 u_\varepsilon \right) ds + \int_0^t \left( \phi \varepsilon L_1 w_\varepsilon \varphi^2 \right) ds + \int_0^t (G(u_\varepsilon) - G(u), \varphi^2 L_1 w_\varepsilon) ds \]

\[ = \int_0^t (\varepsilon L_2 u_\varepsilon \phi^2 L_1 u) ds \]  

(2.7)

By integrating by parts the term \( \varepsilon \int_0^t (L_2 u_\varepsilon, \varphi^2 L_1 u_\varepsilon) ds \), then using inequality (1.5), and inequality (1.6) with \( \theta^2 = -\varphi L_1 \varphi \), we show the minoration:

\[ \varepsilon \int_0^t (L_2 u_\varepsilon, \varphi^2 L_1 u_\varepsilon) ds \geq \frac{\delta \omega}{4} \left\| \phi u_\varepsilon \phi^2 + \varphi |\nabla \overrightarrow{\varepsilon} u_\varepsilon| \right\|_{L^2}^2 - m_1 \sqrt{\varepsilon} 
\]

\[ - m_2 \varepsilon \int_0^t \left\| \phi \varphi^2 L_1 u_\varepsilon \right\|_{L^2}^2 + m_1 \sqrt{\varepsilon} \]

where \( m_1, (i = 1,2) \) is a constant independent of \( \varepsilon \).

Then, it results from (2.7) that:

\[ \varepsilon \frac{\delta \omega}{4} \left\| \phi u_\varepsilon \phi^2 + \varphi |\nabla \overrightarrow{\varepsilon} u_\varepsilon| \right\|_{L^2}^2 + \frac{1}{2} \int_0^t \left\| \phi \varepsilon L_1 w_\varepsilon \right\|_{L^2}^2 ds \leq M_\varepsilon (t) + \varepsilon m_2 \int_0^t \left\| \phi u_\varepsilon \phi^2 + \varphi |\nabla \overrightarrow{\varepsilon} u_\varepsilon| \right\|_{L^2}^2 ds 
\]

where \( M_\varepsilon (t) = \frac{1}{2} \left\| \phi \left[ G(u_\varepsilon) - G(u) \right] \right\|_{L^2}^2 + \frac{1}{2} \int_0^t \left( \varepsilon L_2 u_\varepsilon, \varphi^2 L_1 u \right) ds + m_1 \sqrt{\varepsilon} \).

As \( M_\varepsilon (t) \to 0 \) when \( \varepsilon \to 0^+ \), and \( M_\varepsilon (t) \) is bounded independently of \( \varepsilon \) thanks to (2.3), we conclude thanks to Lebesgue's theorem that \( \left\| \phi \varepsilon L_1 w_\varepsilon \right\|_{L^2}^2 \to 0 \) and the lemma follows.

**Remark.** The proof of the lemma also shows that \( \sqrt{\varepsilon} \phi u_\varepsilon \phi \to 0 \) and \( \sqrt{\varepsilon} \varphi |\nabla \overrightarrow{\varepsilon} u_\varepsilon| \to 0 \) in \( L^\infty (0,T;L^2(\Omega)) \).

### 2.2.2 - CONVERGENCE OF \( u_\varepsilon \) AND \( L_1 u_\varepsilon \):

The results of the subsection 2.2.1 may be improved as follows.

**THEOREM 2.2.4.** With hypothesis \( H_1 \), conditions \( (A) \) and \( (B) \), the solution \( u_\varepsilon \) of problem \( \mathcal{P}_\varepsilon \) verifies:

(i) \( u_\varepsilon \) converges to \( u \) in \( L^\infty (0,T;L^2(\Omega)) \) weak star and in \( L^q (Q) \), \( \forall q < 2 \) where \( u \) is the solution of the problem \( \mathcal{P} \).

\( u_\varepsilon \) converges to \( u \) and \( L_1 u_\varepsilon \) to \( L_1 u \) in \( L^2 (Q') \) where \( Q' \) is an open set of \( Q \) such that \( Q' \cap \mathcal{Y} (\Sigma_\varepsilon) = \phi \), where \( \mathcal{Y} (\Sigma_\varepsilon) \) is a neighborhood of \( \Sigma_\varepsilon \).
for each function \( \varphi \) satisfying conditions (ii):
\[
\sqrt{\varphi} \ u_e \to \sqrt{\varphi} \ u \ in \ L^\infty(0,T;L^2(\Omega))
\]
\[
\sqrt{\varphi} \ L_1 \ u_e \to \sqrt{\varphi} \ L_1 \ u \ weakly \ in \ L^2(\Omega) \ and \ \varphi \ L_1 \ u_e \to \varphi \ L_1 \ u \ strongly \ in \ L^2(\Omega)
\]

(iii) If we also assume hypothesis H2 and that the coefficients \( b_k \) are independent of \( t, u \in L^\infty(0,T;L^2(\Omega)) \) weak-star.

Proof. We remark that existence and uniqueness of \( u \) solution of the problem P is insured under the single hypotheses H1 and H'1 (C. Bardos [1] p.199, by using the transformation \( Gu = -(k+1)u + Mu \)).

To prove points (i) and (ii), we use a method of regularization as in the proof of theorem 2.1.1. We approximate the triplet \( (f,u_0,u_1) \) by a sequence \( (f_{\mu,0,\mu};u_{1,\mu}) \) satisfying H2 such that:
\[
(f_{\mu,0,\mu};u_{1,\mu}) \to (f,u_0,u_1) \ in \ L^2(\Omega) \times H^1_0(\Omega) \times L^2(\Omega)
\]

Let \( w_{\epsilon,\mu} = u_{\epsilon,\mu} - u_{\epsilon} \) and \( w_{\mu} = u_{\mu} - u \). We have:
\[
\epsilon \ L_2 w_{\epsilon,\mu} + L_1 w_{\epsilon,\mu} + G(u_{\epsilon,\mu}) - G(u_{\epsilon}) = f_{\mu} - f
\]
and it results from theorems 2.1.1 and 2.1.2, since \( |G(u_{\epsilon,\mu}) - G(u_{\epsilon})| \leq \ell|w_{\epsilon,\mu}| \), that
\[
|w_{\epsilon,\mu}|^2 + |\sqrt{\varphi} \ L_1 w_{\epsilon,\mu}|^2 \leq k_0 (|f_{\mu} - f|^2 + |u_{0,\mu} - u_0|^2 + |u_{1,\mu} - u_1|^2)
\]

where \( k_0 \) is a positive constant independent of \( \mu \) and \( \epsilon \)
\[
L_1 w_{\mu} + G(u_{\mu}) - G(u) = f_{\mu} - f
\]

from where we deduce by taking the inner product of (2.10) with \( w_{\mu} \), using Green’s formula (1.7) and at last by integrating from 0 to t
\[
\frac{\delta}{2} |w_{\mu}|^2 + \frac{1}{2} \int_{\Sigma} (\sum_{k=1}^n b_k \nu_k)w_{\mu}^2 d\sigma \leq k_1 |u_{0,\mu} - u_0|^2 + \frac{1}{2} |f_{\mu} - f|^2 + k_2 \int_0^t |w_{\mu}|^2 ds.
\]

Then, Gronwall's lemma implies:
\[
|w_{\mu}|^2 \leq k_3 (|f_{\mu} - f|^2 + |u_{0,\mu} - u_0|^2), \ k_3 \ positive \ constant \ independent \ of \ \mu
\]

Now by taking the inner product of (2.10) with \( L_1 w_{\mu} \), we obtain:
\[
|L_1 w_{\mu}|^2 \leq k_4 (|f_{\mu} - f|^2 + |u_{0,\mu} - u_0|^2), \ k_4 \ positive \ constant \ independent \ of \ \mu
\]
At last, by using the results of subsection 2.2.1, for each fixed \( \mu \):

\[
\begin{align*}
    u_\varepsilon, \mu & \rightarrow u_\mu \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak-star} \\
    \sqrt{\varphi} u_\varepsilon, \mu & \rightarrow \sqrt{\varphi} u_\mu \text{ in } L^\infty(0,T;L^2(\Omega)) \\
    \sqrt{\varphi} L_1 u_\varepsilon, \mu & \rightharpoonup \sqrt{\varphi} L_1 u_\mu \text{ weakly in } L^2(Q) \\
    \varphi L_1 u_\varepsilon, \mu & \rightarrow \varphi L_1 u_\mu \text{ in } L^2(Q)
\end{align*}
\]

(2.13)

as \( \varepsilon \to 0_+ \).

As \( u_\varepsilon - u = - w_\varepsilon, \mu + u_\varepsilon, \mu - u_\mu + w_\mu \), one can easily check thanks to (2.8), (2.9), (2.11), (2.12), (2.13) that:

\[
\begin{align*}
    u_\varepsilon & \rightarrow u \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak-star} \\
    \sqrt{\varphi} u_\varepsilon & \rightarrow \sqrt{\varphi} u \text{ in } L^\infty(0,T;L^2(\Omega)) \\
    \sqrt{\varphi} L_1 u_\varepsilon & \rightharpoonup \sqrt{\varphi} L_1 u \text{ weakly in } L^2(Q) \text{ and } \varphi L_1 u_\varepsilon \rightarrow \varphi L_1 u \text{ in } L^2(Q)
\end{align*}
\]

and the point (ii) follows. To achieve the proof of the point (i) we remark that the convergence of \( u_\varepsilon \) and \( L_1 u_\varepsilon \) in \( L^2(Q') \) results from the properties of the functions \( \varphi \). These properties also imply that \( u_\varepsilon \rightarrow u \) a.e. in \( Q \).

As \( |u_\varepsilon - u|^q \) is bounded in \( L^{2/q}(Q) \), \( \forall q \leq 2 \), there is a subsequence of \( u_\varepsilon \) such that \( |u_\varepsilon - u|^q \rightharpoonup 0 \) weakly in \( L^{2/q}(Q) \), \( \forall q < 2 \), from where \( u_\varepsilon \) converges to \( u \) strongly in \( L^q(Q) \), \( \forall q < 2 \).

The point (iii) results from lemma 2.2.1.

### 3. CONVERGENCE OF THE DERIVATIVES OF \( u_\varepsilon \)

In this section, we improve the results of convergence. We aim at obtaining, on the one hand, the strong convergence of \( u_\varepsilon \) in local spaces, on the other hand, the rate of convergence in \( \varepsilon \) of \( \varphi^{3/2}(u_\varepsilon - u) \) in the space \( L^\infty(0,T;L^2(\Omega)) \). This kind of results needs hypotheses of regularity on \( f \), because of the non-regularity of \( u \) under the only assumptions : \( f, f' \in L^2(Q) \), the derivatives of the function \( u \) generally having poles on the part \( \Sigma_0 \) of \( \Sigma \).

So, we impose on \( f \) the hypothesis

$$
H_3 \quad \begin{cases} 
    f \in L^2(0,T;H^1(\Omega)) \\
    f = 0 \text{ on } \gamma = \gamma(\Sigma_0 \cup \Lambda) \cap \Sigma - \text{ where } \gamma(\Sigma_0 \cup \Lambda) \text{ is a neighborhood of } \Sigma_0 \cup \Lambda \text{ in } \Sigma.
\end{cases}
$$

Then, there exists \( \lambda > 0 \), such that

$$
\sum_{k=1}^{n} b_k \nu_k \leq - \lambda \text{ on } (\Sigma-) - \gamma.
$$

(3.1)

With the hypothesis \( H_3 \), we first establish additional a priori estimates which allow us to obtain by compactness arguments the convergence of \( u_\varepsilon \) to \( u \) solution of the problem \( P \).
3.1 - A PRIORI ESTIMATES

THEOREM 3.1.1. We suppose hypotheses $H_1, H_2, H_3$, conditions (A), (B) and $G(0) = 0$. Then for $0 < \varepsilon < \varepsilon_0$, the solution $u_\varepsilon$ satisfies the estimates of theorems 2.1.1, 2.1.2, 2.1.3 and moreover verifies:

for each function $\varphi$ satisfying $\mathcal{A}\varphi$

$$\left| \varphi^{3/2} u_\varepsilon \right|_2^2 + \| \varphi^{3/2} u_\varepsilon \|_2^2 + \varepsilon \| \varphi \|_2^2 \leq C_4 K_4 (f, u_0, u_1, \varepsilon).$$

where

$$K_4^2(f, u_0, u_1, \varepsilon) = \frac{1}{\lambda} \left\| f_1 \right\|_{L^2(\Sigma)}^2 + \left\| f_2 \right\|_2^2 + \left\| f_3 \right\|_2^2 + \| u_0 \|_{H^2(\Omega)}^2 + \| u_0 \|_2^2 + \| f(0) \|_2^2$$

Proof. The smoothness properties of $u_\varepsilon$, under hypothesis $H_2$, allow us to take the inner product of two members of (1.1) with $- \varphi^3 \Delta u_\varepsilon$. It comes:

$$-\varepsilon (u_\varepsilon, \varphi^3 \Delta u_\varepsilon) + \varepsilon |\varphi^{3/2} \Delta u_\varepsilon|_2^2 - (L_1 u_\varepsilon, \varphi^3 \Delta u_\varepsilon) - (G(u_\varepsilon), \varphi^3 \Delta u_\varepsilon) = -(f, \varphi^3 \Delta u_\varepsilon)$$

Green’s formula gives the following transformations:

$$(L_1 u_\varepsilon, \varphi^3 \Delta u_\varepsilon) = \frac{1}{2} \int_\Omega \left( \sum_{k=1}^n \partial_k \varphi^3 \right) \left| \nabla u_\varepsilon \right|_2^2 + \frac{1}{2} \int_\Omega \left( \sum_{k=1}^n b_k \nabla \nabla u_\varepsilon \right) \varphi^3 \left| \nabla u_\varepsilon \right|_2^2$$

$$-\frac{3}{2} \int_{\partial \Omega} \varphi^2 (L_1 \varphi) \left| \nabla u_\varepsilon \right|_2^2$$

where $R(u_\varepsilon) = 3 \int_\Omega \varphi^2 L_1 u_\varepsilon (\nabla \varphi \cdot \nabla u_\varepsilon) dx + \int_\Omega \varphi^3 u_\varepsilon^3 (\nabla a \cdot \nabla u_\varepsilon) dx$

$$+ \sum_{k=1}^n \int_\Omega \varphi^3 \frac{\partial u_\varepsilon}{\partial x_k} (\nabla b_k \cdot \nabla u_\varepsilon) dx - \frac{1}{2} \int_\Omega (a' + \sum_{k=1}^n \frac{\partial b_k}{\partial x_k}) \varphi^3 \left| \nabla u_\varepsilon \right|_2^2$$

as $G(u_\varepsilon) \in L^2(0, T; H^1_0(\Omega))$, thanks to lemma 1.1.

$$-(f, \varphi^3 \Delta u_\varepsilon) = \alpha(f, u_\varepsilon) - \int_\Gamma \varphi^3 f \frac{\partial u_\varepsilon}{\partial n} d\Gamma$$

Then we have:

by taking into account theorem 2.1.2:

$$\int_0^t R(u_\varepsilon) ds \leq K_4^2(f, u_0, u_1, \sqrt{\sigma}) + k_1 \int_0^t \varphi^{3/2} u_\varepsilon^2 ds + k_2 \int_0^t \varphi^{3/2} \left| \nabla u_\varepsilon \right|_2^2 ds$$

(3.6)
thanks to theorem 2.1.1. and lemma 1.1:

\[
\left| \int_0^t (G(u_\varepsilon), \varphi^3 \Delta u_\varepsilon) ds \right| \leq k_3 k_1^2 (f, u_0, u_{1, \varepsilon}) + k_4 \int_0^t \left| \varphi^{3/2} \right| \| \text{grad } u_\varepsilon \|_2^2 ds. \tag{3.7}
\]

and at last:

\[
\int_0^t \left( e (u_{\varepsilon}^n, \varphi^3 \Delta u_\varepsilon) \right) ds \leq \frac{e}{2} \int_0^t \left| \varphi^{3/2} u_{\varepsilon/n} \right|^2 ds + \frac{e}{2} \int_0^t \left| \right. \Delta u_\varepsilon \left. \right|^2 ds \leq k_6 \frac{e}{\lambda} \| f \|_2^2 + k_8 \int_0^t \left( \frac{b_{k+1}}{k} \right) \varphi^3 \| \text{grad } u_\varepsilon \|_2^2 d \Gamma. \tag{3.8}
\]

\[
\int_0^t \left| \frac{\partial u_\varepsilon}{\partial \nu} \right| d \Gamma \leq \frac{k_5}{\lambda} \frac{\| f \|_2^2}{L^2(\Sigma)} - \frac{1}{4} \int_0^t \sum_{k=1}^{n} b_{k+1} \varphi^3 \| \text{grad } u_\varepsilon \|^2 d \Gamma. \tag{3.9}
\]

So, taking into account results (3.3) to (3.9), \( L_1 \varphi \leq 0 \) on \( Q \), \( \left( \frac{b_{k+1}}{k} \right) \leq 0 \) on \( \Sigma \) and the properties of the coefficients, equality (3.2) gives:

\[
\frac{e}{2} \int_0^t \left| \varphi^{3/2} \Delta u_\varepsilon \right|^2 ds + \frac{\delta}{2} \left| \varphi^{3/2} \right| \| \text{grad } u_\varepsilon \|^2 \leq k_6 \frac{e}{\lambda} \| f \|_2^2 + k_8 \int_0^t \left( \frac{b_{k+1}}{k} \right) \varphi^3 \| \text{grad } u_\varepsilon \|^2 d \Gamma.
\]

Now, we consider the method of the differential ratios when the coefficients \( b_k \) depend on \( t \). We have with the same notation as in the proof of the theorem 2.1.3:

\[
e e \left( w_{\varepsilon,n}^n, h^l, v \right) + \left( a(s+h)w_{\varepsilon,h}^l, v \right) + \left( a(s)w_{\varepsilon,h}^l, v \right) + \sum_{k=1}^{n} \left( b_k(s+h) - b_k(s) \right) \frac{\partial w_{\varepsilon,h}^l}{\partial x_k}, v \right) + \left( \frac{b_k(s+h)}{h} - b_k(s) \right) \frac{\partial w_{\varepsilon,h}^l}{\partial x_k}, v \right) + \frac{1}{h} \left( G(u_{\varepsilon,h}^{s+h}) - G(u_{\varepsilon,h}) \right), v \right) = \frac{1}{h} \left( f(s+h) - f(s), v \right) \tag{3.11}
\]

We first obtain by taking \( v = e(w_{\varepsilon,h}^n + 2e a^{-1}(s+h)w_{\varepsilon,h}^n) \) in (3.11) as in the proof of the theorem 2.1.3:

\[
e e^2 \left| w_{\varepsilon,h}^l \right|^2 + e^2 \left| w_{\varepsilon,h}^l \right|^2 \leq k_9 K_3^2 (f, u_0, u_{1, \varepsilon}) \tag{3.12}
\]

and then by putting \( v = \varphi^3 (w_{\varepsilon,h}^n + 2e a^{-1}(s+h)w_{\varepsilon,h}^n) \) in (3.11) and taking into account (3.12), it comes:

\[
\frac{\delta}{6} \left| \varphi^{3/2} u_{\varepsilon,l}^n \right|^2 + \frac{\delta}{2} \left| \varphi^{3/2} \right| \| \text{grad } u_\varepsilon \|^2 \leq k_{11} \left( K_3^2 (f, u_0, u_{1, \varepsilon}) \right) + \int_0^t \left( \left| \varphi^{3/2} u_{\varepsilon,l}^n \right|^2 + \| \text{grad } u_\varepsilon \|^2 \right) ds. \tag{3.13}
\]

It results from (3.10) and (3.13):

\[
\frac{\delta}{6} \left| \varphi^{3/2} u_{\varepsilon,l}^n \right|^2 + \frac{\delta}{2} \left| \varphi^{3/2} \right| \| \text{grad } u_\varepsilon \|^2 \leq k_{11} \left( K_3^2 (f, u_0, u_{1, \varepsilon}) \right) + \int_0^t \left( \left| \varphi^{3/2} u_{\varepsilon,l}^n \right|^2 + \| \text{grad } u_\varepsilon \|^2 \right) ds. \tag{3.14}
\]

and theorem 3.1.1 follows.
3.2 - CONVERGENCE

With hypotheses $H_1, H_2, H_3$, conditions (A), (B) and $G(0) = 0$, the following a priori estimates

$$
|u_\varepsilon|_2 \leq C_1 K_1 (f, u_0, u_1, \varepsilon), \quad \left| \sqrt{\varphi} L_1 u_\varepsilon \right|_2 \leq C_2 K_1 (f, u_0, u_1, \varepsilon)
$$

allow us to extract a subsequence still denoted by $u_\varepsilon$ such that:

$u_\varepsilon \rightharpoonup u$ in $L^\infty(0,T;L^2(\Omega))$ weak star and $\varphi^{3/2} u_\varepsilon \rightharpoonup \varphi^{3/2} u$ in $L^\infty(0,T;H^1_0(\Omega))$ weak star, $\sqrt{\varphi} L_1 u_\varepsilon \rightharpoonup \sqrt{\varphi} L_1 u$ weakly in $L^2(Q)$.

So, in particular, we have: $\varphi^{3/2} u_\varepsilon$ converges to $\varphi^{3/2} u$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. The properties of functions $\varphi$ imply the existence of a new subsequence such that $u_\varepsilon$ converges to $u$ a.e. on $Q$ and so $G(u_\varepsilon)$ converges to $G(u)$ a.e. on $Q$. As $|G(u_\varepsilon)|_2 \leq \varepsilon K_1 (f, u_0, u_1, \varepsilon)$ we finally have:

$$
\lim u_\varepsilon = u \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak star.}
$$

One can check that $u$ is the solution of problem $P$ (for which we also have shown smoothness properties).

Hence we have the

**THEOREM 3.2.1.** **(weak-convergence).** Under hypotheses $H_1, H_2, H_3$, conditions (A), (B) and $G(0) = 0$, the solution $u_\varepsilon$ of $P_\varepsilon$ verifies:

i) $u_\varepsilon \rightharpoonup u$ in $L^\infty(0,T;L^2(\Omega))$ weak star.

ii) for each function $\varphi$ satisfying $\mathcal{A}_2$:

$$
\sqrt{\varphi} L_1 u_\varepsilon \rightharpoonup \sqrt{\varphi} L_1 u \text{ weakly in } L^2(Q), \quad \varphi^{3/2} u_\varepsilon \rightharpoonup \varphi^{3/2} u \text{ in } L^\infty(0,T;H^1_0(\Omega)) \text{ weak-star,}
$$

where $u$ is the solution of the problem $P$.

Of course, the results of strong convergence of theorem 2.2.4 are still valid in the frame of this section. We are now interested by the rate of convergence in $\varepsilon$ of $\varphi^{3/2} (u_\varepsilon - u)$ in $L^\infty(0,T;L^2(\Omega))$. For this, we first improve the estimates satisfied by $u$ in $L^\infty(0,T;H^1_0(\Omega))$ and $u'$ in $L^\infty(0,T;L^2(\Omega))$ which result from the theorem 3.2.1. We obtain the

**LEMMA 3.2.2.** **With the same hypotheses as in theorem 3.2.1, we have**
Proof. We consider the equality
\[ (L_1 u, \varphi^3 (u' - \Delta u)) + (G(u), \varphi^3 (u' - \Delta u)) = (f, \varphi^3 (u' - \Delta u)). \]
Thanks to (3.3), (3.4), (3.5), (3.7), (3.9) where \( u_e \) is replaced by \( u \), inequality
\[
\int_0^t \| R(u) \| ds \leq k_1 K_1^2 \| f(u_0 u_1 \sqrt{\varepsilon}) \| + \frac{\delta}{8} \int_0^t \| \varphi^{3/2} u' \|_2^2 ds + k_2 \int_0^t \| \varphi^{3/2} \|_2^2 \| \text{grad} u \|_2^2 ds,
\]
the fact that \( L_1 \varphi \leq 0 \) on \( Q \), \( \sum_{k=1}^{n} b_k \nu_k \leq 0 \) on \( \Sigma \), and the properties of the coefficients, it comes:
\[
\frac{\delta}{2} \| \varphi^{3/2} \|_2 \| \text{grad} u \|_2^2 + \frac{\delta}{4} \int_0^t \| \varphi^{3/2} u' \|_2^2 ds \leq k_3 \left( \frac{1}{\lambda_1} \| f \|_{L^2(Q)}^2 + \| u_0 \|_2^2 + \| u_1 \|_2^2 \right) + k_4 \int_0^t \| \varphi^{3/2} \|_2 \| \text{grad} u \|_2^2 ds.
\]
And Gronwall’s lemma gives the estimates. (When \( u \) is not smooth enough, the lemma results from the study of the solution of the regularized problem \[
- \eta \Delta v + L_1 v + G(v) = F, \quad \eta > 0,
\]
\[
\Sigma v = 0, \quad v(x,0) = u_0
\]
Now, we may prove the

THEOREM 3.2.3. (rate of convergence). With hypotheses of theorem 3.2.1, for each \( \varepsilon, 0 < \varepsilon < \varepsilon_o \) we have:
\[
\| \varphi^{3/2} (u_\varepsilon - u) \|_2 \leq K_5 \sqrt{\varepsilon} \quad (\text{for each } \varphi \text{ satisfying } \mathcal{A}_1)
\]
\[
\| u_\varepsilon - u \|_{L^2(Q')} \leq K_5 \sqrt{\varepsilon}
\]
where \( Q' \) is an open set of \( Q \) such that \( Q' \cap \gamma(\Sigma_\varepsilon) = \phi \).

Proof. We set \( w_\varepsilon = u_\varepsilon - u \) and we take the inner product of \( \varepsilon L_2 w_\varepsilon + L_1 w_\varepsilon + G(w_\varepsilon) - G(u) = 0 \) with \( \varphi^3 w_\varepsilon \in L^\infty(0,T;\mathcal{H}_0^1(\Omega)) \). It comes:
\[
\varepsilon \int_0^t \| \varphi^{3/2} \|_2 \| \text{grad} w_\varepsilon \|_2^2 ds + \frac{\delta}{2} \| \varphi^{3/2} w_\varepsilon \|_2^2 \leq \varepsilon A_\varepsilon(t) + k_1 \int_0^t \| \varphi^{3/2} w_\varepsilon \|_2^2 ds \quad (3.14)
\]
where \( A_\varepsilon(t) = - (\varphi^3 u_\varepsilon', \varphi^3 w_\varepsilon) + \omega_1 \int_0^t \| \varphi^{3/2} L_1 w_\varepsilon \|_2^2 ds + \int_0^t (\varphi^{3/2} u_\varepsilon', \varphi^{3/2} w_\varepsilon) ds
\]
\[- \int_{Q_t} \varphi^3 (\text{grad} u . \text{grad} w_\varepsilon) dx ds + \int_{Q_t} \varphi^2 w_\varepsilon (\varphi' . u_\varepsilon' + \text{grad} \varphi . \text{grad} u_\varepsilon) dx ds.
\]
As \( \varepsilon |A_\varepsilon(t)| \leq \varepsilon k_2 K_5^2 + \frac{\delta}{6} \| \varphi^{3/2} w_\varepsilon \|_2^2 + \varepsilon \frac{3\omega}{8} \int_0^t \| \varphi^{3/2} \|_2 \| \text{grad} w_\varepsilon \|_2^2 ds + k_3 \int_0^t \| \varphi^{3/2} w_\varepsilon \|_2^2 ds \)
thanks to the theorems 2.1.1, 2.1.2 and lemma 3.2.2, the statement follows by application of Gronwall’s lemma.
Remark. We also have shown that $|\varphi^{3/2}\| \nabla u\|_2 \leqslant K_5$ and $|u_\varepsilon^{3/2}\|_2 \leqslant K_5$.

The following theorem gives results of strong convergence for the derivatives of $u_\varepsilon$.

**THEOREM 3.2.4.** (Strong convergence of the derivatives). With hypotheses of theorem 3.2.1, we have:

(i) $\varphi^{3/2} u_\varepsilon \rightarrow \varphi^{3/2} u$ in $L^2(0,T; \mathcal{H}_0^1(\Omega))$

(ii) $u_\varepsilon \rightarrow u$ in $H^1(Q)$ where $Q^c$ is an open set of $Q$ with $Q^c \cap \gamma(\Sigma_\varepsilon) = \emptyset$.

**Proof.** We consider again (3.14).

As $\lim_{\varepsilon \rightarrow 0} A_\varepsilon(t) = 0$ because $\sqrt{\varphi} w_\varepsilon \rightarrow 0$ in $L^\infty(0,T;L^2(\Omega))$ (theorem 2.2.4)

$\varphi^{3/2} u_\varepsilon \rightarrow \varphi^{3/2} u$ in $L^\infty(0,T;L^2(\Omega))$ weak star (theorem 3.2.1)

$\varphi L^1 w_\varepsilon \rightarrow 0$ in $L^2(Q)$ (theorem 2.2.4)

$\varphi^{3/2} \nabla u_\varepsilon \rightarrow \varphi^{3/2} \nabla u$ in $L^\infty(0,T;L^2(\Omega))$ weak star (theorem 3.2.1)

Gronwall's lemma and Lebesgue's theorem allow us to conclude because $\|A_\varepsilon(t)\|$ is bounded. (ii) results from properties of functions $\varphi$.

**Remark 3.2.5.** With hypothesis $H_1$, conditions (A),(B) and $G(0)=0$, the results of theorem 3.2.3 are still valid if $f \in L^2(0,T;\mathcal{H}_0^1(\Omega))$. It is once more enough to approximate the triplet $(f; A_o; \mu_1)$ in $L^2(0,T;\mathcal{H}_0^1(\Omega)) \times \mathcal{H}_0^1(\Omega) \times L^2(\Omega)$ by a sequence $(f_\mu; A_o; \mu_1; \mu)$ satisfying hypothesis $H_2$.

### 4. APPLICATION OF NON LINEAR INTERPOLATION

The application of non linear interpolation theory (L. Tartar [11]) allows us to explicit $\varphi^{3/2}(u_\varepsilon - u)\|_2$ in $\varepsilon$, with less assumptions than in section 3, in particular without condition on $f$ over $\gamma(\Sigma_\varepsilon \cup \Lambda)$.

We first recall the theorem of non linear interpolation of [11] which will be then applied to our problem. The useful result is the following:

Let $A_o \subseteq A_1$, $B_o \subseteq B_1$ Banach spaces and $T$ a map such that $T(A_1) \subseteq B_1$, $T(A_o) \subseteq B_o$ and:

$\exists \alpha, \beta : 0 < \alpha \leqslant 1, \ 0 < \beta$ such that

$$
\| T_a - T_b \|_{B_1} \leqslant f(\| a \|_{A_1}, \| b \|_{A_1}) \| a - b \|_{A_1}^\alpha, \ \forall a, b \in A_1
$$

$$
\| T_a \|_{B_o} \leqslant g(\| a \|_{A_1}) \| a \|_{A_o}^\beta, \ \forall a \in A_o
$$
Continuous on $\mathbb{R}_+^2$, $g$ continuous on $\mathbb{R}_-^2$.

Then, if $0 \leq \theta < 1$, $1 \leq p \leq \infty$, we have:

$$\| Ta \|_{(B_0^p, B_1^p)_{\eta q}} \leq C h(\| a \|_{A_1}) (1 - \eta \alpha)^{\eta + \eta \alpha}$$

where

$$\frac{1 - \eta}{\eta} = \frac{1 - \theta}{\theta} \frac{\alpha}{\beta}$$

and

$$q = \max(1, (\frac{1 - \theta}{\beta} + \frac{\theta}{\alpha})p)$$

$$h(r) = g(2r)^{1 - \eta} f(r, 2r)^\eta$$

the space $(A_0, A_1)_{\theta, p}$ being defined by:

$$(A_0, A_1)_{\theta, p} = \left\{ a \in A_0 + A_1 \mid t^{-\theta} K(t, a) \in L^p(0, \infty; \frac{dt}{t}) \right\}$$

with the norm

$$\| a \|_{(A_0, A_1)_{\theta, p}} = \| t^{-\theta} K(t, a) \|_{L^p(0, \infty; \frac{dt}{t})}$$

with $K(t, a) = \inf a_0 \in A_0, a_1 \in A_1, a_0 + a_1 = a \| a \|_{A_0} + t \| a \|_{A_1}$

This result applied to our problem gives the:

**THEOREM 4.1.** We suppose hypothesis $H_j$, conditions (A), (B) and $G(0) = 0$,

(i) Let $0 \leq \theta < 1$, if $f \in L^2(0, T; H_j^1(\Omega); L^2(\Omega))$, for each $\varepsilon < \varepsilon_0$, we have:

$$\| \phi^{3/2}(u_\varepsilon - u) \|_{L^2(Q')} \leq K_6 \varepsilon^{2}$$

where $K_6 = C_0 \| f \|_{L^2(0, T; H_j^1(\Omega); L^2(\Omega))} + \| u_0 \|_2 + \| u_1 \|_2$

for each function $\phi$ satisfying $\phi \in H_j$, and:

$$\| u_\varepsilon - u \|_{L^2(Q')} \leq K_6 \varepsilon^{2}$$

where $Q'$ is an open set of $\Omega$ such that $Q' \cap \gamma(\Sigma_e) = \phi$.

(ii) In particular, if $f \in L^2(0, T; H_j^1(\Omega))$, $0 \leq s < \frac{1}{2}$ $\Omega$ regular, we have:

$$\| \phi^{3/2}(u_\varepsilon - u) \|_{L^2(\Omega)} \leq K_6 \varepsilon^{s/2}$$

$$\| u_\varepsilon - u \|_{L^2(\Omega)} \leq K_6 \varepsilon^{s/2}$$

**Proof.** We consider $A_0 = L^2(0, T; H_j^1(\Omega)) \times H_j^1(\Omega) \times L^2(\Omega)$, $A_1 = L^2(\Omega) \times H_j^1(\Omega) \times L^2(\Omega)$

$B_0 = B_1 = L^\infty(0, T; L^2(\Omega))$

$T = T_e : (u_0, u_1) \to \phi^{3/2}(u_\varepsilon - u)$
It results from theorems 1.2 and 2.2.4 that $T_\varepsilon$ maps $A_1$ into $B_1$ and also $A_0$ into $B_0$.

(a) We first consider $T_\varepsilon : A_1 \to B_1$. Let $(f,u_0,u_1) \in A_1$ and $(g,v_0,v_1) \in A_1$, $T_\varepsilon (f,u_0,u_1) = \varphi^{3/2}(u_\varepsilon - u)$ and $T_\varepsilon (g,v_0,v_1) = \varphi^{3/2}(v_\varepsilon - v)$.

If we put $w_\varepsilon = u_\varepsilon - v_\varepsilon$ and $w = u - v$, then :

$$\varepsilon \int_0^t \int_{\Sigma} \frac{\partial u}{\partial t} \cdot f \, d\Gamma d\tau + \int_0^t \int_{\Omega} \nabla u \cdot \nabla f \, dx \, dt$$

from where by recalling the proof of theorem 2.1.1 and taking into account $\|G(u_\varepsilon) - G(v_\varepsilon)\| \leq \|w_\varepsilon\|$, we deduce :

$$\|w_\varepsilon\|^2 \leq k_1 (\|f-g\|^2 + \|u_0-v_0\|^2 + \|u_1-v_1\|^2)$$

(4.1)

$$L_1 w + G(u) - G(v) = f - g$$

(4.2)

We take the inner product of two members of (4.2) with $w$ and we integrate from 0 to $t$. Green's formula (1.7) gives :

$$\frac{\delta}{2} \|w\|^2 + \frac{1}{2} \int_0^t \int_{\Sigma} \left( \sum_{k=1}^n b_k^2 \varepsilon_{k} \right) w^2 d\Gamma d\tau \leq k_1 \|u_0 - v_0\|^2 + \frac{1}{2} \|f-g\|^2 + k_2 \int_0^t \|w\|^2 ds$$

and Gronwall's lemma then implies that :

$$\|w\|^2 \leq k_3 (\|f-g\|^2 + \|u_0-v_0\|^2)$$

(4.3)

at last, (4.1) and (4.3) give the inequality :

$$\|T_\varepsilon (f,u_0,u_1) - T_\varepsilon (g,v_0,v_1)\|_2 \leq k_4 (\|f-g\|^2 + \|u_0-v_0\|^2 + \|u_1-v_1\|^2)^{1/2}$$

$$\forall (f,u_0,u_1) \in A_1 , \forall (g,v_0,v_1) \in A_1$$

(b) If we consider $T_\varepsilon : A_0 \to B_0$, it results from remark 3.2.5 that :

$$\forall (f,u_0,u_1) \in A_0 \quad \|T_\varepsilon (f,u_0,u_1)\|_2 \leq K_5 \varepsilon^{1/2}$$

(c) The hypotheses of the theorem of non linear interpolation are satisfied thanks to (a) and (b), with $\alpha = 1$, $\beta = 1$, $f(r,s) = k_4$, $g(r) = (C_5 \varepsilon)^{1/2}$, $p = 2$

and the application of this theorem allows us to assert that

if $(f,u_0,u_1) \in [A_0,A_1]_\theta$, then $T_\varepsilon (f,u_0,u_1) \in L^\infty(0,T; L^2(\Omega))$ and

$$\|T_\varepsilon (f,u_0,u_1)\|_2 \leq C_6 \varepsilon^{\frac{\theta}{2}} \| (f,u_0,u_1) \|_{[A_0,A_1]_\theta}$$
and point (i) follows as \([A_0,A_1]_\theta = L^2(0,T;[H^1_0(\Omega),L^2(\Omega)]) \times H^1_0(\Omega) \times L^2(\Omega)\).

The result (ii) is an application of point (i) since, for \(0 \leq s < \frac{1}{2}\),
\[ [H^1_0(\Omega) ; L^2(\Omega)]_{1-s} = H^s(\Omega). \]

5. REMARK ABOUT CORRECTORS

We can define under hypothesis \(H_1\) and condition (A) correctors in the sense of J.L. Lions [6].

Let \(g_\epsilon \in L^2(Q)\) given and \(\theta_\epsilon\) defined by

\[
\left\{ \begin{array}{ll}
\epsilon ((\theta_\epsilon + u)'',v) + \epsilon \alpha (\theta_\epsilon + u,v) + (L_1(\theta_\epsilon + u),v) + (G(\theta_\epsilon + u),v) = (f,v) + \epsilon^{1/2}(g_\epsilon,v) \\
\forall v \in H^1_0(\Omega) & \text{a.e. on } t \in ]0,T[ \\
(\theta_\epsilon + u)(x,0) = u_0, & (\theta_\epsilon + u)'(x,0) = u_1.
\end{array} \right.
\]

The theorem 1.2 ensures the existence and uniqueness of \(\theta_\epsilon + u\) such that :

\[
\theta_\epsilon + u \in L^\infty(0,T;H^1_0(\Omega)) \cap L^2(0,T;L^2(\Omega))
\]

Then \(\theta_\epsilon\) is a corrector in the following sense :

THEOREM 5.1. Under hypothesis \(H_1\), condition (A), if \(g_\epsilon \in L^2(Q)\) with \(\|g_\epsilon\|_2\) bounded independently of \(\epsilon\), we have :

\[
\|u_\epsilon - (\theta_\epsilon + u)\|_2 \leq K \sqrt{\epsilon} \text{ where } K \text{ is a positive constant independent of } \epsilon
\]

\[
u_\epsilon - (\theta_\epsilon + u) \rightharpoonup 0 \text{ weakly in } L^2(0,T;H^1_0(\Omega))
\]

\[
u_\epsilon' - (\theta_\epsilon + u)' \rightharpoonup 0 \text{ weakly in } L^2(Q).
\]

Proof. We consider \(w_\epsilon = u_\epsilon - (\theta_\epsilon + u)\) which verifies

\[
\left\{ \begin{array}{ll}
\epsilon (w_\epsilon'',v) + \epsilon \alpha (w_\epsilon,v) + (L_1 w_\epsilon,v) + (G(u_\epsilon) - G(\theta_\epsilon + u),v) = -\epsilon^{1/2}(g_\epsilon,v) \\
w_\epsilon(x,0) = 0, w_\epsilon'(x,0) = 0
\end{array} \right.
\]

and we follow once more the method of the proof of theorem 2.1.1.

We first suppose that \(g'_\epsilon \in L^2(Q)\) and hypothesis \(H_2\).

We obtain by the same arguments, taking into account \(\|G(u_\epsilon) - G(\theta_\epsilon + u)\| \leq \ell \|w_\epsilon\|\) the inequality :
|w_e|_2 + \sqrt{e} \|w'_e\|_2 + \sqrt{\epsilon} \|w_e\|_2 \leq k_1 \|g_e\|_2 \sqrt{\epsilon}

(4.4)

It is enough then to approximative in L^2(Q) × L^2(Q) × H^1_0(Ω) × L^2(Ω) (f; g; u_0; u_1) by
\{(f_\mu, g_{e, \mu}; u_0, u_1, \mu)\} satisfying hypothesis H_2 with g_{e, \mu} \in L^2(Q) to assert that (4.4) is still valid under hypothe-

ses of the theorem which thus is proved.
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